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THE $V-$ INVARIANT χ^2 SEQUENCE SPACES

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Abstract. In this paper we characterize $v-$ invariance of the sequence spaces $\Delta^2\left(\Lambda^2\right)$, and $\Delta^2\left(\chi^2\right)$.

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1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial work on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [5], Moricz [9], Moricz and Rhoades [10], Basarir and Solankan [2], Tripathy [17], Turkmenoglu [19], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n\in N} \left|x_{mn}\right|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} \left|x_{mn} - l\right|^{t_{mn}} = 1\, for\, some\, l \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} \left|x_{mn}\right|^{t_{mn}} = 1\right\},$$

$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left|x_{mn}\right|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [21,22] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [27] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also have examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [28] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and have examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [29] have studied the space $\chi^2_M(p, q, u)$ of double sequences and have given some inclusion relations.

Spaces are strongly summable sequences was discussed by Kuttner [31], Maddox [32], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [8] as an extension of the definition of strongly Cesàro summable sequences. Connor [33] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [34] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [35]-[38], and [39] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton. In their work and throughout this paper, the four dimensional matrices and double sequences have real-valued entries unless specified otherwise. In this paper we extend a few results known in the literature for ordinary(single) sequence spaces to multiply sequence spaces.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see[1]).

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{all\, finite\, sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.
If X is a sequence space, we give the following definitions:

(i)X'= the continuous dual of X;

(ii)$X^{\alpha} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn} x_{mn}| < \infty, \, for\, each\, x \in X \right\}$;

(iii)$X^{\beta} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn} x_{mn}\, is\, convegent, \, for\, each\, x \in X \right\}$;

(iv)$X^{\gamma} = \left\{ a = (a_{mn}) : sup_{mn} \geq 1 \left| \sum_{m,n=1}^{M,N} a_{mn} x_{mn} \right| < \infty, for\, each\, x \in X \right\}$;

(v)$let\, X\, be an FK - space \supset \phi; then\, X^f = \left\{ f(\Im_{mn}) : f \in X' \right\}$;

(vi)$X^{\delta} = \left\{ a = (a_{mn}) : sup_{mn} |a_{mn} x_{mn}|^{1/m+n} < \infty, \, for\, each\, x \in X \right\}$;

$X^{\alpha}.X^{\beta}, X^{\gamma}$ are called $\alpha - (or K\ddot{o}the - Toeplitz)$dual of $X, \beta - (or\, generalized - K\ddot{o}the - Toeplitz)\, dual\, of X, \gamma - dual\, of\, X, \, \delta - dual\, of X\, respectively. X^{\alpha}$ is defined by Gupta and Kamptan [20]. It is clear that $x^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [30] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.
Here c, c_0 and ℓ_{∞} denote the classes of convergent,null and bounded sclar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by BaŞar and Altay in [42] and in the case $0 < p < 1$ by Altay and BaŞar in [43]. The spaces $c(\Delta), c_0(\Delta), \ell_{\infty}(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

2. Definitions and Preliminaries

Definition 1. A sequence X is $v-$ invariant if $X_v = X$ where

$$X_v = \{x = (x_{mn}) : (v_{mn} x_{mn}) \in X\},$$

where $X = \Lambda^2$ and χ^2.

Definition 2. We say that a sequence space $\Delta^2(X)$ is $v-$ invariant if $\Delta_v^2(X) = \Delta^2(X)$.

Defintion 3. Let $A = \left(a_{k,\ell}^{mn}\right)$ denotes a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the $k,\ell-$ th term to Ax is as follows:

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$$

such transformation is said to be nonnegative if $a_{k\ell}^{mn}$ is nonnegative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman [40] and Toeplitz [41]. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which they both added an adiditional assumption of boundedness. This assumption was made because a double sequence which is $P-$ convergent is not necessarily bounded.

3.Main Results

Lemma 4. $sup_{mn}\left|x_{mn}-x_{mn+1}-x_{m+1n+1}+x_{m+1n+1}\right|^{1/m+n}<\infty$ if and only if
(i) $sup_{mn}\left(mn\right)^{-1}\left|x_{mn}\right|^{1/m+n}<\infty$
(ii)sup_{mn}
$\left|x_{mn}-\left(mn\right)\left(mn+1\right)^{-1}x_{mn+1}-\left(mn\right)\left(m+1n\right)^{-1}x_{m+1n}+\left(mn\right)\left(m+1n+1\right)^{-1}x_{m+1n+1}\right|^{1/m+n}$
$<\infty$.
If we consider Lemma (3.1), then we have the following result:

Corollary 5. sup_{mn}
$\left|\left(mn\right)\frac{v_{mn}}{w_{mn}}-\left(mn+1\right)\frac{v_{mn+1}}{w_{mn+1}}-\left(m+1n\right)\frac{v_{m+1n}}{w_{m+1n}}+\left(m+1n+1\right)\frac{v_{m+1n+1}}{w_{m+1n+1}}\right|^{1/m+n}<\infty$
if and only if (i)$sup_{mn}\left|\frac{v_{mn}}{w_{mn}}\right|^{1/m+n}<\infty$ (ii) $sup_{mn}\left(mn\right)\left|\frac{v_{mn}}{w_{mn}}-\frac{v_{mn+1}}{w_{mn+1}}-\frac{v_{m+1n}}{w_{m+1n}}+\frac{v_{m+1n+1}}{w_{m+1n+1}}\right|^{1/m+n}<\infty$.

Theorem 6. $\Delta_w^2\left(\Lambda^2\right)\subset\Delta_v^2\left(\Lambda^2\right)$ if and only if the matrix $A=\left(a_{k\ell}^{mn}\right)$ maps Λ^2 into Λ^2 where

$$a_{k\ell}^{mn}=\begin{cases} v_{i+1j+1}+v_{i+1j}+v_{ij+1}-v_{ij}, & \\ \frac{v_{i+1j+1}}{w_{i+1j+1}} & \text{if } m,n\text{=i+1,j+1} \\ \frac{v_{i+1j}}{w_{i+1j}} & \text{if } m,n\text{=i+1,j} \\ \frac{v_{ij+1}}{w_{ij+1}} & \text{if } m,n \text{ i,j+1} \\ 0 & \text{if } m,n\geq \text{i+2,j+2} \end{cases} \tag{2}$$

Proof: Let $y\in\Lambda^2$. Define $x_{ij}=-\frac{\sum_{m=1}^{i}\sum_{n=1}^{j}|y_{mn}|^{1/m+n}}{w_{pq}}$. Then $w_{ij}x_{ij}-w_{ij+1}x_{ij+1}-w_{i+1j}x_{i+1j}+w_{i+1j+1}x_{i+1j+1}\in\Lambda^2$. Hence $\Delta_w^2x\in\Lambda^2$, so by assumption $\Delta_vx\in\Lambda^2$. It follows that $Ay\in\Lambda^2$. This shows that $A=\left(a_{k\ell}^{mn}\right)$ maps Λ^2 into Λ^2. Let $x\in\Delta_w\left(\Lambda^2\right)$, hence $\left(w_{ij}x_{ij}-w_{ij+1}x_{ij+1}-w_{i+1j}x_{i+1j}+w_{i+1j+1}x_{i+1j+1}\right)\in\Lambda^2$. Then also $y=\left(w_{i-1j-1}x_{i-1j-1}-w_{i-1j}x_{i-1j}-w_{ij-1}x_{ij-1}+w_{ij}x_{ij}\right)\in\Lambda^2$, where $w_{-1}=x_{-1}=0$. By assumption we have $Ay\in\Lambda^2$. Hence $x\in\Delta_v^2\left(\Lambda^2\right)$. This completes the proof.

Theorem 7. Let X and Y be sequence spaces and assume that X is such that a sequence $\begin{pmatrix} x_{11} & x_{12} & x_{13}\cdots & x_{1n} & 0 \\ x_{21} & x_{22} & x_{23}\cdots & x_{2n} & 0 \\ \vdots & & & & \\ x_{m1} & x_{m2} & x_{m3}\cdots & x_{mn} & 0 \\ 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix}$ belongs to X if and only if the sequence $\begin{pmatrix} x_{21} & x_{22} & x_{23}\cdots & x_{2n} & 0 \\ x_{31} & x_{32} & x_{33}\cdots & x_{3n} & 0 \\ \vdots & & & & \\ x_{m1} & x_{m2} & x_{m3}\cdots & x_{mn} & 0 \\ 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix}$.

Then we have $\Delta_w^2(X) \subset \Delta_v^2(Y)$ if and only if the matrix $A = (a_{k\ell}^{mn})$ maps X into Y, where $A = (a_{k\ell}^{mn})$ is defined by equation (3.1).

The proof is very similar to that of Theorem (3.3).

Corollary 8. We have $\Delta_w^2(\Lambda^2) \subset \Delta_v^2(\Lambda^2)$ if and only if

$$sup_{mn}\left|mn\tfrac{v_{mn}}{w_{mn}} - mn+1\tfrac{v_{mn+1}}{w_{mn+1}} - m+1n\tfrac{v_{m+1n}}{w_{m+1n}} + m+1n+1\tfrac{v_{m+1n+1}}{w_{m+1n+1}}\right|^{1/m+n} < \infty.$$

Proof: This follows from Theorem 3.3, the well known characterization of matrices mapping Λ^2 into Λ^2 and corollary 3.2.

Corollary 9. We have $\Delta_w^2(\chi^2) \subset \Delta_v^2(\chi^2)$ if and only if

$$\left((m+n)!\left|mn\tfrac{v_{mn}}{w_{mn}} - mn+1\tfrac{v_{mn+1}}{w_{mn+1}} - m+1n\tfrac{v_{m+1n}}{w_{m+1n}} + m+1n+1\tfrac{v_{m+1n+1}}{w_{m+1n+1}}\right|\right)^{1/m+n} \to$$
0 as $m, n \to \infty$.

Proof: This follows from Theorem 3.5, the well-known characterization of matrices mapping χ^2 into χ^2 and corollary 3.2.

If we consider corollary 3.5 and corollary 3.6, then we have necessary and sufficient conditions for the $v-$invariant of $\Delta^2(\Lambda^2)$ and $\Delta^2(\chi^2)$.

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