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ON A GENERALIZED DOUBLE DIFFERENCE SEQUENCE SPACES DEFINED BY A $\chi-$ SEQUENCE OF MODULUS FUNCTIONS

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Abstract. The idea of single difference sequence spaces was introduced by Kizmaz and this concept was generalized by various authors. In this paper, we define the sequence spaces $\chi^2\left(\Delta_u^\gamma, M_{mn}, p, s\right)$ and $\Lambda^2\left(\Delta_u^\gamma, M_{mn}, p, s\right)$, where $M = (M_{mn})$ is a sequence of modulus functions, and examine some inclusion relations and properties of these spaces.

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1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich[4]. Later on, they were investigated by Hardy[8], Moricz[12], Moricz and Rhoades[13], Basarir and Solankan[2], Tripathy[20], Colak and Turkmenoglu[6], Turkmenoglu[22], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u\left(t\right) := \left\{(x_{mn}) \in w^2 : sup_{m,n\in N}\left|x_{mn}\right|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p\left(t\right) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty}\left|x_{mn} - l\right|^{t_{mn}} = 1\, for\, some\, l \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}\left(t\right) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty}\left|x_{mn}\right|^{t_{mn}} = 1\right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n \to \infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [27,28] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [29] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [30] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [31] and Mursaleen and Edely [32] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [33] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [34] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [35] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.
We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a + b)^p \leq a^p + b^p \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij}(m, n \in \mathbb{N})$ (see[1]).

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2.A sequence $x = (x_{mn})$ is called double entire sequence if $|x_{mn}|^{1/m+n} \to 0$ as $m, n \to \infty$. The double entire sequences will be denoted by Γ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{all\, finite\, sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space (or a metric space) X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})$ $(m, n \in \mathbb{N})$ are also continuous.

Orlicz[16] used the idea of Orlicz function to construct the space $\left(L^M\right)$. Lindenstrauss and Tzafriri [10] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p \, (1 \leq p < \infty)$. subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [17], Mursaleen et al. [14], Bektas and Altin [3], Tripathy et al. [21], Rao and Subramanian [18], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [9].

Recalling [16] and [9], an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [15] and further discussed by Ruckle [19] and Maddox [11], and many others.

An Orlicz function M is said to satisfy the Δ_2- condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u) \, (u \geq 0)$. The Δ_2- condition is equivalent to $M(\ell u) \leq K\ell M(u)$, for all values of u and for $\ell > 1$.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \, for\, some\, \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.
If X is a sequence space, we give the following definitions:

(i) X'= the continuous dual of X;

(ii) $X^{\alpha} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty,\ for\ \ each\ \ x \in X\right\}$;

(iii) $X^{\beta} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn}\,is\ \ convegent,\ \ for\ \ each\ \ x \in X\right\}$;

(iv) $X^{\gamma} = \left\{a = (a_{mn}) : sup_{mn} \geq 1 \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, for\ \ each\ \ x \in X\right\}$;

(v) $let\,X\,beanFK - space \supset \phi;\,then\,X^f = \left\{f(\Im_{mn}) : f \in X'\right\}$;

(vi) $X^{\delta} = \left\{a = (a_{mn}) : sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty,\ for\ \ each\ \ x \in X\right\}$;

$X^{\alpha}.X^{\beta}, X^{\gamma}$ are called $\alpha-$ $(orK\ddot{o}the - Toeplitz)$ dual of $X, \beta -$ $(or\,generalized - K\ddot{o}the - Toeplitz)\,dual\,of\,X, \gamma -\,dual\,of\,X,\,\delta -\,dual\,of\,X\,respectively.X^{\alpha}$ is defined by Gupta and Kamptan [24]. It is clear that $x^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\alpha} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [36] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here w, c, c_0 and ℓ_{∞} denote the classes of all, convergent,null and bounded scalar valued single sequences respectively. The above spaces are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1} |\Delta x_k|$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \Gamma^2$ and χ^2 respectively. $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$

Let $r \in \mathbb{N}$ be fixed, then

$$Z(\Delta^r) = \{(x_{mn}) : (\Delta^r x_{mn}) \in Z\}\ for\,Z = \chi^2, \Gamma^2\,and\,\Lambda^2$$

where $\Delta^r x_{mn} = \Delta^{r-1} x_{mn} - \Delta^{r-1} x_{m,n+1} - \Delta^{r-1} x_{m+1,n} + \Delta^{r-1} x_{m+1,n+1}$.

Now we introduced a generalized difference double operator as follows:

Let $r, \gamma \in \mathbb{N}$ be fixed, then

$$Z\left(\Delta_\gamma^r\right) = \left\{(x_{mn}) : \left(\Delta_\gamma^r x_{mn}\right) \in Z\right\} \; for\, Z = \chi^2, \Gamma^2 \, and \, \Lambda^2$$

where $\Delta_\gamma^r x_{mn} = \Delta_\gamma^{r-1} x_{mn} - \Delta_\gamma^{r-1} x_{m,n+1} - \Delta_\gamma^{r-1} x_{m+1,n} + \Delta_\gamma^{r-1} x_{m+1,n+1}$ and $\Delta_\gamma^0 x_{mn} = x_{mn}$ for all $m, n \in \mathbb{N}$.

The notion of a modulus function was introduced by Nakano [15]. We recall that a modulus f is a function from $[0, \infty) \to [0, \infty)$, such that
(1) $f(x) = 0$ if and only if $x = 0$
(2) $f(x + y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$,
(3) f is increasing,
(4) f is continuous from the right at o. Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from condition (iv) that f is continuous on $[0, \infty)$.

It is immediate from (ii) and (iv) that f is continuous on $[0, \infty)$. Also from codition (ii), we have $f(nx) \leq nf(x)$ for all $n \in \mathbb{N}$ and $n^{-1} f(x) \leq f\left(xn^{-1}\right)$, for all $n \in \mathbb{N}$.
Remark: If f is a modulus function, then the composition $f^s = f \cdot f \cdots f$ $(s\, times)$ is also a modulus function, where s is a positive integer.

Let $p = (p_{mn})$ be a sequence of positive real numbers. We have the following well known inequality, which will be used throughout this paper:

$$|a_{mn} + b_{mn}|^{p_{mn}} \leq D\left(|a_{mn}|^{p_{mn}} + |b_{mn}|^{p_{mn}}\right) \qquad (2)$$

where a_{mn} and b_{mn} are complex numbers, $D = max\left\{1, 2^{H-1}\right\}$ and $H = sup_{mn} p_{mn} < \infty$.

2. Definitions and Notations:

A paranorm on a linear topological space X is a function $g : X \to R$ which satisfies the following axioms: For any $x, y, x_0 \in X$ and $\lambda, \lambda_0 \in \mathbb{C}$, the set of complex numbers,

(i) $g(\theta) = 0$, where $\theta = \begin{pmatrix} 0, & 0, & ...0 \\ 0, & 0, & ...0 \\ . & & \\ . & & \\ . & & \\ 0, & 0, & ...0 \end{pmatrix}$, $the\, zero\, sequence$,

(ii) $g(x) = g(-x)$
(iii) $g(x+y) \leq g(x) + g(y)$ $(subadditivity)$, and
(iv) the scalar multiplication is continuous, that is $\lambda \to \lambda_0, x \to x_0$ imply $\lambda x \to \lambda_0 x_0$; in other rowrds, $|\lambda - \lambda_0| \to 0, g(x - x_0) \to 0.$
A paranormed space is a linear space X with a paranorm g and is written (X, g) , (See [47],p.92).

Any function g which satisfies all the conditions (i)-(iv) together with the condition.
(v) $g(x) = 0$ if and only if $x = \theta$, is called a total paranorm on X, and the pair (X, g) is called a total paranormed space, (See [47], p.92).

Let U be the set of all sequences $u = (u_{mn})$ such that $u_{mn} \neq 0\,(m, n = 1, 2, 3, \cdots)$.

In this paper, we generalize the following sequence spaces:
Let $M = (M_{mn})$ be a sequence of modulus function and γ be a positive integer, and using the notation $\Delta_u^\gamma x_{mn}$ for $u_{mn}\Delta_{x_{mn}}^\gamma$, we define
$\chi^2(\Delta_u^\gamma, M_{mn}, s) =$
$$\left\{x \in w^2 : lim_{m,n\to\infty}(mn)^{-s}\left[M_{mn}\left(\frac{\left((m+n)!\left|\Delta_u^\gamma x_{mn}\right|\right)^{1/m+n}}{\rho}\right)\right] = 0, for\ some\ \rho > 0, s \geq 0,\right\}$$
and
$\Lambda^2(\Delta_u^\gamma, M_{mn}, s) =$
$$\left\{x \in w^2 : sup_{mn}(mn)^{-s}\left[M_{mn}\left(\frac{\left((m+n)!\left|\Delta_u^\gamma x_{mn}\right|\right)^{1/m+n}}{\rho}\right)\right] < \infty, for\ some\ \rho > 0, s \geq 0,\right\}$$
where $\Delta_u^\gamma x_{mn} = \left(\Delta_u^{\gamma-1}x_{mn} - \Delta_u^{\gamma-1}x_{mn+1} - \Delta_u^{\gamma_1}x_{m+1n} + \Delta_u^{\gamma_1}x_{m+1n+1}\right), \Delta_u^0 x_{mn} = (u_{mn}x_{mn}), \Delta_u x_{mn} = (u_{mn}x_{mn} - u_{mn+1}x_{mn+1} - u_{m+1n}x_{m+1n} + u_{m+1n+1}x_{m+1n+1}).$

3.Main Results

We prove the following theorems:
Theorem 3.1 $\Lambda^2(\Delta_u^\gamma, M_{mn}, s)$ is a Banach space with the metric
$$d(x, y) = inf\left\{\rho > 0 : sup_{mn}(mn)^{-s} M_{mn}\left(\frac{\left|\Delta_u^\gamma x_{mn} - \Delta_u^\gamma y_{mn}\right|^{1/m+n}}{\rho}\right) \leq 1\right\}$$
Proof: Let (x^i) be any Cauchy sequence in $\Lambda^2(\Delta_u^\gamma, M_{mn}, s)$ where $x^i = (x_{mn}^i) =$
$$\begin{pmatrix} x_{11}^i, & x_{12}^i, & ...x_{1n}^i \\ x_{21}^i, & x_{22}^i, & ...x_{2n}^i \\ . & & \\ . & & \\ . & & \\ x_{21}^i, & x_{22}^i, & ...x_{2n}^i \end{pmatrix} \in \Lambda^2(\Delta_u^\gamma, M_{mn}, s), for\, each\, i \in \mathbb{N}.$$
Let $r, x_0 > 0$ be fixed. Then for each $\frac{\epsilon}{rx_0} > 0$ there exists a positive integer L such

that $\left(x^i - y^i\right) - \left(x^j_{\Delta^\gamma_u} - y^j_{\Delta^\gamma_u}\right) < \frac{\epsilon}{rx_0}, for\, all\, i, j \geq L$. Using the definition of metric, we have

$sup_{mn} (mn)^{-s} \left[M_{mn}\left(\frac{\left|\left(\Delta^\gamma_u x^i_{mn} - \Delta^\gamma_u y^i_{mn}\right) - \left(\Delta^\gamma_u x^j_{mn} - \Delta^\gamma_u y^j_{mn}\right)\right|^{1/m+n}}{\left(x^i - y^i\right) - \left(x^j_{\Delta^\gamma_u} - y^j_{\Delta^\gamma_u}\right)}\right)\right] \leq 1, for\, all\, m, n \geq 0, and\, for\, all i, j \geq L.$

Therefore one can find that there exists $r > 0$ with $(mn)^{-s} M_{mn}\left(\frac{rx_0}{2}\right) \geq 1$, such that

$(mn)^{-s}\left[M_{mn}\left(\frac{\left|\left(\Delta^\gamma_u x^i_{mn} - \Delta^\gamma_u y^i_{mn}\right) - \left(\Delta^\gamma_u x^j_{mn} - \Delta^\gamma_u y^j_{mn}\right)\right|^{1/m+n}}{\left(x^i - y^i\right) - \left(x^j_{\Delta^\gamma_u} - y^j_{\Delta^\gamma_u}\right)}\right)\right] \leq (mn)^{-s} M_{mn}\left(\frac{rx_0}{2}\right).$

This implies that $\left|\left(\Delta^\gamma_u x^i_{mn} - \Delta^\gamma_u y^i_{mn}\right) - \left(\Delta^\gamma_u x^j_{mn} - \Delta^\gamma_u y^j_{mn}\right)\right|^{1/m+n} \leq \frac{rx_0}{2}\frac{\epsilon}{rx_0} = \frac{\epsilon}{2}$. Since $u_{mn} \neq 0$ for all m, n, we get that

$\left|\left(\Delta^\gamma_u x^i_{mn} - \Delta^\gamma_u y^i_{mn}\right) - \left(\Delta^\gamma_u x^j_{mn} - \Delta^\gamma_u y^j_{mn}\right)\right|^{1/m+n} \leq \frac{\epsilon}{2}, for\, all\, i, j \geq L.$

Hence $\left(\Delta^\gamma_u x^i_{mn} - \Delta^\gamma_u y^i_{mn}\right)$ is a Cauchy sequence in $\mathbb{R}$. Therefore for each $\epsilon\,(0 < \epsilon < 1)$ there exists a positive integer L such that

$\left|\left(\Delta^\gamma_u x^i_{mn} - \Delta^\gamma_u y^i_{mn}\right) - \left(\Delta^\gamma_u x^j_{mn} - \Delta^\gamma_u y^j_{mn}\right)\right|^{1/m+n} \leq \epsilon, for\, all\, i \geq L$. Now, using the continuty of M_{mn} for each mn, we get that

$sup_{mn \geq L} (mn)^{-s}\left[M_{mn}\left(\frac{\left|\left(\Delta^\gamma_u x^i_{mn} - \Delta^\gamma_u y^i_{mn}\right) - lim_{j\to\infty}\left(\Delta^\gamma_u x^j_{mn} - \Delta^\gamma_u y^j_{mn}\right)\right|^{1/m+n}}{\rho}\right)\right] \leq 1$. Thus $sup_{mn \geq L} (mn)^{-s}\left[M_{mn}\left(\frac{\left|\left(\Delta^\gamma_u x^i_{mn} - \Delta^\gamma_u y^i_{mn}\right) - \left(\Delta^\gamma_u x_{mn} - \Delta^\gamma_u y_{mn}\right)\right|^{1/m+n}}{\rho}\right)\right] \leq 1$. Taking infimum of such $\rho' s$ we have

$inf\left\{\rho > 0 : sup_{mn \geq L} (mn)^{-s}\left[M_{mn}\left(\frac{\left|\left(\Delta^\gamma_u x^i_{mn} - \Delta^\gamma_u y^i_{mn}\right) - \left(\Delta^\gamma_u x_{mn} - \Delta^\gamma_u y_{mn}\right)\right|^{1/m+n}}{\rho}\right)\right] \leq 1\right\}$

$\leq \epsilon$, for all $i \geq L$ and $j \to \infty$. Since $\left(x^i\right) \in \Lambda^2\left(\Delta^\gamma_u, M_{mn}, s\right)$, and M_{mn} is an modulus function for each m, n and therefore continuous, we get that $x \in \Lambda^2\left(\Delta^\gamma_u, M_{mn}, s\right)$. This completes the proof.

Theorem 3.2 Let (M_{mn}) be a sequence of modulus function such that M_{mn} satisfies the Δ_2- condition for each mn. Then (i) $\Lambda^2\left(\Delta^\gamma_u, s\right) \subset \Lambda^2\left(\Delta^\gamma_u, M_{mn}, s\right)$,
(ii) $\chi^2\left(\Delta^\gamma_u, s\right) \subset \chi^2\left(\Delta^\gamma_u, M_{mn}, s\right)$.

Proof: (i) Let $x \in \Lambda^2\left(\Delta^\gamma_u, s\right)$, the $\left|\Delta^\gamma_u x_{mn}\right|^{1/m+n} \leq L, for\, all\, m, n$. Therefore $(mn)^{-s}\left[M_{mn}\left(\frac{\left|\Delta^\gamma_u x_{mn}\right|^{1/m+n}}{\rho}\right)\right] \leq (mn)^{-s}\left[M_{mn}\left(\frac{L}{\rho}\right)\right] \leq (mn)^{-s} KHM_{mn}(L)$, for each mn,by the Δ^2- condition. Hence $sup_{mn} (mn)^{-s}\left[M_{mn}\left(\frac{\left|\Delta^\gamma_u x_{mn}\right|^{1/m+n}}{\rho}\right)\right] < \infty$. That is $\Lambda^2\left(\Delta^\gamma_u, s\right) \subset \Lambda^2\left(\Delta^\gamma_u, M_{mn}, s\right)$.

(ii) Let $x \in \chi^2\left(\Delta^\gamma_u, s\right)$, then $\left((m+n)!\left|\Delta^\gamma_u x_{mn}\right|\right)^{1/m+n} \to 0\, as\, m, n \to \infty$. Therefore

$(mn)^{-s}\left[M_{mn}\left(\frac{((m+n)!\left|\Delta_u^\gamma x_{mn}\right|)^{1/m+n}}{\rho}\right)\right] \leq (mn)^{-s}\, KhM_{mn}\left(\frac{((m+n)!\left|\Delta_u^\gamma x_{mn}\right|)^{1/m+n}}{\rho}\right),$
for each m,n by the Δ_2- condition. Hence
$(mn)^{-s}\left[M_{mn}\left(\frac{((m+n)!\left|\Delta_u^\gamma x_{mn}\right|)^{1/m+n}}{\rho}\right)\right] \to 0$ as $m,n \to \infty$. That is $\chi^2\left(\Delta_u^\gamma, s\right) \subset \chi^2\left(\Delta_u^\gamma, M_{mn}, s\right)$. This completes the proof.

Theorem 3.3 Let (M_{mn}) be a sequence of modulus functions. Then
(i)$\Lambda^2\left(\Delta_u^0, M_{mn}, s\right) \subset \Lambda^2\left(\Delta_u^\gamma, M_{mn}, s\right)$, (ii) $\chi^2\left(\Delta_u^0, M_{mn}, s\right) \subset \chi^2\left(\Delta_u^\gamma, M_{mn}, s\right)$.

Proof: It is trivial, so we omit it.

4. Paranormed Double Sequence Spaces

Let $p = (p_{mn})$ be a sequence of positive real numbers, $M = (M_{mn})$ be a sequence of modulus function and γ be a positive integer. We define
$\chi^2\left(\Delta_u^\gamma, M_{mn}, p, s\right) =$
$$\left\{x \in w^2 : \lim_{m,n\to\infty} (mn)^{-s}\left[M_{mn}\left(\frac{((m+n)!\left|\Delta_u^\gamma x_{mn}\right|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}} = 0, for\ some\ \rho > 0, s \geq 0,\right\}$$
and
$\Lambda^2\left(\Delta_u^\gamma, M_{mn}, p, s\right) =$
$$\left\{x \in w^2 : \sup_{mn} (mn)^{-s}\left[M_{mn}\left(\frac{((m+n)!\left|\Delta_u^\gamma x_{mn}\right|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}} < \infty, for\ some\ \rho > 0, s \geq 0,\right\}$$
where $\Delta_u^\gamma x_{mn} = \left(\Delta_u^{\gamma-1} x_{mn} - \Delta_u^{\gamma-1} x_{mn+1} - \Delta_u^{\gamma_1} x_{m+1n} + \Delta_u^{\gamma_1} x_{m+1n+1}\right), \Delta_u^0 x_{mn} = (u_{mn}x_{mn}), \Delta_u x_{mn} = (u_{mn}x_{mn} - u_{mn+1}x_{mn+1} - u_{m+1n}x_{m+1n} + u_{m+1n+1}x_{m+1n+1})$.
If $(M_{mn}) = M$ for all $m, n, s = 0$ and $\gamma = 1$, then these spaces reduce to
$\chi^2\left(\Delta_u^\gamma, M_{mn}, p\right) =$
$$\left\{x \in w^2 : lim_{m,n\to\infty}\left[M_{mn}\left(\frac{((m+n)!|\Delta_u x_{mn}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}} = 0, for\, some\, \rho > 0\right\}$$
$\Lambda^2\left(\Delta_u^\gamma, M_{mn}, p\right) =$
$$\left\{x \in w^2 : sup_{mn}\left[M_{mn}\left(\frac{((m+n)!|\Delta_u x_{mn}|)^{1/m+n}}{\rho}\right)\right]^{p_{mn}} < \infty, for\, some\, \rho > 0\right\}$$
These spaces are paranormed spaces with
$G_u^\gamma(x) = inf\left\{\rho^{p_m/H} > 0 : sup_{mn\geq 1}(mn)^{-s}\left[M_{mn}\left(\frac{\left|\Delta_u^\gamma x_{mn}\right|^{1/m+n}}{\rho}\right)\right]^{p_{mn}/H} \leq 1\right\}$, where
$H = max\left(1, supp_{mn}\right)$
Now, we prove the following theorems

Theorem 4.1 $\Lambda^2\left(\Delta_u^\gamma, M_{mn}, p, s\right)$ is a paranormed space with
$G_u^\gamma(x) = inf\left\{\rho^{p_m/H} > 0 : sup_{mn\geq 1}(mn)^{-s}\left[M_{mn}\left(\frac{\left|\Delta_u^\gamma x_{mn}\right|^{1/m+n}}{\rho}\right)\right]^{p_{mn}/H} \leq 1\right\}$ if and only if $h = inf p_{mn} > 0$, where $H = max\left(1, supp_{mn}\right)$

(ii)$\Lambda^2\left(\Delta_u^\gamma, M_{mn}, p, s\right)$ is a complete paranormed linear metric space if the condition (i) is satisfied.

Proof: (i) Sufficiency: Let $h > 0$. It is trivial that $g(\theta) = 0$ and $G_u^\gamma(-x) = G_u^\gamma(x)$. The inequality $G_u^\gamma(x+y) \leq G_u^\gamma(x) + G_u^\gamma(y)$ follows from the inequality (2), since $p_{mn}/H \leq 1$ for all positive integers m, n. We also may write $G_u^\gamma(\lambda x) \leq max\left(|\lambda|, |\lambda|^{h/H}\right) G_u^\gamma(x)$, since $|\lambda|^{p_{mn}} \leq max\left(|\lambda|^h, |\lambda|^H\right)$ for all positive integers m, n and for any $\lambda \in \mathbb{C}$, the set of complex numbers. Using this inequality, it can be proved that $\lambda x \to \theta$, when x is fixed and $\lambda \to 0$, or $\lambda \to 0$ and $x \to \theta$, or λ is fixed and $x \to \theta$.

Necessity: Let $\Lambda^2\left(\Delta_u^\gamma, M_{mn}, p, s\right)$ be a paranormed space with the paranormed

$G_u^\gamma(x) = inf\left\{\rho^{p_m/H} > 0 : sup_{mn\geq 1}(mn)^{-s}\left[M_{mn}\left(\frac{\left|\Delta_u^\gamma x_{mn}\right|^{1/m+n}}{\rho}\right)\right]^{p_{mn}/H} \leq 1\right\}$, and

suppose that $h = 0$. Since $|\lambda|^{p_m/H} \leq |\lambda|^{h/H} = 1$ for all positive integers m, n and $\lambda \in \mathbb{C}$ such that $0 < |\lambda| \leq 1$, we have

$inf\left\{sup_{mn\geq 1}(mn)^{-s}\left[M_{mn}\left(\frac{|\lambda|^{p_{mn}/H}}{\rho}\right)\right] \leq 1\right\} = 1$. Hence it follows that

$G_u^\gamma(\lambda x) = inf\left\{sup_{mn\geq 1}(mn)^{-s}\left[M_{mn}\left(\frac{|\lambda|^{p_{mn}/H}}{\rho}\right)\right] \leq 1\right\} = 1$

for $x = (\alpha) \in \Lambda^2\left(\Delta_u^\gamma, M_{mn}, p, s\right)$ is a paranormed space with $G_u^\gamma(x)$.

(ii) The proof is clear.

Theorem 4.2 Let $0 < p_{mn} \leq q_{mn} < \infty$ for each mn. Then $\chi^2\left(\Delta_u^\gamma, M_{mn}, p, s\right) \subseteq \chi^2\left(\Delta_u^\gamma, M_{mn}, q, s\right)$

Proof: Let $x \in \chi^2\left(\Delta_u^\gamma, M_{mn}, p, s\right)$. Then there exists some $\rho > 0$ such that

$lim_{m,n\to\infty}(mn)^{-s}\left[M_{mn}\left(\frac{\left((m+n)!\left|\Delta_u^\gamma x_{mn}\right|\right)^{1/m+n}}{\rho}\right)\right]^{p_{mn}} = 0$ This implies that

$(mn)^{-s}\left[M_{mn}\left(\frac{\left((m+n)!\left|\Delta_u^\gamma x_{mn}\right|\right)^{1/m+n}}{\rho}\right)\right]^{p_{mn}} \leq 1$ for sufficiently large m, n, since M_{mn} is non-decreasing for each m, n. Hence

$lim_{m,n\to\infty}(mn)^{-s}\left[M_{mn}\left(\frac{\left((m+n)!\left|\Delta_u^\gamma x_{mn}\right|\right)^{1/m+n}}{\rho}\right)\right]^{q_{mn}} \leq$

$lim_{m,n\to\infty}(mn)^{-s}\left[M_{mn}\left(\frac{\left((m+n)!\left|\Delta_u^\gamma x_{mn}\right|\right)^{1/m+n}}{\rho}\right)\right]^{p_{mn}} = 0$ that is, $x \in \chi^2\left(\Delta_u^\gamma, M_{mn}, q, s\right)$.

This completes the proof.

Theorem 4.3 (i) $1 \leq infp_{mn} \leq p_{mn} \leq 1$. Then $\chi^2\left(\Delta_u^\gamma, M_{mn}, p, s\right) \subseteq \chi^2\left(\Delta_u^\gamma, M_{mn}, s\right)$

(ii) Let $1 \leq p_{mn} \leq supp_{mn} < \infty$. Then $\chi^2\left(\Delta_u^\gamma, M_{mn}, s\right) \subseteq \chi^2\left(\Delta_u^\gamma, M_{mn}, p, s\right)$.

Proof: (i) Let $x \in \chi^2\left(\Delta_u^\gamma, M_{mn}, p, s\right)$.

$lim_{m,n\to\infty}(mn)^{-s}\left[M_{mn}\left(\frac{\left((m+n)!\left|\Delta_u^\gamma x_{mn}\right|\right)^{1/m+n}}{\rho}\right)\right]^{p_{mn}} = 0.$

Since $0 < infp_{mn} \leq p_{mn} \leq 1$, we have

$lim_{m,n\to\infty} (mn)^{-s} \left[M_{mn} \left(\frac{\left((m+n)!\left|\Delta_u^\gamma x_{mn}\right|\right)^{1/m+n}}{\rho} \right) \right] \leq$
$lim_{m,n\to\infty} (mn)^{-s} \left[M_{mn} \left(\frac{\left((m+n)!\left|\Delta_u^\gamma x_{mn}\right|\right)^{1/m+n}}{\rho} \right) \right]^{p_{mn}} = 0$, that is $\chi^2 \left(\Delta_u^\gamma, M_{mn}, s\right)$.
(ii) Let $1 \leq p_{mn}$ for each m, n, and $supp_{mn} < \infty$. Let $x \in \chi^2 \left(\Delta_u^\gamma, M_{mn}, s\right)$, then for each $\epsilon \left(0 < \epsilon < 1\right)$, there exists a positive integer L such that
$(mn)^{-s} \left[M_{mn} \left(\frac{\left((m+n)!\left|\Delta_u^\gamma x_{mn}\right|\right)^{1/m+n}}{\rho} \right) \right] \leq \epsilon$, for all $m, n \geq L$.
Since $1 \leq p_{mn} \leq supp_{mn} < \infty$, we have
$lim_{m,n\to\infty} (mn)^{-s} \left[M_{mn} \left(\frac{\left((m+n)!\left|\Delta_u^\gamma x_{mn}\right|\right)^{1/m+n}}{\rho} \right) \right]^{p_{mn}} \leq$
$lim_{m,n\to\infty} (mn)^{-s} \left[M_{mn} \left(\frac{\left((m+n)!\left|\Delta_u^\gamma x_{mn}\right|\right)^{1/m+n}}{\rho} \right) \right] \leq \epsilon < 1$.
Hence $x \in x \in \chi^2 \left(\Delta_u^\gamma, M_{mn}, p, s\right)$. This completes the proof.
Theorem 4.3 Let (p_{mn}) be double analytic and (M_{mn}) be a sequence of Orlicz functions. Then (i) $\Lambda^2 \left(\Delta_u^0, M_{mn}, p, s\right) \subset \Lambda^2 \left(\Delta_u^\gamma, M_{mn}, p, s\right)$, (ii) $\chi^2 \left(\Delta_u^0, M_{mn}, p, s\right) \subset \chi^2 \left(\Delta_u^\gamma, M_{mn}, p, s\right)$.
Proof: Let $supp_{mn} = H$. If a_{mn} and b_{mn} are complex numbers, then we have
$|a_{mn} + b_{mn}|^{p_{mn}} \leq D \left(|a_{mn}|^{p_{mn}} + |b_{mn}|^{p_{mn}}\right)$ where a_{mn} and b_{mn} are complex numbers, $D = max \left\{1, 2^{H-1}\right\}$ and $H = sup_{mn} p_{mn} < \infty$. Since M_{mn} is non decreasing and convex for each m, n, the results follows from the above inequality. This completes the proof.

References

[1] T. Apostol, Mathematical Analysis, *Addison-wesley , London*, **1**978.

[2] M. Basarir and O. Solancan, On some double sequence spaces, *J. Indian Acad. Math.*, **21(2)** (1999), 193-200.

[3] C. Bektas and Y. Altin, The sequence space $\ell_M (p, q, s)$ on seminormed spaces, *Indian J. Pure Appl. Math.*, **34(4)** (2003), 529-534.

[4] T.J.I'A. Bromwich, An introduction to the theory of infinite series, *Macmillan and Co.Ltd.* ,New York, **(**1965).

[5] J.C. Burkill and H. Burkill, A Second Course in Mathematical Analysis *Cambridge University Press, Cambridge*, New York, **(**1980).

[6] R. Colak and A. Turkmenoglu, The double sequence spaces $\ell^2_\infty(p), c^2_0(p)$ and $c^2(p)$, *(to appear).*

[7] M. Gupta and P.K. Kamthan,Infinite matrices and tensorial transformations, *Acta Math.* , **Vietnam 5** (1980), 33-42.

[8] G.H. Hardy, On the convergence of certain multiple series, *Proc. Camb. Phil. Soc.*, **19** (1917), 86-95.

[9] M.A. Krasnoselskii and Y.B. Rutickii, Convex functions and Orlicz spaces, *Gorningen, Netherlands*, **1961**.

[10] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, *Israel J. Math.*, **10** (1971), 379-390.

[11] I. J. Maddox, Sequence spaces defined by a modulus, *Math. Proc. Cambridge Philos. Soc*, **100(1)** (1986), 161-166.

[12] F. Moricz, Extentions of the spaces c and c_0 from single to double sequences, *Acta. Math. Hungerica*, **57(1-2)**, (1991), 129-136.

[13] F. Moricz and B.E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, *Math. Proc. Camb. Phil. Soc.*, **104**, (1988), 283-294.

[14] M. Mursaleen,M.A. Khan and Qamaruddin, Difference sequence spaces defined by Orlicz functions, *Demonstratio Math.* , **Vol. XXXII** (1999), 145-150.

[15] H. Nakano, Concave modulars, *J. Math. Soc. Japan*, **5**(1953), 29-49.

[16] W. Orlicz, *Ü*ber Raume $\left(L^M\right)$ *Bull. Int. Acad. Polon. Sci. A*, (1936), 93-107.

[17] S.D. Parashar and B. Choudhary, Sequence spaces defined by Orlicz functions, *Indian J. Pure Appl. Math.* , **25(4)**(1994), 419-428.

[18] K. Chandrasekhara Rao and N. Subramanian, The Orlicz space of entire sequences, *Int. J. Math. Math. Sci.*, **68**(2004), 3755-3764.

[19] W.H. Ruckle, FK spaces in which the sequence of coordinate vectors is bounded, *Canad. J. Math.*, **25**(1973), 973-978.

[20] B.C. Tripathy, On statistically convergent double sequences, *Tamkang J. Math.*, **34(3)**, (2003), 231-237.

[21] B.C. Tripathy,M. Et and Y. Altin, Generalized difference sequence spaces defined by Orlicz function in a locally convex space, *J. Analysis and Applications*, **1(3)**(2003), 175-192.

[22] A. Turkmenoglu, Matrix transformation between some classes of double sequences, *Jour. Inst. of math. and Comp. Sci. (Math. Seri.)*, **12(1)**, (1999), 23-31.

[23] A. Wilansky, Summability through Functional Analysis, *North-Holland Mathematics Studies, North-Holland Publishing, Amsterdam*, **Vol.85**(1984).

[24] P.K. Kamthan and M. Gupta, Sequence spaces and series, Lecture notes, Pure and Applied Mathematics, *65 Marcel Dekker, In c., New York* , **1**981.

[25] M. Gupta and P.K. Kamthan, Infinite Matrices and tensorial transformations, *Acta Math. Vietnam* **5**, (1980), 33-42.

[26] N. Subramanian, R. Nallswamy and N.Saivaraju, Characterization of entire sequences via double Orlicz space, *Internaional Journal of Mathematics and Mathemaical Sciences*, **Vol.2007**(2007), Article ID 59681, 10 pages.

[27] A. Gökhan and R. Colak, The double sequence spaces $c_2^P(p)$ and $c_2^{PB}(p)$, *Appl. Math. Comput.*, **157(2)**, (2004), 491-501.

[28] A. Gökhan and R. Colak, Double sequence spaces ℓ_2^∞, *ibid.*, **160(1)**, (2005), 147-153.

[29] M. Zeltser, Investigation of Double Sequence Spaces by Soft and Hard Analitical Methods, Dissertationes Mathematicae Universitatis Tartuensis 25, *Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu*, **2001**.

[30] M. Mursaleen and O.H.H. Edely, Statistical convergence of double sequences, *J. Math. Anal. Appl.*, **288(1)**, (2003), 223-231.

[31] M. Mursaleen, Almost strongly regular matrices and a core theorem for double sequences, *J. Math. Anal. Appl.*, **293(2)**, (2004), 523-531.

[32] M. Mursaleen and O.H.H. Edely,Almost convergence and a core theorem for double sequences, *J. Math. Anal. Appl.*, **293(2)**, (2004), 532-540.

[33] B. Altay and F. Basar, Some new spaces of double sequences, *J. Math. Anal. Appl.*, **309(1)**, (2005), 70-90.

[34] F. Basar and Y. Sever, The space $\mathcal{L}_p$ of double sequences, *Math. J. Okayama Univ*, **51**, (2009), 149-157.

[35] N. Subramanian and U.K. Misra, The semi normed space defined by a double gai sequence of modulus function, *Fasciculi Math.*, **46**, (2010).

[36] H. Kizmaz, On certain sequence spaces, *Cand. Math. Bull.*, **24(2)**, (1981), 169-176.

[37] N. Subramanian and U.K. Misra, Characterization of gai sequences via double Orlicz space, *Southeast Asian Bulletin of Mathematics*, **(revised)**.

[38] N. Subramanian, B.C. Tripathy and C. Murugesan, The double sequence space of Γ^2, *Fasciculi Math.* , **40**, (2008), 91-103.

[39] N.Subramanian, B.C.Tripathy and C.Murugesan, The Cesáro of double entire sequences, *International Mathematical Forum*, **4 no.2**(2009), 49-59.

[40] N. Subramanian and U.K. Misra, The Generalized double of gai sequence spaces, *Fasciculi Math.*, **43**, (2010).

[41] N. Subramanian and U.K. Misra, Tensorial transformations of double gai sequence spaces, *International Journal of Computational and Mathematical Sciences*, **3:4**, (2009), 186-188.

[42] Erwin Kreyszig, Introductory Functional Analysis with Applications, *John Wiley and Sons Inc.* , **1**978.

[43] B. Kuttner, Note on strong summability, *J. London Math. Soc.*, **21**, (1946).

[44] I. J. Maddox, Spaces of strongly summable sequences, *Quart. J. Math. Oxford Ser-2*, **18**, (1967), 345-355.

[45] J. S. Cannor, On Stron matrix summability with respect to a modulus and statistical convergence, *Canad Math. Bull.*, **32(2)**, (1989), 194-198.

[46] I. Leindler, Uber die la Vallee-Pousinche Summierbarkeit Allgemeiner Orthogonalreihen, *Acta Math. Hung.* , **16**(1965), 375-378.

[47] I.J.Maddox, Elements of Functional Analysis, *2nd Edition, Cambridge University Press*, **1970**.

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