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THE $\left(\frac{G'}{G}\right)$-EXPANSION METHOD FOR SOLVING COUPLED POTENTIAL KDV AND COUPLED MKDV EQUATIONS

A. Jabbari and H. Kheiri

Abstract. In this paper, the $(\frac{G'}{G})$-expansion method is used to seek more general exact solutions of coupled potential KdV and coupled MKdV equations. As a result, hyperbolic function solutions, trigonometric function solutions and rational function solutions with free parameters are obtained. When the parameters are taken as special values the solitary wave solutions are also derived from the travelling wave solutions. It is shown that the proposed method is more powerful and more general.

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1. Introduction

The search of exact solutions to coupled systems of nonlinear partial differential equations is of great importance, because these systems appear in complex physics phenomena, mechanics, chemistry, biology and engineering. A variety of powerful and direct methods have been developed in this direction. The principal objective of this paper, is to present the application of $(\frac{G'}{G})$-expansion method in solving coupled systems of two equations.

Recently, Wang et al. [1] proposed the (G'/G)-expansion method and showed that it is powerful for finding analytic solutions of PDEs. Next, Bekir [2] applied the method to some nonlinear evolution equations gaining traveling wave solutions. More recently, Zhang et al. [3] proposed a generalized $(\frac{G'}{G})$-expansion method to improve and extend Wang et al.'s work [1] for solving variable coefficient equations and high dimensional equations. Kheiri et al. applied this method for solving the combined and the double combined sinh-cosh-Gordon equations and the double sinh-Gordon and generalized form of the double sinh-Gordon equations [4, 5]. Also Zhang [3] solved the equations with the balance numbers of which are not positive integers, by this method. The $(\frac{G'}{G})$-expansion method is based on the explicit linearization of nonlinear differential equations for traveling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. Computations are performed with a computer algebra system such as Maple to deduce the solutions of the nonlinear equations in an explicit form. The solution process of the method is direct, effective and convenient due to solving the auxiliary equation of second-order differential equation with constant coefficients. In this work, we apply the $(\frac{G'}{G})$-expansion method to solve coupled potential KdV and coupled MKdV equations.

2.Description of the $\left(\frac{G'}{G}\right)$-expansion method

We suppose that the given nonlinear partial differential equation for $u(x,t)$ to be in the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \tag{1}$$

where P is a polynomial in its arguments. The essence of the $(\frac{G'}{G})$-expansion method can be presented in the following steps:

step 1. Seek traveling wave solutions of Eq. (1) by taking $u(x,t) = u(\xi)$, $\xi = x - ct$, and transform Eq. (1) to the ordinary differential equation

$$Q(u, u', u'', \ldots) = 0, \tag{2}$$

where prime denotes the derivative with respect to ξ.

step 2. If possible, integrate Eq. (2) term by term one or more times. This yields constant(s) of integration.

step 3. Introduce the solution $u(\xi)$ of Eq. (2) in the finite series form

$$u(\xi) = \sum_{i=0}^{m} a_i \Big(\frac{G'(\xi)}{G(\xi)}\Big)^i, \tag{3}$$

where a_i are constants with $a_m \neq 0$ to be determined, m is a positive integer to be determined. The function $G(\xi)$ is the solution of the auxiliary linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{4}$$

where λ and μ are real constants to be determined.

step 4. Determine m. This, usually, can be accomplished by balancing the linear term(s) of highest order with the highest order nonlinear term(s) in Eq. (2).

step 5. Substituting (3) together with (4) into Eq. (2) yields an algebraic equation involving powers of $(\frac{G'}{G})$. Equating the coefficients of each power of $(\frac{G'}{G})$ to zero gives a system of algebraic equations for a_i, λ, μ and c. Then, we solve the system with the aid of a computer algebra system (CAS), such as Maple, to determine these constants. On the other hand, depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, the solutions of Eq. (4) are well known for us. So, we can obtain exact solutions of the given Eq. (1).

3. Coupled potential KdV equations

We begin our analysis by working on the coupled potential KdV equations

$$
\begin{aligned}
u_t &= \frac{1}{2}u_{xxx} + \frac{1}{2}v_{xxx} + 2{u_x}^2 + {v_x}^2, \\
v_t &= \frac{1}{2}u_{xxx} + \frac{1}{2}v_{xxx} + 2{v_x}^2 + {u_x}^2.
\end{aligned} \tag{5}
$$

By using the traveling wave variable

$$u(x,t) = U(\xi), \quad v(x,t) = V(\xi), \tag{6}$$

where $\xi = x - ct$, we find

$$
\begin{aligned}
cU' + \frac{1}{2}U''' + \frac{1}{2}V''' + 2(U')^2 + (V')^2 = 0, \\
cV' + \frac{1}{2}U''' + \frac{1}{2}V''' + 2(V')^2 + (U')^2 = 0,
\end{aligned} \tag{7}
$$

Balancing U''' with $(U')^2$ gives $m = 1$. Similarly, for $v(x,t)$ we find n=1.
The $(\frac{G'}{G})$-expansion method admits the use of the expansion

$$u(\xi) = a_0 + a_1\Big(\frac{G'}{G}\Big), \quad a_1 \neq 0, \tag{8}$$

$$v(\xi) = b_0 + b_1\Big(\frac{G'}{G}\Big), \quad b_1 \neq 0. \tag{9}$$

Substituting (8) and (9) into (7), collecting all terms with the same powers of $(\frac{G'}{G})$ and setting each coefficient to zero, we obtain the following system of algebraic equations

$$
\begin{aligned}
\Big(\frac{G'}{G}\Big)^0 &: -\frac{1}{2}a_1\lambda^2\mu - a_1\mu^2 - b_1\mu^2 - \frac{1}{2}b_1\lambda^2\mu + 2a_1^2\mu^2 - ca_1\mu + b_1^2\mu^2 = 0, \\
\Big(\frac{G'}{G}\Big)^1 &: -\frac{1}{2}a_1\lambda^3 - 4a_1\lambda\mu - \frac{1}{2}b_1\lambda^3 - 4b_1\lambda\mu - ca_1\lambda + 4a_1^2\lambda\mu + 2b_1^2\lambda\mu = 0, \\
\Big(\frac{G'}{G}\Big)^2 &: -\frac{7}{2}a_1\lambda^2 - 4a_1\mu - \frac{7}{2}b_1\lambda^2 - 4b_1\mu + 2a_1^2\lambda^2 + 4a_1^2\mu + b_1^2\lambda^2 + 2b_1^2\mu - ca_1 = 0, \\
\Big(\frac{G'}{G}\Big)^3 &: -6a_1\lambda - 6b_1\lambda + 4a_1^2\lambda + 2b_1^2\lambda = 0, \\
\Big(\frac{G'}{G}\Big)^4 &: -3a_1 + b_1^2 - 3b_1 + 2a_1^2 = 0,
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\left(\frac{G'}{G}\right)^0 &: -\frac{1}{2}a_1\lambda^2\mu - a_1\mu^2 - b_1\mu^2 - \frac{1}{2}b_1\lambda^2\mu + a_1^2\mu^2 + 2b_1^2\mu^2 - cb_1\mu = 0,\\
\left(\frac{G'}{G}\right)^1 &: -\frac{1}{2}a_1\lambda^3 - 4a_1\lambda\mu - \frac{1}{2}b_1\lambda^3 - 4b_1\lambda\mu + 2a_1^2\lambda\mu + 4b_1^2\lambda\mu - cb_1\lambda = 0,\\
\left(\frac{G'}{G}\right)^2 &: -\frac{7}{2}a_1\lambda^2 - 4a_1\mu - \frac{7}{2}b_1\lambda^2 - 4b_1\mu + a_1^2\lambda^2 + 2a_1^2\mu + 2b_1^2\lambda^2 + 4b_1^2\mu - cb_1 = 0,\\
\left(\frac{G'}{G}\right)^3 &: -6a_1\lambda - 6b_1\lambda + 2a_1^2\lambda + 4b_1^2\lambda = 0,\\
\left(\frac{G'}{G}\right)^4 &: -3a_1 + 2b_1^2 - 3b_1 + a_1^2 = 0.
\end{aligned}
$$

Solving this system by Maple gives

$$a_0 = a_0, \qquad a_1 = 2, \qquad b_0 = b_0, \qquad b_1 = 2, \qquad c = 4\mu - \lambda^2 \tag{11}$$

Substituting the solution set (11) and the corresponding solutions of (4) into (8) and (9), we deduce the following traveling wave solutions

When $\lambda^2 - 4\mu > 0$**,** we obtain the hyperbolic function traveling wave solutions

$$
\begin{aligned}
U_1(\xi) &= A + \sqrt{\lambda^2 - 4\mu}\left(\frac{C_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right),\\
V_1(\xi) &= B + \sqrt{\lambda^2 - 4\mu}\left(\frac{C_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right),
\end{aligned}
\tag{12}
$$

When $\lambda^2 - 4\mu < 0$**,** we obtain the trigonometric function traveling wave solutions

$$
\begin{aligned}
U_2(\xi) &= A + \sqrt{4\mu - \lambda^2}\left(\frac{-C_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right),\\
V_2(\xi) &= B + \sqrt{4\mu - \lambda^2}\left(\frac{-C_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right),
\end{aligned}
\tag{13}
$$

where $A = a_0 - \lambda$ and $B = b_0 - \lambda$ are arbitrary constants and $\xi = x - (4\mu - \lambda^2)t$, for (12) and (13).

The last results lead to the relation $v(x,t) = u(x,t) + \alpha, \quad \alpha = B - A$, where α is an arbitrary constant.

When $\lambda^2 - 4\mu = 0$**,** we obtain the rational function traveling wave solutions

$$U_3(\xi) = \frac{a_0 C_1 + a_0 C_2 \xi - \lambda C_1 + 2C_2 - C_2 \lambda \xi}{C_1 + C_2 \xi},$$

$$V_3(\xi) = \frac{b_0C_1 + b_0C_2\xi - \lambda C_1 + 2C_2 - C_2\lambda\xi}{C_1 + C_2\xi}, \tag{14}$$

where $\xi = x - (4\mu - \lambda^2)t$.
In solutions (12)-(14), C_1 and C_2 are left as free parameters. It is obvious that hyperbolic, trigonometric and rational solutions were obtained by using the $(\frac{G'}{G})$-expansion method, whereas only hyperbolic solutions were obtained in [6] by using the tanh method.
In particular, if in (12), we take $C_1 \neq 0$ and $C_2 = 0$, then we obtain

$$\begin{aligned} u_1(x,t) &= A + \sqrt{-c}\tanh[\frac{\sqrt{-c}}{2}(x - ct)], \\ v_1(x,t) &= B + \sqrt{-c}\tanh[\frac{\sqrt{-c}}{2}(x - ct)], \end{aligned} \tag{15}$$

where A and B are arbitrary constants.
The last results lead to the relation $v(x,t) = u(x,t) + \alpha, \quad \alpha = B - A$, where α is an arbitrary constant.
Also, if we take $C_1 = 0$ and $C_2 \neq 0$, the solutions in terms of coth can be derived. We observe that the results (20) in Wazwaz [6] are particular cases of our results (12).

4. Coupled MKdV equations

We continue our analysis and study the coupled MKdV equations

$$\begin{aligned} u_t &= \frac{1}{2}u_{xxx} - 3u^2u_x + \frac{3}{2}v_{xx} + 3(uv)_x - 3\delta u_x = 0, \\ v_t &= -v_{xxx} - 3vv_x - 3u_xv_x + 3u^2v_x + 3\delta v_x = 0. \end{aligned} \tag{16}$$

By using the traveling wave variable

$$u(x,t) = U(\xi), \quad v(x,t) = V(\xi), \tag{17}$$

where $\xi = x + ct$, we find

$$\begin{aligned} &cU' - \frac{1}{2}U''' + 3U^2U' - \frac{3}{2}V'' - 3(UV)' + 3\delta U' = 0, \\ &cV' + V''' + 3VV' + 3U'V' - 3U^2V' - 3\delta V' = 0. \end{aligned} \tag{18}$$

Balancing the order of U^2U' with the order of U''' and the order of U^2V' with V''' in Eq. (18), we find $m = 1$ and $n = 1$. So the solutions take the form

$$u(\xi) = a_0 + a_1\Big(\frac{G'}{G}\Big), \qquad a_1 \neq 0, \tag{19}$$

$$v(\xi) = b_0 + b_1\Big(\frac{G'}{G}\Big), \qquad b_1 \neq 0. \tag{20}$$

On substituting (19) and (20) into (18), collecting all terms with the same powers of $(\frac{G'}{G})$ and setting each coefficient to zero, we obtain the following system of algebraic equations

$$\begin{aligned}
\Big(\frac{G'}{G}\Big)^0 &: -\frac{1}{2}a_1\lambda^2\mu - \frac{3}{2}b_1\lambda\mu + 3a_1\mu b_0 + a_1\mu^2 + 3a_0b_1\mu - ca_1\mu - 3a_0^2a_1\mu - 3\delta a_1\mu = 0,\\
\Big(\frac{G'}{G}\Big)^1 &: \frac{1}{2}a_1\lambda^3 - \frac{3}{2}b_1\lambda^2 - 3b_1\mu + 4a_1\lambda\mu + 3a_1\lambda b_0 + 6a_1\mu b_1 + 3a_0b_1\lambda - ca_1\lambda\\
&\quad -3a_0^2a_1\lambda - 6a_1^2a_0\mu - 3\delta a_1\lambda = 0,\\
\Big(\frac{G'}{G}\Big)^2 &: \frac{7}{2}a_1\lambda^2 + 4a_1\mu - \frac{9}{2}b_1\lambda + 3a_0b_1 - ca_1 - 3a_0^2a_1 - 3a_1^3\mu - 3\delta a_1 + 3a_1b_0\\
&\quad +6a_1\lambda b_1 - 6a_1^2a_0\lambda = 0,\\
\Big(\frac{G'}{G}\Big)^3 &: -3b_1 + 6a_1\lambda + 6a_1b_1 - 6a_1^2a_0 - 3a_1^3\lambda = 0,\\
\Big(\frac{G'}{G}\Big)^4 &: 3a_1 - 3a_1^3 = 0,\\
\Big(\frac{G'}{G}\Big)^0 &: -2b_1\mu^2 - b_1\lambda^2\mu - cb_1\mu - 3b_0b_1\mu + 3a_1\mu^2b_1 + 3a_0^2b_1\mu + 3\delta b_1\mu = 0, \qquad (21)\\
\Big(\frac{G'}{G}\Big)^1 &: -b_1\lambda^3 - 8b_1\lambda\mu - 3b_1^2\mu - cb_1\lambda - 3b_0b_1\lambda + 6a_1\lambda b_1\mu + 3a_0^2b_1\lambda\\
&\quad +6a_1a_0b_1\mu + 3\delta b_1\lambda = 0,\\
\Big(\frac{G'}{G}\Big)^2 &: -7b_1\lambda^2 - 8b_1\mu - cb_1 - 3b_1^2\lambda - 3b_0b_1 + 3a_0^2b_1 + 3\delta b_1 + 3a_1\lambda^2b_1 + 6a_1\mu b_1\\
&\quad +6a_1a_0b_1\lambda + 3a_1^2b_1\mu = 0,\\
\Big(\frac{G'}{G}\Big)^3 &: -3b_1^2 - 12b_1\lambda + 6a_1\lambda b_1 + 6a_1a_0b_1 + 3a_1^2b_1\lambda = 0,\\
\Big(\frac{G'}{G}\Big)^4 &: -6b_1 + 3a_1b_1 + 3a_1^2b_1 = 0.
\end{aligned}$$

Solving the resulting system, we have the following set of solution:

$$a_0 = a_0, \quad a_1 = 1, \quad b_0 = -\frac{\lambda^2}{2} + \lambda a_0 + \delta, \quad b_1 = -\lambda + 2a_0, \quad c = \frac{\lambda^2}{2} + \mu - 3\lambda a_0 + 3{a_0}^2. \tag{22}$$

Substituting the solutions set (22) and the corresponding solutions of (4) into (19) and (20), we deduce the following traveling wave solutions:
When $\lambda^2 - 4\mu > 0$**,** we obtain the hyperbolic function traveling wave solutions

$$\begin{aligned} U_1(\xi) &= a_0 - \frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right), \\ V_1(\xi) &= \delta - \frac{\lambda\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right) \\ &\quad + a_0\sqrt{\lambda^2-4\mu}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right). \end{aligned} \tag{23}$$

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function traveling wave solutions

$$\begin{aligned} U_2(\xi) &= a_0 - \frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-c_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right), \\ V_2(\xi) &= \delta - \frac{\lambda\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-c_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right) \\ &\quad + a_0\sqrt{4\mu-\lambda^2}\left(\frac{-c_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}\right). \end{aligned} \tag{24}$$

When $\lambda^2 - 4\mu = 0$, we obtain the rational function traveling wave solutions

$$\begin{aligned} U_3(\xi) &= -\frac{1}{2}\frac{-2a_0C_1 - 2a_0C_2\xi + \lambda C_1 - 2C_2 + C_2\lambda\xi}{(C_1 + C_2\xi)}, \\ V_3(\xi) &= -\frac{-\delta C_1 - \delta C_2\xi + \lambda C_2 - 2a_0C_2}{C_1 + C_2\xi}, \end{aligned} \tag{25}$$

where $\xi = x - (\frac{\lambda^2}{2} + \mu - 3\lambda a_0 + 3{a_0}^2)t$, for (23)-(25).
In solutions (23)-(25), C_1 and C_2 are left as free parameters.
In particular, if we take $C_1 \neq 0$, $C_2 = 0$, $\lambda = 0$ and $\mu > 0$, we have

$$\begin{aligned} U_1(\xi) &= a_0 + \sqrt{(-c + 3{a_0}^2)}\tanh[\sqrt{(-c + 3{a_0}^2)}(x + ct)], \\ V_1(\xi) &= \delta + 2a_0\sqrt{(-c + 3{a_0}^2)}\tanh[\sqrt{(-c + 3{a_0}^2)}(x + ct)], \end{aligned} \tag{26}$$

Also, if we take $C_1 = 0$ and $C_2 \neq 0$, the solutions in terms of coth can be derived. Note that hyperbolic, trigonometric and rational solutions were obtained by using the $(\frac{G'}{G})$-expansion method, whereas only hyperbolic solutions were obtained in [7] by using the tanh-coth method.

5.Conclusions

In this paper, the $(\frac{G'}{G})$-expansion method is used to obtain general exact solutions of the coupled potential KdV and coupled MKdV equations. As a result, hyperbolic function solutions, trigonometric function solutions and rational solutions with arbitrary parameters are obtained. The arbitrary parameters in the obtained solutions imply that these solutions have rich local structures. It is illustrated that the $(\frac{G'}{G})$-expansion method is direct, effective and can be used for many other NLEEs in mathematical physics.

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Azizeh Jabbari and Hossein Kheiri
Faculty of mathematical sciences
University of Tabriz, Tabriz, Iran
email: *azizeh.jabbari@gmail.com, h-kheiri@tabrizu.ac, kheirihossein@yahoo.com*