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FIXED POINT THEOREMS BY ALTERING DISTANCES FOR OCCASIONALLY WEAKLY COMPATIBLE HYBRID D-MAPPINGS AND APPLICATIONS

Valeriu Popa and Alina-Mihaela Patriciu

Abstract. The purpose of this paper is to prove a general fixed point theorem by altering distance for two owc pairs hybrid mappings and to reduce the study of fixed points of pairs of mappings satisfying a contractive condition of integral type at the study of fixed point in metric spaces by altering distances satisfying an implicit relation, which generalize some results from [1], [6], [9], [21], [27] and other papers.

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1. Introduction and Preliminaries

Let (X,d) be a metric space and $B(X)$ the set of all nonempty bounded subsets of X. As in [11] and [12] we define the functions $\delta(A,B)$ and $D(A,B)$, where $A,B \in B(X)$ by

$$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\},$$
$$D(A,B) = \inf\{d(a,b) : a \in A, b \in B\}, A, B \in B(X).$$

If A consists of a single point a, we write $\delta(A,B) = \delta(a,B)$. If B consists also of a single point b, we write $\delta(A,B) = d(a,b)$.

It follows immediately from definition of δ that

$$\delta(A,B) = \delta(B,A),$$
$$\delta(A,C) \leq \delta(A,B) + \delta(B,C),$$
$$\delta(A,B) = 0 \textit{ then } A = B = \{a\}, \textit{ for } A,B,C \in B(X).$$

Definition 1.1 *A sequence $\{A_n\}$ of a nonempty subset of X is said to be convergent to a set A of X [11], [12] if* *(i) each point* $a \in A$ *is the limit of a convergent sequence* $\{a_n\}$ *where* $a_n \in A_n$ *for all* $n \in N$.

(ii) for arbitrary $\varepsilon > 0$, *there exists an integer* $m > 0$ *such that* $\{A_n\} \subset A_\varepsilon$ *for* $n > m$, *where* A_ε *denote the set of all points* $x \in X$ *for which there exists a point* $a \in A$, *depending on* x, *such that* $d(x, a) < \varepsilon$.

A is said to be the limit of the sequence $\{A_n\}$.

Lemma 1.1 (Fisher, [11]) *If* $\{A_n\}$ *and* $\{B_n\}$ *are sequences in* $B(X)$ *converging to* A *and* B, *respectively, in* $B(X)$, *then the sequence* $\delta(A_n, B_n)$ *converges to* $\delta(A, B)$.

Lemma 1.2 (Fisher and Sessa, [12]) *Let* $\{A_n\}$ *be a sequence in* $B(X)$ *and* $y \in X$ *such that* $\delta(A_n, y) \to 0$, *then the sequence* $\{A_n\}$ *converges to the set* $\{y\}$ *in* X.

Let A and S be self mappings of a metric space (X, d). Jungck [14] defined A and S to be compatible if $\lim d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim Sx_n = \lim Tx_n = t$ for some $t \in X$.

A point $x \in X$ is a coincidence point of A and S if $Ax = Sx$. We denote $C(A, S)$ the set of all coincidence points of A and S.

In [24], Pant defined A and S to be pointwise R-weakly commuting mappings if for all $x \in X$, there exists $R > 0$ such that $d(SAx, ASx) \leq R \cdot d(Ax, Sx)$. It is proved in [25] that pointwise R-weakly commuting is equivalent to the commuting at coincidence points.

Definition 1.2 *A and S is said to be weakly compatible [15] if* $ASu = SAu$ *for* $u \in C(A, S)$.

Definition 1.3 *A and S is said to be occasionally weakly compatible [5] if* $ASu = SAu$ *for some* $u \in C(A, S)$.

Remark 1.1 *If A and S are weakly compatible and* $C(A, S) \neq 0$ *then A and S are owc, but the converse is not true (Example [5]).*

Some fixed point theorems for owc mappings are proved in [2], [3], [18] and other papers.

Definition 1.4 *Let* $f : X \to X$ *and* $F : X \to B(X)$ *be. Then:*

1) a point $x \in X$ *is said to be a coincidence point of f and F if* $fx \in Fx$. *We denote* $C(f, F)$ *the set of all coincidence points of f and F.*

2) a point $x \in X$ *is said to be a strict coincidence point of f and F if* $\{fx\} = Fx$.

3) a point $x \in X$ *is a fixed point of F if* $x \in Fx$.

4) a point $x \in X$ is a strict fixed point of F if $\{x\} = Fx$.

Definition 1.5 *The mappings $f : X \to X$ and $F : X \to B(X)$ is said to be δ - compatible [16] if $\lim \delta(Ffx_n, fFx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $fFx_n \in B(X)$, $fx_n \to t$, $Fx_n \to \{t\}$ for some $t \in X$.*

Definition 1.6 *The pair $f : X \to X$ and $F : X \to B(X)$ is weakly compatible [17] if for each $x \in C(f, F)$, $fFx = Ffx$.*

If the pair (f, F) is δ - compatible, then (f, F) is weakly compatible but the converse is not true, in general [17].

Definition 1.7 *Let S and T be two single valued self mappings of a metric space (X, d). We say that S and T satisfy property $(E.A)$ [1] if there exists a sequence $\{x_n\}$ in X such that $\lim Tx_n = \lim Sx_n = t$ for some $t \in X$.*

Remark 1.2 *It is obvious that two self mappings T and S of a metric space (X, d) will be noncompatible if there exists at least one sequence $\{x_n\}$ such that either $\lim Sx_n = \lim Tx_n = t$ for some $t \in X$ but $\lim d(STx_n, TSx_n)$ is either nonzero or does not exists. Therefore, two noncompatible mappings of a metric space (X, d) satisfy property $(E.A)$.*

Recently, Djoudi and Khemis [9] introduced a generalization of pair of mappings satisfying property $(E.A)$.

Definition 1.8 *The mappings $I : X \to X$ are $F : X \to B(X)$ are said to be D - mappings if there exists a sequence $\{x_n\}$ in X such that $\lim Ix_n = t$ and $\lim Fx_n = \{t\}$ for some $t \in X$.*

Obvious, two mappings which are not δ - compatible are D - mappings.

Some fixed point theorems for D - mappings are proved in [6], [7] and in other papers.

Definition 1.9 *The hybrid pair $f : X \to X$ are $F : X \to B(X)$ are strict occasionally weakly compatible (sowc) if there exists $x \in X$ such that $\{fx\} = Fx$ implies $fFx = Ffx$.*

Remark 1.3 *If the pair (f, F) is weakly compatible and $C(f, F) \neq \phi$, then the pair (f, F) is sowc. There exists sowc pairs which are not weakly compatible ([4], Example 1.12).*

Let $\varphi : R_+ \to R_+$ satisfying the following conditions:
(φ_1): φ is continuous,
(φ_2): φ is nondecreasing on R_+,
(φ_3): $0 < \varphi(t) < t$ for $t > 0$.
The following theorem is proved in [1].

Theorem 1.1 *Let A, B, S and T self mappings of a metric space (X, d) such that:*
(1.1) $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,
(1.2) $d(Ax, By) \leq \varphi(\max\{d(Sx, Ty), d(Ty, By), d(Sx, By)\})$,
(1.3) (A, S) *and* (B, T) *are weakly compatible,*
(1.4) (A, S) *or* (B, T) *satisfy property* $(E.A)$.
If one $A(X)$, $B(X)$, $S(X)$, $T(X)$ is closed in X, then A, B, S, T have a unique common fixed point.

For D - mappings the following theorems are recently proved.

Theorem 1.2 ([9]) *Let $F, G : X \to B(X)$ and $I, J : X \to X$ such that*
(1.5) $F(X) \subset J(X)$ *and* $G(X) \subset I(X)$,
(1.6) $\delta(Fx, Gy) < \max\{cd(Ix, Jy), c\delta(Ix, Fx), c\delta(Jy, Fx), aD(Ix, Gy) + bD(Jy, Fx)\}$,
for all $x, y \in X$ where $0 \leq c < 1$, $0 \leq a + b < 1$, holds whenever the right side of (1.6) is positive. If the pairs (F, I) and (G, J) are weak compatible and D - mappings and either $F(X)$ or $G(X)$ (respectively, $I(X)$ or $J(X)$) is closed, then I, J, F and G have a unique common fixed point in X.

Theorem 1.3 ([6]) *Let (X, d) be a metric space, $F, G : X \to B(X)$ and $I, J : X \to X$ satisfying the following conditions:*
(1.7) $F(X) \subset J(X)$ *and* $G(X) \subset I(X)$,
(1.8) $\delta(Fx, Gy) < \alpha \max\{d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Fx)\} + (1-\alpha)(aD(Ix, Gy) + bD(Jy, Fx))$, *for all $x, y \in X$, where $0 \leq \alpha < 1$, $a \geq 0$, $b \geq 0$, $a + b < 1$ holds whenever the right side of (1.8) is positive. If either*
(1.9) F and J are weakly compatible D- mappings and (G, J) are weakly compatible and $F(X)$ or $J(X)$ is closed, or
(1.10) G and J are weakly compatible D- mappings and (F, I) are weakly compatible and $G(X)$ or $I(X)$ is closed,
then there exists an unique fixed point t in X such that $Ft = Gt = \{t\} = \{It\} = \{Jt\}$.

2. Contractive conditions of integral type

In [8], Branciari established the following result

Theorem 2.1 *Let (X, d) be a complete metric space, $c \in (0, 1)$ and $f : X \to X$ be a mapping such that*

$$\int_0^{d(fx,fy)} h(t)dt \leq c \int_0^{d(x,y)} h(t)dt, \tag{1}$$

where $h : [0, \infty) \to [0, \infty)$ is a Lebesgue measurable mapping which is summable (i.e. with a finite integral) on each compact subset of $[0, \infty)$, such that for $\varepsilon > 0$, $\int_0^{\varepsilon} h(t)dt > 0$. Then f has a unique fixed point $z \in X$ such that for each $x \in X$, $\lim f^n x = z$.

Quite recently, Kumar et al. [21] extended Theorem 2.1 for two compatible mappings satisfying the following conditions:

Theorem 2.2 *Let $f, g : (X, d) \to (X, d)$ compatible mappings such that*
(i) $f(X) \subset g(X)$,
(ii) *g is continuous, and*

$$\int_0^{d(fx,fy)} h(t)dt \leq c \int_0^{d(gx,gy)} h(t)dt, \tag{2}$$

for all $x, y \in X$, $c \in (0, 1)$,
where h is as in Theorem 2.1. Then f and g have a unique common fixed point.

Definition 2.1 *Let X be a nonempty set. A symmetric on X is a non-negative real valued function D on $X \times X$ such that*
(i) $D(x, y) = 0$ *if and only if* $x = y$,
(ii) $D(x, y) = D(y, x)$ *for all* $x, y \in X$.

Some fixed point theorems in metric and symmetric spaces for compatible, weak compatible and occasionally compatible mappings satisfying a contractive condition of integral type are studied in [2], [3], [20], [23], [33] and other papers.

Let (X, d) be a metric space and $D(x, y) = \int_0^{d(x,y)} h(t)dt$, where $h(t)$ is as in Theorem 2.1. It is proved in [23], [29] that the study of the fixed points for mappings satisfying a contractive condition of integral type is reduced to the study of fixed points in symmetric spaces. The method is not applicable for hybrid pair of mappings.

Definition 2.2 *An altering distance is a mapping* $\psi : [0, \infty) \to [0, \infty)$ *which satisfies the following conditions:*
(i) ψ *is increasing and continuous,*
(ii) $\psi(t) = 0$ *if and only if* $t = 0$.

Fixed point problem involving altering distances have been studied in [10], [19], [22], [27], [31], [32] and other papers. In [22], the following theorem is proved.

Theorem 2.3 *Let* (X, d) *be a metric space and* $T : X \to B(X)$ *a multivalued mapping satisfying the following inequality*

$$\begin{aligned}\psi(\delta(Tx, Ty)) \quad &\leq \quad a\psi(d(x, y)) + b[\psi(\delta(x, Tx)) + \psi(\delta(y, Ty))] + \\ &\quad +c \min\{\psi(D(x, Ty)), \psi(D(y, Tx))\},\end{aligned}$$

for all $x, y \in X$, *where* $a > 0$, $b, c \geq 0$ *such that* $a + 2b < 1$ *and* $a + c < 1$, *then* T *has a unique fixed point.*

In [10], Theorem 2.3 is generalized for two pairs of hybrid weakly compatible mappings.

Theorem 2.4 *Let* (X, d) *be a metric space,* $I, J : X \to X$, $T, S : X \to B(X)$ *such that:*

$$\begin{aligned}\psi(\delta(Tx, Sy)) \leq a\psi(d(Ix, Jy)) + b[\psi(\delta(Ix, Tx)) + \psi(\delta(Jy, Sy))]+ \\ +c \min\{\psi(D(Ix, Sy)), \psi(D(Jy, Tx))\},\end{aligned} \tag{3}$$

for all $x, y \in X$, $x \neq y$, *where* $a + 2b + c < 1$, $a, b, c \geq 0$.
Suppose in addition that either
I) (T, I) *is compatible,* I *is continuous and* (J, S) *is weakly compatible,*
II) (S, J) *is compatible,* J *is continuous and* (I, T) *is weakly compatible.*
Then I, J, S *and* T *have a unique common fixed point.*

Lemma 2.1 *The function* $\psi(x) = \int_0^x h(t)dt$, *where* h *is as in Theorem 2.1, is an altering distance.*

Proof. By definitions of ψ and h it follows that $\psi(x)$ is increasing and $\psi(x) = 0$ if and only if $x = 0$.
By Lemma 2.5 [23], $\psi(t)$ is continuous.
In [27] a general fixed point theorem for compatible mappings satisfying an implicit relation is proved. In [13], the results from [27] are improved relaxing compatibility to weak compatibility.

The purpose of this paper is to prove a general fixed point theorem by altering distance for two D - mappings pairs and to reduce the study of fixed point of pairs of mappings satisfying a contractive condition of integral type at the study of fixed points in metric spaces by altering distance satisfying an implicit relation, which generalize some results from [1], [6], [9], [21], [27].

3. Implicit relations

Let $\mathcal{F}_D$ be the set of all real continuous mappings $\phi(t_1, ..., t_6) : R_+^6 \to R$ satisfying the following conditions:

(ϕ_1): ϕ is nonincreasing in variables t_5 and t_6,

(ϕ_2): $\phi(t, 0, 0, t, t, 0) \leq 0$ or $\phi(t, 0, t, 0, 0, t) \leq 0$ implies $t = 0$,

(ϕ_3): $\phi(t, t, 0, 0, t, t) \geq 0$, $\forall t > 0$.

Example 3.1 $\phi(t_1, ..., t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, where $0 \leq c < 1$, $a \geq 0$, $b \geq 0$ and $a + b < 1$.

(ϕ_1): Obviously.

(ϕ_2): $\phi(t, 0, 0, t, t, 0) = t(1 - \max\{a, c\}) \leq 0$ implies $t = 0$.

Similarly, $\phi(t, 0, t, 0, 0, t) = t(1 - \max\{b, c\}) \leq 0$ implies $t = 0$,

(ϕ_3): $\phi(t, t, 0, 0, t, t) = t(1 - \max\{c, a + b\}) \geq 0$, $\forall t > 0$.

Example 3.2 $\phi(t_1, ..., t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)(at_5 + bt_6)$, where $0 \leq \alpha < 1$, $a \geq 0$, $b \geq 0$ and $a + b < 1$.

(ϕ_1): Obviously.

(ϕ_2): $\phi(t, 0, 0, t, t, 0) = (1 - \alpha)(1 - a)t \leq 0$ implies $t = 0$. $\phi(t, 0, t, 0, 0, t) = t(1 - \alpha)(1 - b) \leq 0$ implies $t = 0$,

(ϕ_3): $\phi(t, t, 0, 0, t, t) = t(1 - (a + b)) \geq 0$, $\forall t > 0$.

Example 3.3 $\phi(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c\min\{t_5, t_6\}$, where $a \geq 0$, $b \geq 0$, $c \geq 0$, $b < 1$ and $a + c < 1$.

Example 3.4 $\phi(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6)$, where $a \geq 0$, $b \geq 0$, $c \geq 0$, $b + c < 1$ and $a + 2c < 1$.

Example 3.5 $\phi(t_1, ..., t_6) = t_1 - \max\left\{t_2, \frac{1}{2}(t_3 + t_4), \frac{1}{2}[(t_5 + t_6)k]\right\}$, where $0 \leq k < 1$.

Example 3.6 $\phi(t_1, ..., t_6) = t_1 - \max\left\{k_1 t_2, \frac{k_2}{2}(t_3 + t_4), \frac{t_5 + t_6}{2}\right\}$, where $0 \leq k_1 < 1$, $1 \leq k_2 < 2$.

Example 3.7 $\phi(t_1, ..., t_6) = t_1 - \max\left\{k_1(t_2 + t_3 + t_4), \frac{k_2}{2}(t_5 + t_6)\right\}$, where $0 \le k_1 < 1$, $0 \le k_2 < 1$.

Example 3.8 $\phi(t_1, ..., t_6) = t_1 - h\max\{t_2, t_3, t_4, t_5, t_6\}$, where $0 \le h < 1$.

Example 3.9 $\phi(t_1, ..., t_6) = t_1^2 - at_2^2 - t_3t_4 - bt_5^2 - ct_6^2$, where a, b, $c \ge 0$ and $a + b + c < 1$.

Example 3.10 $\phi(t_1, ..., t_6) = t_1^3 - k(t_2^3 + t_3^3 + t_4^3 + t_5^3 + t_6^3)$, where $0 \le k < \frac{1}{3}$.

Example 3.11 $\phi(t_1, ..., t_6) = t_1^3 - at_1^2t_2 - bt_1t_3t_4 - ct_5^2t_6 - dt_5t_6^2$, where a, b, c, $d \ge 0$ and $a + c + d < 1$.

Example 3.12 $\phi(t_1, ..., t_6) = t_1^3 - \frac{t_3^2t_4^2 + t_5^2t_6^2}{1 + t_2 + t_3 + t_4}$.

Example 3.13 $\phi(t_1, ..., t_6) = t_1 - \varphi\left(\max\left\{t_2, t_3, t_4, t_5, \frac{t_6}{2}\right\}\right)$.

Example 3.14 $\phi(t_1, ..., t_6) = t_1 - \varphi\left(\max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}\right)$.

Example 3.15 $\phi(t_1, ..., t_6) = t_1 - \varphi\left(\max\left\{t_2, t_3, t_4, \frac{k}{2}(t_5 + t_6)\right\}\right)$, where $0 \le k < 1$.

4. Main results

Theorem 4.1 *Let $I : X \to X$ and $F : X \to B(X)$ sowc mappings. If I and F have a unique point of strict coincidence $\{z\} = \{Ix\} = Fx$, then z is the unique common fixed point of I and F which is a strict fixed point for F.*

Proof. Since I and F are sowc, there exists $x \in X$ such that $\{z\} = \{Ix\} = Fx$ implies $IFx = FIx$. Then $\{Iz\} = \{IIx\} = IFx = FIx = Fz = \{u\}$. Hence, u is a point of strict coincidence of I and F. By hypothesis $z = u$, hence, $\{z\} = \{Iz\} = Fz$ and z is a common fixed point of I and F which is a strict fixed point for F. Suppose that $v \ne z$ is another common fixed point of I and F which is a strict fixed point of F. Hence $\{v\} = \{Iv\} = Fv$. Therefore, v is a point of strict coincidence of I and F and by hypothesis, $v = z$.

Theorem 4.2 *Let $I, J : X \to X$, $F, G : X \to B(X)$ such that*

$$\begin{gathered}\phi(\psi(\delta(Fx, Gy)), \psi(d(Ix, Jy)), \psi(\delta(Ix, Fx)), \\ \psi(\delta(Jy, Gy)), \psi(D(Ix, Gy)), \psi(D(Jy, Fx))) < 0\end{gathered} \tag{4}$$

for all $x, y \in X$, where ψ is an altering distance and ϕ satisfies property (ϕ_3). Suppose that there exist $x, y \in X$ such that $\{u\} = \{Ix\} = Fx$ and $\{v\} = \{Jy\} = Gy$. Then, u is the unique point of strict coincidence of I and F and v is the unique point of strict coincidence of J and G.

Proof. First we prove that $Ix = Jy$. Suppose that $Ix \neq Jy$. By (4) we obtain

$$\phi(\psi(d(Ix, Jy)), \psi(d(Ix, Jy)), 0, 0, \psi(d(Ix, Jy)), \psi(d(Ix, Jy))) < 0,$$

a contradiction of (ϕ_3). Hence $Ix = Jy$ and $\{u\} = \{Ix\} = \{Jy\} = Fx = Gy$.

Suppose that there exists $z \in X$ such that $\{w\} = \{Iz\} = Fz$. Then by (4) we obtain

$$\phi(\psi(d(Iz, Jy)), \psi(d(Iz, Jy)), 0, 0, \psi(d(Iz, Jy)), \psi(d(Iz, Jy))) < 0,$$

a contradiction of (ϕ_3). Hence $Iz = Jy$. Therefore $\{w\} = \{Iz\} = \{Jy\} = Gy = Fx = \{Ix\} = \{u\}$. Hence u is the unique point of strict coincidence of I and F. Similarly, v is the unique point of strict coincidence of J and G.

Theorem 4.3 *Let (X, d) be a metric space, $I, J : X \to X$ and $F, G : X \to B(X)$ satisfying the following conditions:*

(4.1) the inequality (4) holds for all $x, y \in X$, where ψ is an altering distance and $\phi \in \mathcal{F}_D$;

(4.2) $F(X) \subset J(X)$ and $G(X) \subset I(X)$.

If the pairs (F, I) and (G, J) are D - mappings and $F(X)$ (resp. $J(X)$) or $G(X)$ (resp. $I(X)$), is a closed set of X, then

(4.3) I and F have a strict coincidence point,

(4.4) J and G have a strict coincidence point.

Moreover, if the pairs (I, F) and (J, G) are sowc, then I, J, F, G have a unique common fixed point which is a strict fixed point for F and G.

Proof. Since the pairs (F, I) are D - mappings then there exists a sequence $\{x_n\}$ in X such that $\lim Ix_n = \lim Fx_n = \{t\}$ for some $t \in X$. Since $F(X)$ is closed and $F(X) \subset J(X)$, there exists $u \in X$ such that $t = Ju$. By (4) we have

$$\begin{gathered}\phi(\psi(\delta(Fx_n, Gu)), \psi(d(Ix_n, Ju)), \psi(\delta(Ix_n, Fx_n)),\\ \psi(\delta(Ju, Gu)), \psi(D(Ix_n, Gu)), \psi(D(Ju, Fx_n))) < 0.\end{gathered}$$

Letting n tend to infinity we obtain

$$\phi(\psi(\delta(Ju, Gu)), 0, 0, \psi(\delta(Ju, Gu)), \psi(D(Ju, Gu)), 0) \leq 0.$$

By (ϕ_2) it follows that $\psi(\delta(Ju, Gu)) = 0$ which implies $\{Ju\} = Gu$. Hence J and G have a strict coincidence point. Since $G(X) \subset I(X)$, there exists a point $v \in X$ such that $\{Iv\} = Gu$. Then by (4) and (ϕ_1) we have successively:

$$\begin{gathered}\phi(\psi(\delta(Fv, Gu)), \psi(d(Iv, Ju)), \psi(\delta(Iv, Fv)),\\ \psi(\delta(Ju, Gu)), \psi(\delta(Iv, Gu)), \psi(\delta(Ju, Fv))) < 0,\end{gathered}$$

$$\phi(\psi(\delta(Fv, Gu)), 0, \psi(d(Fv, Gu)), 0, 0, \psi(\delta(Fv, Gu))) < 0.$$

By (ϕ_2) we obtain $\psi(\delta(Fv, Gu)) = 0$ which implies $Fv = Gu = \{Iv\}$. Hence F and I have a strict coincidence point. Therefore, $\{t\} = \{Ju\} = Gu = Fv = \{Iv\}$. By Theorem 4.2, t is the unique point of strict coincidence of I and F, also t is the unique point of strict coincidence of J and G. If the pairs (I, F) and (J, G) are sowc then by Theorem 4.1 t is the unique fixed point of I and F and for J and G which is a strict fixed point for F and G.

If $I = J$ and $F = G$ we obtain

Theorem 4.4 *Let (X, d) be a metric space, $I : X \to X$ and $F : X \to B(X)$ satisfying the following conditions:*

a) $F(X) \subset I(X)$,

b) $$\begin{gathered}\phi(\psi(\delta(Fx, Fy)), \psi(d(Ix, Iy)), \psi(\delta(Ix, Fx)),\\ \psi(\delta(Iy, Fy)), \psi(D(Ix, Fy)), \psi(D(Iy, Fx))) < 0,\end{gathered}$$

for all $x, y \in X$, where ψ is an altering distance and $\phi \in \mathcal{F}_D$.

If (I, F) are D - mappings and $F(X)$ (or $I(X)$) is a closed set of X, then

c) I and F have a strict coincidence point.

Moreover, if the pair (I, F) is sowc, then I and F have a unique common fixed point which is a strict fixed point for F.

If f, g, I, J are single valued mappings we obtain

Theorem 4.5 *Let (X, d) be a metric space, $I, J, f, g : X \to X$ satisfying the following conditions:*

a) $f(X) \subset J(X)$, $g(X) \subset I(X)$,

b) $$\begin{gathered}\phi(\psi(d(fx, gy)), \psi(d(Ix, Jy)), \psi(d(Ix, fx)),\\ \psi(d(Jy, gy)), \psi(d(Ix, gy)), \psi(d(Jy, fx))) < 0\end{gathered}$$

for all $x, y \in X$, where ψ is an altering distance and $\phi \in \mathcal{F}_D$.

If the pair (f, I) or (g, J) have property $(E.A)$ and $f(X)$ (resp. $J(X)$) or $g(X)$ (resp. $I(X)$) are closed sets of X, then

c) I and f have a coincidence point,

d) J and g have a coincidence point.

Moreover, if the pairs (I,f) and (J,g) are owc, then I,J,f and g have a unique common fixed point.

If $\psi(t)=t$, then by Theorem 4.3 we obtain the following theorem

Theorem 4.6 (Theorem 3.3 [27]) *Let (X,d) be a metric space, $I,J:X\to X$ and $F,G:X\to B(X)$ satisfying the following conditions:*
a) $F(X)\subset J(X)$ *and* $G(X)\subset I(X)$,
b) $\phi(\delta(Fx,Gy),d(Ix,Jy),\delta(Ix,Fx),\delta(Jy,Gy),D(Ix,Gy),D(Jy,Fx))<0$
for all $x,y\in X$, where $\phi\in\mathcal{F}_D$.
If the pairs (I,F) or (J,G) are D - mappings and $F(X)$ (resp. $J(X)$) or $G(X)$ (resp. $I(X)$) are closed sets in X, then
c) I and F have a strict coincidence point,
d) J and G have a strict coincidence point.
Moreover, if the pairs (F,I) and (G,J) are sowc, then I,J,F and G have a unique common fixed point which is a strict fixed point for F and G.

Remark 4.1 *a) By Theorem 4.6 and Example 3.1 we obtain a generalization of Theorem 1.2.*
b) By Theorem 4.6 and Example 3.2 we obtain a generalization of Theorem 1.3.
c) By Theorem 4.6 and Examples 3.3 - 3.15 we obtain new results.

If $\psi(t)=t$, then by Theorem 4.4 we obtain

Theorem 4.7 *Let (X,d) be a metric space, $I:X\to X$ and $F:X\to B(X)$ satisfying the following conditions:*
a) $F(X)\subset I(X)$,
b) $\phi(\delta(Fx,Fy),d(Ix,Iy),\delta(Ix,Fx),\delta(Iy,Fy),D(Ix,Fy),D(Iy,Fx))<0$
for all $x,y\in X$ and $\phi\in\mathcal{F}_D$.
If (I,F) are D - mappings and $F(X)$ or $I(X)$ is a closed set of X, then
c) I and F have a strict coincidence point.
Moreover, if the pair (F,I) is sowc, then I and F have a unique common fixed point which is a strict fixed point for F.

Remark 4.2 *a) By Theorem 4.7 and Example 3.2 we obtain a generalization of Corollary 3.1 [6].*
b) By Theorem 4.7 and Example 3.1, 3.3 - 3.15 we obtain new results.

If $\psi(t)=t$, by Theorem 4.5 we obtain

Theorem 4.8 *Let (X,d) be a metric space and $I,J,f,g : X \to X$ single valued mappings satisfying the following conditions:*

a) $f(X) \subset J(X)$ *and* $g(X) \subset I(X)$,

b) $\phi(d(fx,gy),d(Ix,Jy),d(Ix,fx),d(Jy,gy),d(Ix,gy),d(Jy,fx)) < 0$

for all $x,y \in X$ *and* $\phi \in \mathcal{F}_D$.

If the pair (f,I) or (g,J) have property $(E.A)$ and $f(X)$ (resp. $J(X)$) or $g(X)$ (resp. $I(X)$) are closed subsets of X, then

c) I *and* f *have a coincidence point,*

d) J *and* g *have a coincidence point.*

Moreover, if the pairs (I,f) and (J,g) are owc, then I,J,f and g have a unique common fixed point.

Remark 4.3 *By Theorem 4.8 and Example 3.13 we obtain a generalization of Theorem 1.1 because*

$$\varphi(\max\{d(Sx,Ty),d(Sx,By),d(Ty,By)\} <$$
$$< \varphi\left(\max\left\{d(Sx,Ty),d(Ax,Sx),d(Ty,By),d(Sx,By),d(Sy,By),\tfrac{d(Ty,Ax)}{2}\right\}\right).$$

We denote $Fix(f) = \{x \in X : x = fx\}$ and $sFix(F) = \{x \in X : \{x\} = Fx\}$.

Theorem 4.9 *Let $I,J : X \to X$ and $F,G : X \to B(X)$ be single valued, respectively multivalued mapping. If the inequality (4) holds for all $x,y \in X$ and $\phi \in \mathcal{F}_D$, then:*

$$[Fix(I) \cap Fix(J)] \cap sFix(F) = [Fix(I) \cap Fix(J)] \cap sFix(G).$$

Proof. Let $u \in [Fix(I) \cap Fix(J)] \cap sFix(F)$. Then $\{u\} = \{Iu\} = \{Ju\} = Fu$. Then by (4) we have

$$\phi(\psi(\delta(u,Gu)),0,0,\psi(\delta(u,Gu)),\psi(\delta(u,Gu),0) < 0$$

which implies by (ϕ_2) that $\delta(u,Gu) = 0$ i.e. $\{u\} = Gu$, hence $u \in sFix(G)$, hence

$$[Fix(I) \cap Fix(J)] \cap sFix(F) \subset [Fix(I) \cap Fix(J)] \cap sFix(G).$$

Similarly,

$$[Fix(I) \cap Fix(J)] \cap sFix(G) \subset [Fix(I) \cap Fix(J)] \cap sFix(F).$$

Theorem 4.3 and Theorem 4.9 imply the next one

Theorem 4.10 *Let $I, J : X \to X$ be self mappings of a metric space (X, d) and $F_n : X \to B(X), n \in N^*$ be a sequence of set valued mappings such that*

a) $F_2(X) \subset I(X)$ and $F_1(X) \subset J(X)$,

b) the pairs (F_1, I) or (F_2, J) are D - mappings and either $F_2(X)$ or $F_1(X)$ (resp. $I(X)$ and $J(X)$) is a closed set of X,

c) the inequality

$$\phi(\psi(\delta(F_n x, F_{n+1} y)), \psi(d(Ix, Jy)), \psi(\delta(Ix, F_n x)),$$
$$\psi(\delta(Jy, F_{n+1} y)), \psi(D(Ix, F_{n+1} y), \psi(D(Jy, F_n x))) < 0$$

holds for all $x, y \in X$ and $n \in N^*$,

d) (I, F_1) are sowc D - mappings and (F_2, J) are sowc, or

e) (J, F_2) are sowc D - mappings and (F_1, I) are sowc.

Then, there exists a unique common fixed point of I, J, $\{F_n\}_{n \in N^*}$ which is a strict fixed point for $\{F_n\}, n \in N^*$.

Remark 4.4 *1. By Theorem 4.10 and Example 3.2 we obtain a generalization of Theorem 3.5 [6].*

2. If $\psi(t) = t$ then by Theorem 4.10 we obtain Theorem 3.6 [27].

5. Applications

Theorem 5.1 *Let (X, d) be a metric space, $I, J : X \to X$ and $F, G : X \to B(X)$ such that:*

a) $F(X) \subset J(X)$ and $G(X) \subset I(X)$,

b) $$\phi(\int_0^{\delta(Fx,Gy)} h(t)dt, \int_0^{d(Ix,Jy)} h(t)dt, \int_0^{\delta(Ix,Fx)} h(t)dt,$$
$$\int_0^{\delta(Jy,Gy)} h(t)dt, \int_0^{D(Ix,Gy)} h(t)dt, \int_0^{D(Jy,Fx)} h(t)dt) < 0$$

for all $x, y \in X$, where $\phi \in \mathcal{F}_D$ and $h(t)$ is as in Theorem 2.1.

If the pair (I, F) (or (J, G)) is D - mappings and $F(X)$ (resp. $J(X)$) or $G(X)$ (resp. $I(X)$) is a closed set of X, then

c) I and F have a strict coincidence point,

d) J and G have a strict coincidence point.

Moreover, if the pairs (I, F) and (J, G) are sowc, then I, J, F and G have a unique common fixed point which is a strict fixed point for F and G.

Proof. As in Lemma 2.1 we have

$$\begin{aligned} \psi(\delta(Fx, Gy)) &= \int_0^{\delta(Fx,Gy)} h(t)dt, \ \psi(d(Ix, Jy)) = \int_0^{d(Ix,Jy)} h(t)dt, \\ \psi(\delta(Ix, Fx)) &= \int_0^{\delta(Ix,Fx)} h(t)dt, \ \psi(\delta(Jy, Gy)) = \int_0^{\delta(Jy,Gy)} h(t)dt, \end{aligned}$$

$$\psi(D(Ix,Gy)) = \int_0^{D(Ix,Gy)} h(t)dt, \ \psi(D(Jy,Fx)) = \int_0^{D(Jy,Fx)} h(t)dt.$$

Then by b) we obtain

$$\phi(\psi(\delta(Fx,Gy)),\psi(d(Ix,Jy)),\psi(\delta(Ix,Fx)),$$
$$\psi(\delta(Jy,Gy)),\psi(D(Ix,Gy)),\psi(D(Jy,Fx))) < 0$$

for all $x, y \in X$ and $\phi \in \mathcal{F}_D$, which is the inequality (4) because by Lemma 2.1, $\psi(t)$ is an altering distance. Hence the conditions of Theorem 4.3 are satisfied and Theorem 5.1 it follows from Theorem 4.3.

If $F = G$ and $I = J$ by Theorem 4.1 we obtain

Theorem 5.2 *Let (X,d) be a metric space, $I : X \to X$, $F : X \to B(X)$ such that:*

a) $F(X) \subset I(X)$,

b) the inequality

$$\phi(\int_0^{\delta(Fx,Fy)} h(t)dt, \int_0^{d(Ix,Jy)} h(t)dt, \int_0^{\delta(Ix,Fx)} h(t)dt,$$
$$\int_0^{\delta(Jy,Fy)} h(t)dt, \int_0^{D(Ix,Fy)} h(t)dt, \int_0^{D(Iy,Fx)} h(t)dt) < 0$$

holds for all $x, y \in X$, $\phi \in \mathcal{F}_D$ and $h(t)$ as in Theorem 2.1.

If (I,F) are D - mappings and $F(X)$ or $I(X)$ is a closed set of X, then

c) I and F have a strict coincidence point.

Moreover, if the pair (I,F) is sowc, then I and F have a unique common fixed point which is a strict fixed point for F.

Corollary 5.1 *Let (X,d) be a metric space, $I : X \to X$, $F : X \to B(X)$ such that:*

a) $F(X) \subset I(X)$,

b) the inequality

$$\int_0^{\delta(Fx,Fy)} h(t)dt < a\int_0^{d(Ix,Jy)} h(t)dt + b\left[\int_0^{\delta(Ix,Fx)} h(t)dt + \int_0^{\delta(Jy,Fy)} h(t)dt\right] +$$
$$+c\min\left[\int_0^{D(Ix,Fy)} h(t)dt, \int_0^{D(Iy,Fx)} h(t)dt)\right]$$

holds for all $x, y \in X$, $h(t)$ as in Theorem 2.1, $a > 0$, $b \geq 0$, $c \geq 0$ and $a+2b+c < 1$.

If F and I are D - mappings and $F(X)$ or $I(X)$ is a closed set of X, then

c) F and I have a strict point of coincidence.

Moreover, if the pair (I,F) is sowc, then I and F have a unique common fixed point which is a strict fixed point for F.

Remark 5.1 *If in Corollary 5.1 F and I are single valued mappings and $a = b = c$ we obtain a generalization of Theorem 2.2.*

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Valeriu Popa
Department of Mathematics and Informatics
Faculty of Sciences
"Vasile Alecsandri" University of Bacău
157 Calea Mărăşeşti, Bacău, 600115, Romania
email: *vpopa@ub.ro*

Alina-Mihaela Patriciu
Department of Mathematics and Informatics
Faculty of Sciences
"Vasile Alecsandri" University of Bacău
157 Calea Mărăşeşti, Bacău, 600115, Romania
email: *alina.patriciu@ub.ro*