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ON SOME FENG QI TYPE Q-INTEGRAL INEQUALITIES

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Abstract. In this paper are given several Feng Qi type q-integral inequalities, by using elementary analytic methods in Quantum Calculus.

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1. Introduction

In [7] the following problem was posed: Under what conditions does the inequality

$$\int_a^b [f(x)]^t dx \geq \Big[\int_a^b f(x)dx\Big]^{t-1} \tag{1.1}$$

holds for $t > 1$? In [1] has been proved the following: Let $[a, b]$ be a closed interval of $\mathbb{R}$ and let $p \geq 1$ be a real number. For any real continuous function f on $[a, b]$, differentiable on $]a, b[$, such that $f(a) \geq 0$, and $f'(x) \geq p$ for all $x \in]a, b[$, we have that

$$\int_a^b [f(x)]^{p+2} dx \geq \frac{1}{(b-a)^{p-1}}\Big[\int_a^b f(x)dx\Big]^{p+1}. \tag{1.2}$$

In [2] has obtained the q-analogue of the previous result as follows. Let $p \geq 1$ be a real number and f a function defined on $[a, b]_q$ (see below for the definitions and notation), such that $f(a) \geq 0$, and $D_q f(x) \geq p$ for all $x \in (a, b]_q$. Then

$$\int_a^b [f(x)]^{p+2} d_q x \geq \frac{1}{(b-a)^{p-1}}\Big[\int_a^b f(qx)d_q x\Big]^{p+1}. \tag{1.3}$$

The aim of this paper is to extend this result. This paper will also provide some more sufficient conditions such that inequalities presented in [6] are valid.

2. Notations and preliminaries

For the convenience of the reader, we provide a summary of notations and definitions used in this paper. For details, one may refer to [3] and [5].

Let $q \in (0,1)$. The q-analog of the derivative of a function f, denoted by $D_q f$ is given by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, x \neq 0. \tag{2.4}$$

If $f'(0)$ exists, then $D_q f(0) = f'(0)$. As q tends to 1^-, the q-derivative reduces to the usual derivative.

The q-Jackson integral from 0 to $a \in \mathbb{R}$ is defined by (see [4])

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^{\infty} f(aq^n) q^n. \tag{2.5}$$

The q-Jackson integral on a general interval $[a,b]$ may be defined by (see [5])

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \tag{2.6}$$

For any function f one has

$$D_q\Big(\int_a^x f(t) d_q t \Big) = f(x). \tag{2.7}$$

For $b > 0$ and $a = bq^n$, $n \in \mathbb{N}$ denote

$$[a,b]_q = \{bq^k : 0 \leq k \leq n\}; \qquad (a,b]_q = [q^{-1}a, b]_q.$$

3. Main Results

In order to prove our main results we need the following Lemma from [2].

Lemma 3.1 *Let $p \geq 1$ and g be a non-negative monotonic function on $[a,b]_q$. Then*

$$pg^{p-1}(qx) D_q g(x) \leq D_q[g^p(x)] \leq pg^{p-1}(x) D_q g(x), \;\; x \in (a,b]_q. \tag{3.8}$$

Theorem 3.2 *If f is a non-negative increasing function on $[a,b]_q$ and satisfies*

$$(\alpha-1) f^{\alpha-2}(qx) D_q f(x) \geq \frac{\beta(\beta-1)}{(b-a)^{2\beta-\alpha-1}} (x-a)^{\beta-2} f^{(\beta-1)}(x) \tag{3.9}$$

for $\alpha \geq 1$ and $\beta \geq 2$, then

$$\int_a^b [f(x)]^{\alpha} d_q x \geq \frac{1}{(b-a)^{2\beta-\alpha-1}} \Big[\int_a^b f(x) d_q x \Big]^{\beta}. \tag{3.10}$$

Proof. For $x \in [a,b]_q$, let

$$F(x) = \int_a^x [f(t)]^\alpha d_q t - \frac{1}{(b-a)^{2\beta-\alpha-1}} \cdot \Big[\int_a^x f(t) d_q t\Big]^\beta$$

and $h(x) = \int_a^x f(t) d_q t$. By virtue of Lemma 3.1, it follows that

$$\begin{aligned} D_q F(x) &= f^\alpha(x) - \frac{1}{(b-a)^{2\beta-\alpha-1}} D_q(h^\beta(x)) \\ &\geq f^\alpha(x) - \frac{\beta}{(b-a)^{2\beta-\alpha-1}} h^{\beta-1}(x) D_q h(x) \\ &\geq f^\alpha(x) - \frac{\beta}{(b-a)^{2\beta-\alpha-1}} h^{\beta-1}(x) f(x) \\ &= f(x) F_1(x), \end{aligned}$$

where $F_1(x) = f^{\alpha-1}(x) - \frac{\beta}{(b-a)^{2\beta-\alpha-1}} h^{\beta-1}(x)$.

By Lemma 3.1 we have

$$\begin{aligned} D_q F_1(x) &= D_q(f^{\alpha-1}(x)) - \frac{\beta}{(b-a)^{2\beta-\alpha-1}} D_q(h^{\beta-1}(x)) \\ &\geq (\alpha-1) f^{\alpha-2}(qx) D_q f(x) - \frac{\beta(\beta-1)}{(b-a)^{2\beta-\alpha-1}} h^{\beta-2}(x) f(x). \end{aligned}$$

Since f is a non-negative and increasing function, then

$$h(x) = \int_a^x f(t) d_q t \leq f(x)(x-a),$$

hence

$$D_q F_1(x) \geq (\alpha-1) f^{\alpha-2}(qx) D_q f(x) - \frac{\beta(\beta-1)}{(b-a)^{2\beta-\alpha-1}} (x-a)^{\beta-2} f^{\beta-1}(x)$$

which means that $D_q F_1(x) \geq 0$. Since $F_1(a) = f^{\alpha-1}(a) \geq 0$, we obtain $F_1(x) \geq 0$. Since $F(a) = 0$ and $D_q F(x) \geq 0$ it follows that $F(x) \geq 0$, for all $x \in [a,b]_q$, which completes the proof. □

Theorem 3.3 *If f is a non-negative and increasing function on $[bq^m, b]_q$, $m \in \mathbb{N}$ and satisfies*

$$(\alpha-1) D_q f(x) \geq \frac{\beta(\beta-1)}{(b-a)^{2\beta-\alpha-1}} \cdot (x-a)^{\beta-2} \cdot f^{\beta-\alpha+1}(q^m x) \tag{3.11}$$

on $[a,b]_q$ for $\alpha \geq 1$ and $\beta \geq 2$, then

$$\int_a^b [f(x)]^\alpha d_q x \geq \frac{1}{(b-a)^{2\beta-\alpha-1}} \Big[\int_a^b f(q^m x) d_q x\Big]^\beta. \tag{3.12}$$

Proof. For $x \in [a,b]_q$ let
$$F(x) = \int_a^x [f(t)]^\alpha d_q t - \frac{1}{(b-a)^{2\beta-\alpha-1}} \Big[\int_a^x f(q^m t) d_q t \Big]^\beta$$
and $h(x) = \int_a^x f(q^m t) d_q t$. Utilizing Lemma 3.1 gives that
$$\begin{aligned} D_q F(x) &= f^\alpha(x) - \frac{1}{(b-a)^{2\beta-\alpha-1}} D_q(h^\beta(x)) \\ &\geq f^\alpha(x) - \frac{\beta}{(b-a)^{2\beta-\alpha-1}} h^{\beta-1}(x) D_q h(x) \\ &\geq f^\alpha(x) - \frac{\beta}{(b-a)^{2\beta-\alpha-1}} h^{\beta-1}(x) f(x) \\ &= f(x) F_1(x), \end{aligned}$$
where $F_1(x) = f^{\alpha-1}(x) - \frac{\beta}{(b-a)^{2\beta-\alpha-1}} h^{\beta-1}(x)$. By Lemma 3.1, we obtain that
$$\begin{aligned} D_q F_1(x) &= D_q[f^{\alpha-1}(x)] - \frac{\beta}{(b-a)^{2\beta-\alpha-1}} D_q(h^{\beta-1}(x)) \\ &\geq (\alpha-1) f^{\alpha-2}(qx) D_q f(x) - \frac{\beta(\beta-1)}{(b-a)^{2\beta-\alpha-1}} h^{\beta-2}(x) f(q^m x). \end{aligned}$$
Since f is a non-negative and increasing function, then
$$h(x) = \int_a^x f(q^m x) d_q t \leq f(q^m x)(x-a),$$
hence
$$\begin{aligned} D_q F_1(x) &\geq (\alpha-1) f^{\alpha-2}(qx) D_q F(x) - \frac{\beta(\beta-1)}{(b-a)^{2\beta-\alpha-1}} (x-a)^{\beta-2} f^{\beta-1}(q^m x) \\ &= f^{\alpha-2}(qx) \Big[(\alpha-1) D_q f(x) - \frac{\beta(\beta-1)}{(b-a)^{2\beta-\alpha-1}} (x-a)^{\beta-2} f^{\alpha-\beta+1}(q^m x) \Big], \end{aligned}$$
which means that $D_q F_1(x) \geq 0$. Since $F_1(a) = f^{\alpha-1}(a) \geq 0$, we obtain $F_1(x) \geq 0$. Since $F(a) = 0$ and $D_q F(x) \geq 0$ it follows that $F(x) \geq 0$, for all $x \in [a,b]_q$, as claimed. □

Corollary 3.4 *If f is a non-negative increasing function on $[a,b]_q$ and satisfies*
$$f^p(qx) D_q f(x) \geq \frac{p}{(b-a)^{2\beta-\alpha-1}} f^p(x)(x-a)^{p-1} \tag{3.13}$$
for $p \geq 0$, then
$$\int_a^b f^{p+2}(x) d_q(x) \geq \frac{1}{(b-a)^{p-1}} \Big[\int_a^b f(x) d_q(x) \Big]^{p+1}. \tag{3.14}$$

Proof. In Theorem 3.2 put $\alpha = p+2, \beta = p+1$. □

At the end of the notes we pose the following problem. Under what conditions does the inequality

$$\int_a^b [f(x)]^\alpha \leq \frac{1}{(b-a)^\alpha}\Big[\int_a^b x^\beta f(x) d_q x\Big]^\beta$$

holds for $\alpha \geq 1, \beta \geq 1$?

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