

YANG–BAXTER OPERATORS FROM ALGEBRA STRUCTURES AND LIE SUPER-ALGEBRA STRUCTURES

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ABSTRACT. The concept of symmetry plays an important role in solving equations and systems of equations. We will present solutions for the (constant and spectral-parameter) Yang-Baxter equations and Yang-Baxter systems arising from algebra structures and discuss about their symmetries. In the last section, we present enhanced versions of Theorem 1 (from [20]), solutions for the classical Yang-Baxter equation, and solutions for the Yang-Baxter systems from Lie (super)algebras.

2000 *Mathematics Subject Classification*: 16T25, 17B60, 17B63, 17C90

1. INTRODUCTION AND PRELIMINARIES

The concept of symmetry appears in almost every scientific and artistic area. It plays an important role in solving equations and systems of equations. Sometimes an equation has a high degree of symmetry, but it is not perfectly symmetric. Similar situations, appearing in a geometric framework, were considered by [6], where concepts like “distance from symmetry in shape”, “closest symmetric shape” and “symmetry distance” were introduced.

In this paper we reveal abstract situations in which equations and some of their solutions are symmetric or almost symmetric. Besides these forms of symmetry, we encounter the super-symmetry. The super-symmetry property is a concept of interest at CERN, and it states that any particle has an associated super-symmetric particle (also called its super-partner). While this property has not been confirmed yet, there exist plenty of examples and applications of super-symmetric structures in particle physics and quantum groups.

The quantum Yang–Baxter equation (QYBE) first appeared in theoretical physics [22] and statistical mechanics [1, 2]. It plays a crucial role in analysis of integrable systems, in quantum and statistical mechanics, in knot theory, and also in the theory of quantum groups. On the other hand, the theory of integrable Hamiltonian systems makes great use of the solutions of the one-parameter form of the Yang-Baxter

equation, since coefficients of the power series expansion of such a solution give rise to commuting integrals of motion. The two-parameter form of the QYBE is related to Yang's paper [22], and its solutions are referred to as a colored *Yang-Baxter operator*. *Yang-Baxter systems* emerged from the study of quantum integrable systems, as generalizations of the QYBE related to nonultralocal models.

This paper presents results on Yang-Baxter operators from algebra structures and related topics (colored Yang-Baxter operators, Yang-Baxter systems, Yang-Baxter operator from Lie super-algebras). In the last section, we present enhanced versions of Theorem 1 (from [20]), solutions for the classical Yang-Baxter equation (CYBE), and solutions for a *WXZ*-system.

The Yang-Baxter equations (QYBE, CYBE, set-theoretical Yang-Baxter equation, etc) have some kind of symmetries, which can be used to find solutions for them; many times, it is possible to obtain larger classes of solutions by some sort of deformation (quantization) of those solutions. The Yang-Baxter equations are related to symmetric spaces [3], Boolean algebras [16], Jordan algebras [11], etc. The following is a short bibliography on QYBE [13, 12, 7, 20, 15], and Yang-Baxter systems [10, 9, 4, 17, 21].

Throughout this paper k is a field.

All tensor products appearing in this paper are defined over k .

For V a k -space, we denote by $\tau : V \otimes V \rightarrow V \otimes V$ the twist map defined by $\tau(v \otimes w) = w \otimes v$, and by $I : V \rightarrow V$ the identity map of the space V .

We use the following notations concerning the Yang-Baxter equation.

If $R : V \otimes V \rightarrow V \otimes V$ is a k -linear map, then $R^{12} = R \otimes I$, $R^{23} = I \otimes R$, $R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau)$.

Definition 1.1. *An invertible k -linear map $R : V \otimes V \rightarrow V \otimes V$ is called a Yang-Baxter operator if it satisfies the equation*

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23} \tag{1.1}$$

Remark 1.2. *The equation (1.1) is usually called the braid equation. It has some kind of symmetry.*

Remark 1.3. *The operator R satisfies (1.1) if and only if $R \circ \tau$ satisfies the constant QYBE (if and only if $\tau \circ R$ satisfies the constant QYBE):*

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12} \tag{1.2}$$

This equation is symmetric with regard to “=”.

Remark 1.4. (i) $\tau : V \otimes V \rightarrow V \otimes V$ is an example of a Yang-Baxter operator.

(ii) An exhaustive list of invertible solutions for (1.2) in dimension 2 is given in [8] and in the appendix of [10].

(iii) Finding all Yang-Baxter operators in dimension greater than 2 is an unsolved problem.

Let A be a (unitary) associative k -algebra, and $\alpha, \beta, \gamma \in k$. We define the k -linear map: $R_{\alpha, \beta, \gamma}^A : A \otimes A \rightarrow A \otimes A$, $R_{\alpha, \beta, \gamma}^A(a \otimes b) = \alpha ab \otimes 1 + \beta 1 \otimes ab - \gamma a \otimes b$.

Theorem 1.1. (S. Dăscălescu and F. F. Nichita, [5]) Let A be an associative k -algebra with $\dim A \geq 2$, and $\alpha, \beta, \gamma \in k$. Then $R_{\alpha, \beta, \gamma}^A$ is a Yang-Baxter operator if and only if one of the following holds:

(i) $\alpha = \gamma \neq 0$, $\beta \neq 0$;

(ii) $\beta = \gamma \neq 0$, $\alpha \neq 0$;

(iii) $\alpha = \beta = 0$, $\gamma \neq 0$.

If so, we have $(R_{\alpha, \beta, \gamma}^A)^{-1} = R_{\frac{1}{\beta}, \frac{1}{\alpha}, \frac{1}{\gamma}}^A$ in cases (i) and (ii), and $(R_{0, 0, \gamma}^A)^{-1} = R_{0, 0, \frac{1}{\gamma}}^A$ in case (iii).

Remark 1.5. The Yang-Baxter equation plays an important role in knot theory. Turaev has described a general scheme to derive an invariant of oriented links from a Yang-Baxter operator, provided this one can be “enhanced”. In [14], we considered the problem of applying Turaev’s method to the Yang-Baxter operators derived from algebra structures presented in the above theorem. We concluded that Turaev’s procedure invariably produces from any of those enhancements the Alexander polynomial of knots.

We now present the matrix form of the operator obtained in the case (i) of the previous theorem, $R = R_{\alpha, \beta, \alpha}^A : A \otimes A \rightarrow A \otimes A$, $R(a \otimes b) = \alpha ab \otimes 1 + \beta 1 \otimes ab - \alpha a \otimes b$.

We consider the algebra $A = \frac{k[X]}{(X^2 - mX - n)}$, where m, n are scalars. Then A has the basis $\{1, x\}$, where x is the image of X in the factor ring.

In matrix form, this operator reads:

$$\begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta - \alpha & \alpha & 0 \\ 0 & 0 & \beta & 0 \\ (\alpha + \beta)n & \beta m & \alpha m & -\alpha \end{pmatrix} \tag{1.3}$$

Let us observe that $R' = R \circ \tau$ is a solution for the equation (1.2). It is convenient to get rid of the auxiliary parameters and to consider the simplest form of R' :

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1-q & q & 0 \\ \eta & 0 & 0 & -q \end{pmatrix} \quad (1.4)$$

where $\eta \in \{0, 1\}$, and $q \in k - \{0\}$. The matrix form (1.4) was obtained as a consequence of the fact that isomorphic algebras produce isomorphic Yang-Baxter operators.

If $q = 1$, $\eta = 0$, the above matrix is symmetric with regard to the first diagonal, and can be obtained from the self-inverse operators described in [15]. Thus, the general case could be seen as a deformation of that case.

2. THE TWO-PARAMETER FORM OF THE QYBE

Formally, a colored Yang-Baxter operator is defined as a function $R : X \times X \rightarrow \text{End}_k(V \otimes V)$, where X is a set and V is a finite dimensional vector space over a field k . Thus, for any $u, v \in X$, $R(u, v) : V \otimes V \rightarrow V \otimes V$ is a linear operator. We consider three operators acting on a triple tensor product $V \otimes V \otimes V$, $R^{12}(u, v) = R(u, v) \otimes I$, $R^{23}(v, w) = I \otimes R(v, w)$, and similarly $R^{13}(u, w)$ as an operator that acts non-trivially on the first and third factor in $V \otimes V \otimes V$.

If R satisfies the two-parameter form of the QYBE:

$$R^{12}(u, v)R^{13}(u, w)R^{23}(v, w) = R^{23}(v, w)R^{13}(u, w)R^{12}(u, v) \quad (2.5)$$

$\forall u, v, w \in X$, then it is called a *colored Yang-Baxter operator*.

Theorem 2.1. (F. F. Nichita and D. Parashar, [17]) *Let A be an associative k -algebra with $\dim A \geq 2$, and $X \subset k$. Then, for any two parameters $p, q \in k$, the function $R : X \times X \rightarrow \text{End}_k(A \otimes A)$ defined by*

$$R(u, v)(a \otimes b) = R_{p,q}(u, v)(a \otimes b) = p(u-v)1 \otimes ab + q(u-v)ab \otimes 1 - (pu - qv)b \otimes a, \quad (2.6)$$

satisfies the colored QYBE (2.5).

Remark 2.6. 1. *The solution (2.6) is related to Yang's paper [22].*

2. *If $pu \neq qv$ and $qu \neq pv$ then the operator (2.6) is invertible. Moreover, the following formula holds:*

$$R^{-1}(u, v)(a \otimes b) = \frac{p(u-v)}{(qu-pv)(pu-qv)}ba \otimes 1 + \frac{q(u-v)}{(qu-pv)(pu-qv)}1 \otimes ba - \frac{1}{(pu-qv)}b \otimes a.$$

3. It follows an almost symmetric relation:

$$R_{q,p}(u, v) = (qu - pv)(pu - qv)R_{p,q}^{-1}(u, v) \circ \tau.$$

4. Let us consider the Theorem 2.6. If we let $v=0$ and $u=1$, we obtain the operator $R(a \otimes b) = p1 \otimes ab + qab \otimes 1 - pb \otimes a$, which satisfies the constant QYBE (1.2). Notice that $\tau \circ R$ is the Yang-Baxter operator from the Theorem 1.1, case (i).

5. The system of equations (2.7–2.11) is related to the above theorem. It is an open problem to classify its solutions. This system of equations has some remarkable symmetry properties which can be used to find some solutions for it. For example, the equations (2.8) and (2.11) are in some sense dual to each other. Likewise, (2.9) and (2.10) are in some sense dual to each other.

$$\begin{aligned} &(\beta(v, w) - \gamma(v, w))(\alpha(u, v)\beta(u, w) - \alpha(u, w)\beta(u, v) + (\alpha(u, v) - \gamma(u, v))) \\ &(\alpha(v, w)\beta(u, w) - \alpha(u, w)\beta(v, w)) = 0 \end{aligned} \quad (2.7)$$

$$\begin{aligned} &\beta(v, w)(\beta(u, v) - \gamma(u, v))(\alpha(u, w) - \gamma(u, w)) \\ &+(\alpha(v, w) - \gamma(v, w))(\beta(u, w)\gamma(u, v) - \beta(u, v)\gamma(u, w)) = 0 \end{aligned} \quad (2.8)$$

$$\begin{aligned} &\alpha(u, v)\beta(v, w)(\alpha(u, w) - \gamma(u, w)) + \alpha(v, w)\gamma(u, w)(\gamma(u, v) - \alpha(u, v)) \\ &+\gamma(v, w)(\alpha(u, v)\gamma(u, w) - \alpha(u, w)\gamma(u, v)) = 0 \end{aligned} \quad (2.9)$$

$$\begin{aligned} &\alpha(u, v)\beta(v, w)(\beta(u, w) - \gamma(u, w)) + \beta(v, w)\gamma(u, w)(\gamma(u, v) - \beta(u, v)) \\ &+\gamma(v, w)(\beta(u, v)\gamma(u, w) - \beta(u, w)\gamma(u, v)) = 0 \end{aligned} \quad (2.10)$$

$$\begin{aligned} &\alpha(u, v)(\alpha(v, w) - \gamma(v, w))(\beta(u, w) - \gamma(u, w)) + (\beta(u, v) - \gamma(u, v)) \\ &(\alpha(u, w)\gamma(v, w) - \alpha(v, w)\gamma(u, w)) = 0 \end{aligned} \quad (2.11)$$

We now consider the algebra $A = \frac{k[X]}{(X^2 - \sigma)}$, where σ is a scalar. Then A has the basis $\{1, x\}$, where x is the image of X in the factor ring. We consider the basis $\{1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x\}$ of $A \otimes A$ and represent the operator (2.6) in this basis:

$$\begin{aligned} R(u, v)(1 \otimes 1) &= (qu - pv)1 \otimes 1 \\ R(u, v)(1 \otimes x) &= p(u - v)1 \otimes x + (q - p)ux \otimes 1 \\ R(u, v)(x \otimes 1) &= (q - p)v1 \otimes x + q(u - v)x \otimes 1 \\ R(u, v)(x \otimes x) &= \sigma(p + q)(u - v)1 \otimes 1 - (pu - qv)x \otimes x \end{aligned}$$

In matrix form, this operator reads

$$R(u, v) = \begin{pmatrix} qu - pv & 0 & 0 & \sigma(q + p)(u - v) \\ 0 & p(u - v) & (q - p)v & 0 \\ 0 & (q - p)u & q(u - v) & 0 \\ 0 & 0 & 0 & qv - pu \end{pmatrix} \quad (2.12)$$

3. YANG-BAXTER SYSTEMS

It is convenient to describe the Yang-Baxter systems in terms of the Yang-Baxter commutators.

Let V, V', V'' be finite dimensional vector spaces over the field k , and let $R : V \otimes V' \rightarrow V \otimes V', S : V \otimes V'' \rightarrow V \otimes V''$ and $T : V' \otimes V'' \rightarrow V' \otimes V''$ be three linear maps. The *Yang-Baxter commutator* is a map $[R, S, T] : V \otimes V' \otimes V'' \rightarrow V \otimes V' \otimes V''$ defined by

$$[R, S, T] := R^{12}S^{13}T^{23} - T^{23}S^{13}R^{12}. \quad (3.13)$$

Note that $[R, R, R] = 0$ is just a short-hand notation for writing the constant QYBE (1.2).

A system of linear maps $W : V \otimes V \rightarrow V \otimes V, Z : V' \otimes V' \rightarrow V' \otimes V', X : V \otimes V' \rightarrow V \otimes V'$, is called a *WXZ-system* if the following conditions hold:

$$[W, W, W] = 0 \quad [Z, Z, Z] = 0 \quad [W, X, X] = 0 \quad [X, X, Z] = 0 \quad (3.14)$$

It was observed that *WXZ*-systems with invertible W, X and Z can be used to construct dually paired bialgebras of the FRT type leading to quantum doubles. The above is one type of a constant Yang-Baxter system that has recently been studied in [17] and also shown to be closely related to entwining structures [4]. Thus, the *WXZ*-systems, because of their symmetry, are related to “gluing procedures” (obtaining a bigger object from two objects of the same kind). In the next theorem if we let $\mu, \lambda \rightarrow 1$, we obtain the trivial solution of a *WXZ*-system.

Theorem 3.2. (*F. F. Nichita and D. Parashar, [17]*) *Let A be a k -algebra, and $\lambda, \mu \in k$. The following is a *WXZ*-system:*

$$\begin{aligned} W : A \otimes A &\rightarrow A \otimes A, & W(a \otimes b) &= ab \otimes 1 + \lambda 1 \otimes ab - b \otimes a, \\ Z : A \otimes A &\rightarrow A \otimes A, & Z(a \otimes b) &= \mu ab \otimes 1 + 1 \otimes ab - b \otimes a, \\ X : A \otimes A &\rightarrow A \otimes A, & X(a \otimes b) &= ab \otimes 1 + 1 \otimes ab - b \otimes a. \end{aligned}$$

4. APPLICATIONS AND CONCLUSIONS

4.1 IMPROVING THEOREMS

Using the techniques from above we now present enhanced versions of Theorem 1 (from [20]).

Theorem 4.3. (*F. F. Nichita and B. P. Popovici, [18]*) *Let $V = W \oplus kc$ be a k -space, and $f, g : V \otimes V \rightarrow V$ k -linear maps such that $f, g = 0$ on $V \otimes c + c \otimes V$. Then, $R : V \otimes V \rightarrow V \otimes V$, $R(v \otimes w) = f(v \otimes w) \otimes c + c \otimes g(v \otimes w)$ is a solution for QYBE (1.2).*

4.2 LIE SUPER-ALGEBRAS

In particle physics, super-symmetry is a symmetry that relates elementary particles of one spin to other particles that differ by half a unit of spin and are known as super-partners.

According to the spin-statistic theorem, bosonic fields commute while fermionic fields anti-commute. Combining the two kinds of fields into a single algebra requires the introduction of a \mathbb{Z}_2 -grading under which the bosons are the even elements and the fermions are the odd elements. Such an algebra is called a Lie super-algebra.

Let $(L, [,])$ be a Lie super-algebra over k , and $Z(L) = \{z \in L : [z, x] = 0 \ \forall x \in L\}$.

For $z \in Z(L)$, $|z| = 0$ and $\alpha \in k$ we define:

$$\phi_\alpha^L : L \otimes L \longrightarrow L \otimes L$$

$$x \otimes y \mapsto \alpha[x, y] \otimes z + (-1)^{|x||y|} y \otimes x .$$

Its inverse is:

$$\phi_\alpha^{L^{-1}} : L \otimes L \longrightarrow L \otimes L$$

$$x \otimes y \mapsto z \otimes [x, y] \otimes z + \frac{1}{\alpha} (-1)^{|x||y|} y \otimes x .$$

Theorem 4.4. (*F. F. Nichita and B. P. Popovici, [19]*)

Let $(L, [,])$ be a Lie superalgebra and $z \in Z(L)$, $|z| = 0$, and $\alpha \in k$. Then: ϕ_α^L is a YB operator.

4.3 CYBE

Theorem 5.5. *Let $(L, [, \cdot])$ be a Lie algebra and $z \in Z(L)$. Then:*

$$r : L \otimes L \longrightarrow L \otimes L, \quad x \otimes y \mapsto [x, y] \otimes z - \alpha x \otimes y$$

satisfies the classical Yang-Baxter equation:

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$$

Proof. It is left for the reader.

4.4 POISSON SUPERALGEBRA

In mathematics, a Poisson superalgebra is a Z_2 -graded generalization of a Poisson algebra. Specifically, a Poisson superalgebra is an (associative) superalgebra A with a Lie superbracket $\{, \} : A \otimes A \rightarrow A$, such that $(A, \{, \})$ is a Lie superalgebra and the operator $\{x, \} : A \rightarrow A$ is a superderivation of A :

$$\{x, yz\} = \{x, y\}z + (-1)^{|x||y|}y\{x, z\}.$$

This is one possible way of "super"izing the Poisson algebra. This gives the classical dynamics of fermion fields and classical spin-1/2 particles. The other is to define an antibracket algebra instead. This is used in the BRST and Batalin-Vilkovisky formalism.

Theorem 5.6. *Let A be a Poisson superalgebra with a unity, $1 = 1_A$, for the product $*$, such that $\{x, 1_A\} = 0 \quad \forall x \in A$. Then, we have the following WXZ -system:*

$$\begin{aligned} W(x \otimes y) &= \{x, y\} \otimes 1 + (-1)^{|x||y|} x \otimes y; \\ X(x \otimes y) &= 1 \otimes \{x, y\} + (-1)^{|x||y|} x \otimes y; \\ Z(x \otimes y) &= 1 \otimes x * y + x * y \otimes 1 - y \otimes x. \end{aligned}$$

Proof. The condition $[W, W, W] = 0$ follows from Theorem 4.10, and $[Z, Z, Z] = 0$ follows from Theorem 1.1.

The condition $[X, X, Z] = 0$ is satisfied because $\{x, \} : A \rightarrow A$ is a superderivation of A . $[W, X, X] = 0$ is a consequence of the Lie superalgebra axioms.

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