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INCLUSION THEOREMS INVOLVING WRIGHT'S GENERALIZED HYPERGEOMETRIC FUNCTIONS AND HARMONIC UNIVALENT FUNCTIONS

Om P. Ahuja and Poonam Sharma[†]

Abstract. The purpose of this paper is to apply Wright generalized hypergeometric (Wgh) functions in defining a linear operator and obtain some inclusion relationships between the classes of harmonic univalent functions under this linear operator whenever certain Wgh inequalities with its validity conditions hold. Results for special cases of Wgh functions are also mentioned.

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1. Introduction and Preliminaries

If u and v are real valued harmonic functions in a simply connected domain D in the complex plane C, then a continuous function $f = u + iv$ is called a complex valued harmonic function in D. Clunie and Sheil-Small [9] introduced a class $\mathcal{HS}$ of complex valued harmonic functions f which are univalent and sense-preserving in the open unit disk:

$$\triangle = \{z \,:\, z \in C, |z| < 1\}$$

and with a normalized representation given by

$$f(z) = h\,(z) + \overline{g\,(z)}$$

where

$$h\,(z) = \sum_{n=1}^{\infty} h_n z^n,\ (h_1 = 1),\ \ g\,(z) = \sum_{n=1}^{\infty} g_n z^n,\ |g_1| < 1 \tag{1}$$

are analytic and univalent in $\triangle$.

For $0 \le \lambda \le 1, 0 \le \gamma < 1$ and $0 \le k < \infty$, let $\mathcal{HQ}(k;\gamma;\lambda)$ denote a class of functions $f = h + \overline{g} \in \mathcal{HS}$ satisfying the condition:

$$\Re\left\{\frac{zh'(z)-\overline{zg'(z)}}{(1-\lambda)z+\lambda\left(h(z)+\overline{g(z)}\right)}\right\}\geq k\left|\frac{zh'(z)-\overline{zg'(z)}}{(1-\lambda)z+\lambda\left(h(z)+\overline{g(z)}\right)}-1\right|+\gamma; \quad (2)$$

and by $\mathcal{HCV}(k;\gamma)$ denote a class of functions $f=h+\overline{g}\in\mathcal{HS}$, satisfying for any real $\phi, 0\leq\gamma<1$ and $0\leq k<\infty$, the condition:

$$\Re\left\{1+\left(1+ke^{i\phi}\right)\frac{z^2h''(z)+\overline{2zg'(z)+z^2g''(z)}}{zh'(z)-\overline{zg'(z)}}\right\}\geq\gamma. \quad (3)$$

The families $\mathcal{HQ}(k;\gamma;\lambda)$ and $\mathcal{HCV}(k;\gamma)$ were studied, respectively, in [5] and [12] for results in entirely different directions and with different objectives. Note that both these families are comprehensive sets that contain various subclasses of $\mathcal{HS}$. For example, $\mathcal{HN}(\gamma)=\mathcal{HQ}(0;\gamma;0)$, $\mathcal{HS}^*(\gamma)=\mathcal{HQ}(0;\gamma;1)$ and $\mathcal{HCV}(k;\gamma)=\mathcal{HK}(\gamma)$ were investigated for different objectives, respectively, in [8], [11] and [7]. Denote by $\mathcal{HR}(\gamma)$, a subclass of $f=h+\overline{g}\in\mathcal{HS}$ if and only if $zh'(z)-\overline{zg'(z)}\in\mathcal{HN}(\gamma)$. Class $\mathcal{HR}(\gamma)$ was also studied in [8]. For additional subclasses of $\mathcal{HQ}(k;\gamma;\lambda)$ and $\mathcal{HCV}(k;\gamma)$, one may also refer to the references listed in [5] and [12]. Denote by $\mathcal{THS}$ the subclass of functions $f=h+\overline{g}\in\mathcal{HS}$ such that

$$h(z)=z-\sum_{n=2}^{\infty}|h_n|z^n \;\; g(z)=\sum_{n=1}^{\infty}|g_n|z^n, \quad (4)$$

and let $\mathcal{THQ}(k;\gamma;\lambda)=\mathcal{HQ}(k;\gamma;\lambda)\cap\mathcal{THS}$ and $\mathcal{THCV}(k;\gamma)=\mathcal{HCV}(k;\gamma)\cap\mathcal{THS}$. Also, denote $\mathcal{THS}^*(k;\gamma)=\mathcal{HS}^*(k;\gamma)\cap\mathcal{THS}$, $\mathcal{THN}(\gamma)=\mathcal{HN}(\gamma)\cap\mathcal{THS}$ and $\mathcal{THR}(\gamma)=\mathcal{HR}(\gamma)\cap\mathcal{THS}$. Note that various subclasses of $\mathcal{TH}Q(k;\gamma;\lambda)$ and $\mathcal{THCV}(k;\gamma)$ were earlier defined and studied in [6], [7], [8], [5], [16], [12], and others.

Several inclusion properties involving hypergeometric functions and harmonic univalent functions have recently been studied by the first author in [1], [3], [4] and [2]. Involvement of the Wright's generalized hypergeometric function (Wgh) in the harmonic univalent functions has recently been investigated in [15], [17] and [14].

Let $A_i>0$ $(i=1,2,...,p)$ and $B_i>0$ $(i=1,2,...,q)$ such that $1+\sum_{i=1}^{q}B_i-\sum_{i=1}^{p}A_i\geq 0$. Following the definition and terminology in [23] ([19] and [22]), a Wright's generalized hypergeometric (Wgh) function for non-negative integers p and q, $a_i\in C$ $\left(\frac{a_i}{A_i}\neq 0,-1,-2,...;i=1,2,...,p\right)$ and
$b_i\in C$ $\left(\frac{b_i}{B_i}\neq 0,-1,-2,...;i=1,2,...,q\right)$ is defined by

$${}_p\psi_q\left[\begin{array}{c}(a_1,A_1),...,(a_p,A_p)\\(b_1,B_1),...,(b_q,B_q)\end{array};z\right] = {}_p\psi_q\left[\begin{array}{c}(a_i,A_i)_{1,p}\\(b_i,B_i)_{1,q}\end{array};z\right] \quad (5)$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + nA_i)\ z^n}{\prod_{i=1}^{q} \Gamma(b_i + nB_i)\ n!}. \quad (6)$$

Note that Wgh is an entire function if $1+\sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i > 0$. On the other hand, Wgh is an analytic function in $|z| < \frac{\prod_{i=1}^{q} B_i^{B_i}}{\prod_{i=1}^{p} A_i^{A_i}}$ if $1+\sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i = 0$. However, if $1+\sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i = 0$ and $|z| = \frac{\prod_{i=1}^{q} B_i^{B_i}}{\prod_{i=1}^{p} A_i^{A_i}}$, then Wgh function is analytic for $\Re\left\{\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i\right\} + \frac{p-q}{2} > \frac{1}{2}$ (for details one may refer to [13]).

Wgh functions have an increasingly significant role in various types of applications (see [19], [20]). Generalized hypergeometric functions, generalized Mittag-Leffler functions and Bessel-Maitland (Wright generalized Bessel) functions are some special cases of Wgh functions; one may refer to [21], [22].

In view of the convergence conditions obtained in [13], we consider Wgh functions ${}_p\psi_q \left[\begin{matrix} (a_i, A_i)_{1,p} \\ (b_i, B_i)_{1,q} \end{matrix} ; z \right]$ and ${}_r\psi_s \left[\begin{matrix} (c_i, C_i)_{1,r} \\ (d_i, D_i)_{1,s} \end{matrix} ; z \right]$ (defined above by (5)), for positive integers $A_i, B_i, C_i,$ and D_i with the conditions $\prod_{i=1}^{q} B_i^{B_i} \geq \prod_{i=1}^{p} A_i^{A_i}$ (in case $1+\sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i = 0$), $\prod_{i=1}^{s} D_i^{D_i} \geq \prod_{i=1}^{r} C_i^{C_i}$ (in case $1+\sum_{i=1}^{s} D_i - \sum_{i=1}^{r} C_i = 0$) and for $a_i \neq 0, -1, -2, ...; i = 1, 2, ..., p,\ b_i \neq 0, -1, -2, ...; i = 1, 2, ..., q,\ c_i \neq 0, -1, -2, ...; i = 1, 2, ..., r,\ d_i \neq 0, -1, -2, ...; i = 1, 2, ..., s,$ Following [4] and [6], for $f = h + \overline{g} \in \mathcal{HS}$ of the form (1), we define a linear operator:

$$\mathbf{I} \equiv \mathbf{I} \left(\begin{matrix} (a_i, A_i)_{1,p} \\ (b_i, B_i)_{1,q} \end{matrix} ; \begin{matrix} (c_i, C_i)_{1,r} \\ (d_i, D_i)_{1,s} \end{matrix} \right) : \mathcal{HS} \to \mathcal{HS}$$

by

$$\mathbf{I}f(z) = h(z) * \mathbf{H}(z) + \overline{g(z) * \mathbf{G}(z)} \quad (7)$$

$$= z + \sum_{n=2}^{\infty} \theta_n\ h_n\ z^n + \sum_{n=1}^{\infty} \overline{\zeta_n\ g_n\ z^n} \quad (8)$$

where

$$\mathbf{H}(z) = z\ \frac{\prod_{i=1}^{q} \Gamma(b_i)}{\prod_{i=1}^{p} \Gamma(a_i)}\ {}_p\psi_q \left[\begin{matrix} (a_i, A_i)_{1,p} \\ (b_i, B_i)_{1,q} \end{matrix} ; z \right] = \sum_{n=1}^{\infty} \theta_n\ \ z^n, \quad (9)$$

$$\mathbf{G}(z) = z\ \frac{\prod\limits_{i=1}^{s}\Gamma\left(d_i\right)}{\prod\limits_{i=1}^{r}\Gamma\left(c_i\right)}\ {}_r\psi_s\left[\begin{array}{c}(c_i, C_i)_{1,r}\\(d_i, D_i)_{1,s}\end{array};z\right] = \sum_{n=1}^{\infty}\zeta_n\ \ z^n \tag{10}$$

and for $n \in N = \{1, 2, 3, ...\}$,

$$\theta_n = \frac{\prod\limits_{i=1}^{p}\frac{\Gamma(a_i+(n-1)A_i)}{\Gamma(a_i)}}{\prod\limits_{i=1}^{q}\frac{\Gamma(b_i+(n-1)B_i)}{\Gamma(b_i)}\ (n-1)!},\ \zeta_n = \frac{\prod\limits_{i=1}^{r}\frac{\Gamma(c_i+(n-1)C_i)}{\Gamma(c_i)}}{\prod\limits_{i=1}^{s}\frac{\Gamma(d_i+(n-1)D_i)}{\Gamma(d_i)}\ (n-1)!}. \tag{11}$$

The purpose of this paper is to find some inclusion relationships between the classes mentioned above under the linear operator $\mathbf{I}$ defined by (7) whenever certain Wgh inequalities with its validity conditions hold.

2. Lemmas

In order to obtain our main results, we need following Lemmas.

Lemma 1 *[5] If $f = h + \overline{g} \in \mathcal{HS}$ where h and g are defined by (1) and if the condition*

$$\sum_{n=2}^{\infty}\frac{n\,(k+1) - \lambda\,(k+\gamma)}{1-\gamma}|h_n| + \sum_{n=1}^{\infty}\frac{n\,(k+1) + \lambda\,(k+\gamma)}{1-\gamma}|g_n| \leq 1 \tag{12}$$

is satisfied for each k $(0 \leq k < \infty)$, γ $(0 \leq \gamma < 1)$ and $\lambda\,(0 \leq \lambda \leq 1)$, then f is sense preserving and harmonic in $\triangle$ and $f \in \mathcal{HQ}(k;\gamma;\lambda)$. Furthermore, $f \in \mathcal{THQ}(k;\gamma;\lambda)$ if and only if (12) holds.

Lemma 2 *[12] If $f = h + \overline{g} \in \mathcal{HS}$ where h and g are defined by (1) and if the condition*

$$\sum_{n=2}^{\infty}\frac{n\left\{n\,(k+1) - (k+\gamma)\right\}}{1-\gamma}|h_n| + \sum_{n=1}^{\infty}\frac{n\left\{n\,(k+1) + (k+\gamma)\right\}}{1-\gamma}|g_n| \leq 1 \tag{13}$$

is satisfied for each k $(0 \leq k < \infty)$, γ $(0 \leq \gamma < 1)$ and $\lambda\,(0 \leq \lambda \leq 1)$, then f is sense preserving and harmonic mapping in $\triangle$ and $f \in \mathcal{HCV}(k;\gamma)$. Furthermore, $f \in \mathcal{THCV}(k;\gamma)$ if and only if (13) holds.

Lemma 3 *[8] If $f = h + \overline{g} \in \mathcal{THS}$ where h and g are defined by (4), then the* *condition*

$$\sum_{n=2}^{\infty} n^2|h_n| + \sum_{n=1}^{\infty} n^2|g_n| \leq 1-\gamma \tag{14}$$

holds for γ $(0 \leq \gamma < 1)$ *if and only if* $f \in \mathcal{THR}(\gamma)$.

We next prove the following Lemma to get our main results.

Lemma 4 *Let for positive integers* A_i , B_j *and for* $-n \neq a_i, b_j \in C$, $n \in N_0 = N \cup \{0\}$, $i = 1, 2, ..., p$; $j = 1, 2, ..., q$, θ_n *be defined by (11). Then for* $n \in N$,

$$|\theta_n| \leq \frac{\prod\limits_{i=1}^{p} \frac{\Gamma(|a_i|+(n-1)A_i)}{\Gamma(|a_i|)}}{\prod\limits_{j=1}^{q} \frac{\Gamma(\Re(b_j)+(n-1)B_j)}{\Gamma(\Re(b_j))}\ (n-1)!}. \tag{15}$$

Proof. Using the formula ([18], p. 240, Eq. (1.26))

$$\frac{\Gamma(a+nA)}{\Gamma(a)} = \left(\frac{a}{A}\right)_n \left(\frac{a+1}{A}\right)_n ... \left(\frac{a+A-1}{A}\right)_n (A)^{nA}, \ n = 0, 1, 2, ...$$

and

$$\left(\Re\left(\lambda\right)\right)_n \leq \left|\left(\lambda\right)_n\right| \leq \left(\left|\lambda\right|\right)_n,$$

where $(\lambda)_n$ is the Pochhammer symbol, we get for $n \in N$,

$$\begin{aligned}
&\left|\frac{\Gamma\left(b_j+(n-1)B_j\right)}{\Gamma\left(b_j\right)}\right| \\
= \ & \left|\left(\frac{b_j}{B_j}\right)_{n-1}\right| \left|\left(\frac{b_j+1}{B_j}\right)_{n-1}\right| ... \left|\left(\frac{b_j+B_j-1}{B_j}\right)_{n-1}\right| (B_j)^{(n-1)B_j} \\
\geq \ & \left(\frac{\Re\left(b_j\right)}{B_j}\right)_{n-1} \left(\frac{\Re\left(b_j+1\right)}{B_j}\right)_{n-1} ... \left(\frac{\Re\left(b_j+B_j-1\right)}{B_j}\right)_{n-1} (B_j)^{(n-1)B_j} \\
= \ & \frac{\Gamma\left(\Re\left(b_j\right)+(n-1)B_j\right)}{\Gamma\left(\Re\left(b_j\right)\right)}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{\Gamma\left(a_i+(n-1)A_i\right)}{\Gamma\left(a_i\right)}\right| \\
= \ & \left|\left(\frac{a_i}{A_i}\right)_{n-1}\right| \left|\left(\frac{a_i+1}{A_i}\right)_{n-1}\right| ... \left|\left(\frac{a_i+A_i-1}{A_i}\right)_{n-1}\right| (A_i)^{(n-1)A_i} \\
\leq \ & \frac{\Gamma\left(|a_i|+(n-1)A_i\right)}{\Gamma\left(|a_i|\right)}.
\end{aligned}$$

Hence, we evidently obtain from (11) that

$$|\theta_n| = \frac{\prod\limits_{i=1}^{p}\left|\frac{\Gamma(a_i+(n-1)A_i)}{\Gamma(a_i)}\right|}{\prod\limits_{j=1}^{q}\left|\frac{\Gamma(b_j+(n-1)B_j)}{\Gamma(b_j)}\right|}\frac{1}{(n-1)!} \leq \frac{\prod\limits_{i=1}^{p}\frac{\Gamma(|a_i|+(n-1)A_i)}{\Gamma(|a_i|)}}{\prod\limits_{j=1}^{q}\frac{\Gamma(\Re(b_j)+(n-1)B_j)}{\Gamma(\Re(b_j))}\,(n-1)!}.$$

This proves the result (15) of Lemma 4.

3. Main Results

Theorem 1 *Let for $f \in \mathcal{THS}$ and for positive integers A_i, B_i, C_i, and D_i, the operator* $\mathbf{I}$ *be defined by (7) with $\Re(b_i) > 0 \ (i = 1, 2, .., p)$, $\Re(d_i) > 0 \ (i = 1, 2, .., q)$. If, under the validity condition (in case $\prod\limits_{i=1}^{q} B_i^{B_i} = \prod\limits_{i=1}^{p} A_i^{A_i}$, $1+\sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i = 0$ and $\prod\limits_{i=1}^{s} D_i^{D_i} = \prod\limits_{i=1}^{r} C_i^{C_i}$, $1+\sum_{i=1}^{s} D_i - \sum_{i=1}^{r} C_i = 0$):*

$$\sum_{i=1}^{q}\Re(b_i) - \sum_{i=1}^{p}|a_i| + \frac{p-q}{2} > \frac{3}{2}, \ \sum_{i=1}^{s}\Re(d_i) - \sum_{i=1}^{r}|c_i| + \frac{r-s}{2} > \frac{3}{2}, \tag{16}$$

for $j = 0, 1$,

$$_p\psi_q\left[\begin{array}{c}(|a_i| + jA_i, A_i)_{1,p} \\ (\Re(b_i) + jB_i, B_i)_{1,q}\end{array};1\right] := \ _p\Psi_q^j, \ _r\psi_s\left[\begin{array}{c}(|c_i| + jC_i, C_i)_{1,r} \\ (\Re(d_i) + jD_i, D_i)_{1,s}\end{array};1\right] := \ _r\Psi_s^j, \tag{17}$$

satisfy for $0 \leq \gamma < 1, 0 \leq k < \infty$, the Wgh inequality

$$\frac{\prod\limits_{i=1}^{q}\Gamma(\Re(b_i))}{\prod\limits_{i=1}^{p}\Gamma(|a_i|)}\left\{(k+1)\ {}_p\Psi_q^1 + (1-\gamma)\ {}_p\Psi_q^0\right\} +$$

$$\frac{\prod\limits_{i=1}^{s}\Gamma(\Re(d_i))}{\prod\limits_{i=1}^{r}\Gamma(|c_i|)}\left\{(k+1)\ \ {}_r\Psi_s^1 + (2k+1+\gamma)\ \ {}_r\Psi_s^0\right\} \leq 2-\gamma, \tag{18}$$

then $\mathbf{I}\mathcal{THN}(\gamma) \subset \mathcal{HCV}(k;\gamma)$.

Proof. Let $f = h + \overline{g}$ where h and g are of the form (4) belongs to $\mathcal{THN}(\gamma)$, then, using special form of Lemma 1 for $\lambda = 0, k = 0$, we get the necessary coefficient inequality:

$$|h_n| \leq \frac{1-\gamma}{n} \ (n \geq 2), \ |g_n| \leq \frac{1-\gamma}{n} \ (n \geq 1). \tag{19}$$

To show $\mathbf{I}f \in \mathcal{HCV}(k;\gamma)$, we need to show by Lemma 2, that

$$S_1 := \sum_{n=2}^{\infty} \frac{n\{n(k+1)-(k+\gamma)\}}{1-\gamma} |h_n \ \theta_n| + \sum_{n=1}^{\infty} \frac{n\{n(k+1)+(k+\gamma)\}}{1-\gamma} |g_n \ \zeta_n| \leq 1. \tag{20}$$

In view of Lemma 4, we denote

$$|\theta_n| \leq \frac{\prod_{i=1}^{p} \frac{\Gamma(|a_i|+(n-1)A_i)}{\Gamma(|a_i|)}}{\prod_{i=1}^{q} \frac{\Gamma(\Re(b_i)+(n-1)B_i)}{\Gamma(\Re(b_i))} \ (n-1)!} := \vartheta_n, |\ \zeta_n| \leq \frac{\prod_{i=1}^{r} \frac{\Gamma(|c_i|+(n-1)C_i)}{\Gamma(|c_i|)}}{\prod_{i=1}^{s} \frac{\Gamma(\Re(d_i)+(n-1)D_i)}{\Gamma(\Re(d_i))} \ (n-1)!} := \eta_n, \tag{21}$$

and hence, for some $j \in N_0$, we obtain identities:

$$\sum_{n=1+j}^{\infty} (n-j)_j \ \vartheta_n = \frac{\prod_{i=1}^{q} \Gamma(\Re(b_i))}{\prod_{i=1}^{p} \Gamma(|a_i|)} \ {}_p\Psi_q^j, \tag{22}$$

$$\sum_{n=1+j}^{\infty} (n-j)_j \ \eta_n = \frac{\prod_{i=1}^{s} \Gamma(\Re(d_i))}{\prod_{i=1}^{r} \Gamma(|c_i|)} \ {}_r\Psi_s^j. \tag{23}$$

We note that (16) ensures the convergence of ${}_p\Psi_q^j$ and ${}_r\Psi_s^j$ for $j = 0, 1$. Thus, by (19), (21) and using identities (22), (23) for $j = 0, 1$, we obtain

$$\begin{aligned} S_1 &\leq \sum_{n=2}^{\infty} \{n(k+1)-(k+\gamma)\} \vartheta_n + \sum_{n=1}^{\infty} \{n(k+1)+(k+\gamma)\} \eta_n \\ &= \sum_{n=2}^{\infty} \{(n-1)(k+1)+(1-\gamma)\} \vartheta_n + \sum_{n=1}^{\infty} \{(n-1)(k+1)+(2k+1+\gamma)\} \eta_n \\ &= \frac{\prod_{i=1}^{q} \Gamma(\Re(b_i))}{\prod_{i=1}^{p} \Gamma(|a_i|)} \Big\{(k+1) \ {}_p\Psi_q^1 + (1-\gamma) \ {}_p\Psi_q^0\Big\} - (1-\gamma) \end{aligned}$$

$$+\frac{\prod_{i=1}^{s}\Gamma\left(\Re\left(d_i\right)\right)}{\prod_{i=1}^{r}\Gamma\left(\left|c_i\right|\right)}\left\{(k+1)\quad {}_r\Psi_s^1+(2k+1+\gamma)\quad {}_r\Psi_s^0\right\}\leq 1,$$

if (18) holds. This proves Theorem 1.

In a similar manner, on applying Lemma 1 for $\lambda = 1$, we can prove following result:

Theorem 2 *Under the same hypothesis and the same validity condition of Theorem 1, if Wgh inequality*

$$\frac{\prod_{i=1}^{q}\Gamma\left(\Re\left(b_i\right)\right)}{\prod_{i=1}^{p}\Gamma\left(\left|a_i\right|\right)}\left\{{}_p\Psi_q^1+\ {}_p\Psi_q^0\right\}+\frac{\prod_{i=1}^{s}\Gamma\left(\Re\left(d_i\right)\right)}{\prod_{i=1}^{r}\Gamma\left(\left|c_i\right|\right)}\left\{\ {}_r\Psi_s^1+\ {}_r\Psi_s^0\right\}\leq 2, \tag{24}$$

holds, then for $0 \leq \gamma < 1, 0 \leq k < \infty$, $\mathbf{I}\mathcal{THS}^*(k;\gamma) \subset \mathcal{HCV}(k;\gamma)$.

Further, on applying Lemma 2 and Lemma 1 for $\lambda = 1$, we get following theorem, the proof is based on the similar lines of the proof of Theorem 1, hence, we omit its proof.

Theorem 3 *Under the same hypothesis of Theorem 1, if, under the validity condition (in case* $\prod_{i=1}^{q} B_i^{B_i} = \prod_{i=1}^{p} A_i^{A_i}$, $1+\sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i = 0$ *and* $\prod_{i=1}^{s} D_i^{D_i} = \prod_{i=1}^{r} C_i^{C_i}$, $1+\sum_{i=1}^{s} D_i - \sum_{i=1}^{r} C_i = 0$*)*

$$\sum_{i=1}^{q}\Re\left(b_i\right)-\sum_{i=1}^{p}\left|a_i\right|+\frac{p-q}{2}>\frac{1}{2},\sum_{i=1}^{s}\Re\left(d_i\right)-\sum_{i=1}^{r}\left|c_i\right|+\frac{r-s}{2}>\frac{1}{2}, \tag{25}$$

${}_p\Psi_q^0$, ${}_r\Psi_s^0$ *defined in (17) (for* $j=0$*), satisfy Wgh inequality*

$$\frac{\prod_{i=1}^{q}\Gamma\left(\Re\left(b_i\right)\right)}{\prod_{i=1}^{p}\Gamma\left(\left|a_i\right|\right)}\quad {}_p\Psi_q^0+\frac{\prod_{i=1}^{s}\Gamma\left(\Re\left(d_i\right)\right)}{\prod_{i=1}^{r}\Gamma\left(\left|c_i\right|\right)}\quad {}_r\Psi_s^0\leq 2, \tag{26}$$

then for $0 \leq \gamma < 1, 0 \leq k < \infty$, $\mathbf{I}\mathcal{THCV}(k;\gamma) \subset \mathcal{HCV}(k;\gamma)$ *and* $\mathbf{I}\mathcal{THS}^*(k;\gamma) \subset \mathcal{HS}^*(k;\gamma)$.

Theorem 4 *Under the same hypothesis of Theorem 1, if, under the validity condition (in case* $\prod_{i=1}^{q} B_i^{B_i} = \prod_{i=1}^{p} A_i^{A_i}$, $1+\sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i = 0$ *and* $\prod_{i=1}^{s} D_i^{D_i} = \prod_{i=1}^{r} C_i^{C_i}$, $1+\sum_{i=1}^{s} D_i - \sum_{i=1}^{r} C_i = 0$*)*

$$\sum_{i=1}^{q} \Re(b_i) - \sum_{i=1}^{p} |a_i| + \frac{p-q}{2} > \frac{1}{2},\ \sum_{i=1}^{s} \Re(d_i) - \sum_{i=1}^{r} |c_i| + \frac{r-s}{2} > \frac{1}{2}, \tag{27}$$

for $j = 0, 1,$

$$_{p+1}\psi_{q+1}\left[\begin{array}{c} (|a_i| + jA_i, A_i)_{1,p}, (1+j,1) \\ (\Re(b_i) + jB_i, B_i)_{1,q}, (2+j,1) \end{array};1\right] \ :=\ {}_{p+1}\Psi^j_{q+1}, \tag{28}$$

$$_{r+1}\psi_{s+1}\left[\begin{array}{c} (|c_i| + jC_i, C_i)_{1,r}, (1+j,1) \\ (\Re(d_i) + jD_i, D_i)_{1,s}, (2+j,1) \end{array};1\right] \ :=\ {}_{r+1}\Psi^j_{s+1} \tag{29}$$

satisfy for $0 \le \gamma < 1, 0 \le k < \infty,$ *Wgh inequality*

$$\frac{\prod_{i=1}^{q} \Gamma(\Re(b_i))}{\prod_{i=1}^{p} \Gamma(|a_i|)} \left\{(1+k)\ {}_{p+1}\Psi^1_{q+1} + (1-\gamma)\ {}_{p+1}\Psi^0_{q+1}\right\} +$$

$$\frac{\prod_{i=1}^{s} \Gamma(\Re(d_i))}{\prod_{i=1}^{r} \Gamma(|c_i|)} \left\{(1+k)\ {}_{r+1}\Psi^1_{s+1} + (2k+1+\gamma)\ {}_{r+1}\Psi^0_{s+1}\right\} \le 2-\gamma, \tag{30}$$

then $\mathbf{I}\mathcal{THN}(\gamma) \subset \mathcal{HS}^*(k;\gamma)$.

Proof. Let $f \in \mathcal{THN}(\gamma)$ where h and g are of the form (4), then on using Lemma 1 for special values of the parameters that is for $\lambda = 0, k = 0$, we have (19). To prove the result, again by Lemma 1 (for $\lambda = 1$), we need to show

$$S_2 := \sum_{n=2}^{\infty} \frac{n(k+1) - (k+\gamma)}{1-\gamma} |h_n\ \theta_n| + \sum_{n=1}^{\infty} \frac{n(k+1) + (k+\gamma)}{1-\gamma} |g_n\ \zeta_n| \le 1. \tag{31}$$

Hence, on using (19), (21) and identities

$$\sum_{n=1+j}^{\infty} (n-j)_j\ \frac{\vartheta_n}{n} \ =\ \frac{\prod_{i=1}^{q} \Gamma(\Re(b_i))}{\prod_{i=1}^{p} \Gamma(|a_i|)}\ {}_{p+1}\Psi^j_{q+1}, \tag{32}$$

$$\sum_{n=1+j}^{\infty}(n-j)_j\ \frac{\eta_n}{n} = \frac{\prod\limits_{i=1}^{s}\Gamma(\Re(d_i))}{\prod\limits_{i=1}^{r}\Gamma(|c_i|)}\ {}_{r+1}\Psi_{s+1}^{j}, \tag{33}$$

for $j=0,1$, we get

$$\begin{aligned} S_2 \ &\leq\ (1+k)\sum_{n=2}^{\infty}(n-1)\frac{\vartheta_n}{n}+(1-\gamma)\sum_{n=2}^{\infty}\frac{\vartheta_n}{n}+(1+k)\sum_{n=2}^{\infty}(n-1)\frac{\eta_n}{n}+\\ &\quad (2k+1+\gamma)\sum_{n=1}^{\infty}\frac{\eta_n}{n}\\ &= \frac{\prod\limits_{i=1}^{q}\Gamma(\Re(b_i))}{\prod\limits_{i=1}^{p}\Gamma(|a_i|)}\left\{(1+k)\ {}_{p+1}\Psi_{q+1}^{1}+(1-\gamma)\ {}_{p+1}\Psi_{q+1}^{0}\right\}-(1-\gamma)\\ &\quad +\frac{\prod\limits_{i=1}^{s}\Gamma(\Re(d_i))}{\prod\limits_{i=1}^{r}\Gamma(|c_i|)}\left\{(1+k)\ {}_{r+1}\Psi_{s+1}^{1}+(2k+1+\gamma)\ {}_{r+1}\Psi_{s+1}^{0}\right\}\leq 1, \end{aligned}$$

if (30) holds. This proves Theorem 4.

We obtain following theorem which is a direct consequence of Theorem 4.

Theorem 5 *Under the same hypothesis of Theorem 4, if Wgh inequality (30) holds, then for* $0\leq\gamma<1, 0\leq k<\infty$, $\mathbf{I}\mathcal{THR}(\gamma)\subset\mathcal{HCV}(k;\gamma)$.

Similar to the proof of Theorem 4, we can also prove the following result:

Theorem 6 *Under the same hypothesis of Theorem 1, if, under the validity condition (in case* $\prod\limits_{i=1}^{q}B_i^{B_i}=\prod\limits_{i=1}^{p}A_i^{A_i}$, $1+\sum_{i=1}^{q}B_i-\sum_{i=1}^{p}A_i=0$ *and* $\prod\limits_{i=1}^{s}D_i^{D_i}=\prod\limits_{i=1}^{r}C_i^{C_i}$, $1+\sum_{i=1}^{s}D_i-\sum_{i=1}^{r}C_i=0$*)*

$$\sum_{i=1}^{q}\Re(b_i)-\sum_{i=1}^{p}|a_i|+\frac{p-q}{2}>-\frac{1}{2},\sum_{i=1}^{s}\Re(d_i)-\sum_{i=1}^{r}|c_i|+\frac{r-s}{2}>-\frac{1}{2}, \tag{34}$$

${}_{p+1}\Psi_{q+1}^{j}$, ${}_{r+1}\Psi_{s+1}^{j}$ *defined in (28) and (29) for* $j=0$, *satisfy Wgh inequality*

$$\frac{\prod\limits_{i=1}^{q}\Gamma(\Re(b_i))}{\prod\limits_{i=1}^{p}\Gamma(|a_i|)}\ {}_{p+1}\Psi_{q+1}^{0}+\frac{\prod\limits_{i=1}^{s}\Gamma(\Re(d_i))}{\prod\limits_{i=1}^{r}\Gamma(|c_i|)}\ {}_{r+1}\Psi_{s+1}^{0}\quad\leq 2, \tag{35}$$

then for $0 \leq \gamma < 1, 0 \leq k < \infty$, $\mathbf{I}\mathcal{THCV}(k;\gamma) \subset \mathcal{HS}^*(k;\gamma)$.

Theorem 7 *Under the same hypothesis of Theorem 1, and if under the validity condition (in case* $\prod_{i=1}^{q} B_i^{B_i} = \prod_{i=1}^{p} A_i^{A_i}$, $1+\sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i = 0$ *and* $\prod_{i=1}^{s} D_i^{D_i} = \prod_{i=1}^{r} C_i^{C_i}$, $1+\sum_{i=1}^{s} D_i - \sum_{i=1}^{r} C_i = 0$*)*

$$\sum_{i=1}^{q} \Re(b_i) - \sum_{i=1}^{p} |a_i| + \frac{p-q}{2} > -\frac{1}{2}, \sum_{i=1}^{s} \Re(d_i) - \sum_{i=1}^{r} |c_i| + \frac{r-s}{2} > -\frac{1}{2}, \tag{36}$$

for $j = 0, 1$,

$${}_{p+2}\psi_{q+2}\left[\begin{array}{c} (|a_i| + jA_i, A_i)_{1,p}, (1+j,1), (1+j,1) \\ (\Re(b_i) + jB_i, B_i)_{1,q}, (2+j,1), (2+j,1) \end{array}; 1\right] \; : \; = \; {}_{p+2}\Psi^j_{q+2}, \tag{37}$$

$${}_{r+2}\psi_{s+2}\left[\begin{array}{c} (|c_i| + jC_i, C_i)_{1,r}, (1+j,1), (1+j,1) \\ (\Re(d_i) + D_i, D_i)_{1,s}, (2+j,1), (2+j,1) \end{array}; 1\right] \; : \; = \; {}_{r+2}\Psi^j_{s+2} \tag{38}$$

satisfy Wgh inequality

$$\frac{\prod_{i=1}^{q} \Gamma(\Re(b_i))}{\prod_{i=1}^{p} \Gamma(|a_i|)} \left[(1+k)\ {}_{p+2}\Psi^1_{q+2} + (1-\gamma)\ {}_{p+2}\Psi^0_{q+2}\right] +$$

$$\frac{\prod_{i=1}^{s} \Gamma(\Re(d_i))}{\prod_{i=1}^{r} \Gamma(|c_i|)} \left[(1+k)\ {}_{r+2}\Psi^1_{s+2} + (2k+1+\gamma)\ {}_{r+2}\Psi^0_{s+2}\right] \leq 2-\gamma, \tag{39}$$

then for $0 \leq \gamma < 1, 0 \leq k < \infty$, $\mathbf{I}\mathcal{THR}(\gamma) \subset \mathcal{HS}^*(k;\gamma)$.

Proof. If $f = h + \overline{g}$ where h and g are of the form (4), belongs to $\mathcal{THR}(\gamma)$, then by Lemma 3, we get

$$|h_n| \leq \frac{1-\gamma}{n^2}\ (n \geq 2), \ |g_n| \leq \frac{1-\gamma}{n^2}\ (n \geq 1). \tag{40}$$

To prove the result, we need to show (31). Hence, using (40), (21) and the identities

$$\sum_{n=1+j}^{\infty} \frac{(n-j)_j}{n^2} \vartheta_n \;=\; \frac{\prod_{i=1}^{q} \Gamma(\Re(b_i))}{\prod_{i=1}^{p} \Gamma(|a_i|)}\ {}_{p+2}\Psi^j_{q+2}, \tag{41}$$

$$\sum_{n=1+j}^{\infty} \frac{(n-j)_j}{n^2}\eta_n = \frac{\prod_{i=1}^{s} \Gamma(\Re(d_i))}{\prod_{i=1}^{r} \Gamma(|c_i|)} \; {}_{r+2}\Psi_{s+2}^{j}, \tag{42}$$

for $j = 0, 1$, we get

$$\begin{aligned} S_2 \leq\; & (1+k)\sum_{n=2}^{\infty}(n-1)\frac{\vartheta_n}{n^2} + (1-\gamma)\sum_{n=2}^{\infty}\frac{\vartheta_n}{n^2} + (1+k)\sum_{n=2}^{\infty}(n-1)\frac{\eta_n}{n^2} + \\ & (2k+1+\gamma)\sum_{n=1}^{\infty}\frac{\eta_n}{n^2} \\ =\; & \frac{\prod_{i=1}^{q}\Gamma(\Re(b_i))}{\prod_{i=1}^{p}\Gamma(|a_i|)}\left[(1+k)\;{}_{p+2}\Psi_{q+2}^{1} + (1-\gamma)\;{}_{p+2}\Psi_{q+2}^{0}\right] - (1-\gamma) \\ & + \frac{\prod_{i=1}^{s}\Gamma(\Re(d_i))}{\prod_{i=1}^{r}\Gamma(|c_i|)}\left[(1+k)\;{}_{r+2}\Psi_{s+2}^{1} + (2k+1+\gamma)\;{}_{r+2}\Psi_{s+2}^{0}\right] \leq 1, \end{aligned}$$

if (39) holds. This proves Theorem 7.

Remark 1 Taking $k = 0$ in the Main Results, we can directly obtain sufficient Wgh inequalities for $\mathbf{I}f \in \mathcal{HK}(\gamma)$ $(\mathcal{HS}^*(\gamma))$ whenever $f \in \mathcal{THN}(\gamma)$ $(\mathcal{THR}(\gamma))$ $(\mathcal{THS}^*(\gamma))$ $(\mathcal{THK}(\gamma))$.

3. Results on Some Special Cases

In particular, if we take $A_i = 1,\ i = 1,2,3,..,p;\ B_i = 1,\ i = 1,2,3,..,q$ and $C_i = 1,\ i = 1,2,3,..,r;\ D_i = 1,\ i = 1,2,3,..,s$, the operator $\mathbf{I}$ defined by (7) reduces to the operator $\mathbf{J}$ which involve generalized hypergeometric (gh) functions ${}_pF_q\left[\begin{matrix}(a_i)_{1,p}\\(b_i)_{1,q}\end{matrix};z\right]$ and ${}_rF_s\left[\begin{matrix}(c_i)_{1,r}\\(d_i)_{1,s}\end{matrix};z\right]$ for $p \leq q+1$ and $r \leq s+1$; and for $f = h + \overline{g} \in \mathcal{HS}$, it is defined by

$$\mathbf{J}(z) = z\;{}_pF_q\left[\begin{matrix}(a_i)_{1,p}\\(b_i)_{1,q}\end{matrix};z\right] * h(z) + \overline{z\;{}_rF_s\left[\begin{matrix}(c_i)_{1,r}\\(d_i)_{1,s}\end{matrix};z\right] * g(z)} \in \mathcal{HS}. \tag{43}$$

The gh function ${}_pF_q$ is defined by

$$ {}_pF_q\left[\begin{array}{c}(a_i)_{1,p}\\(b_i)_{1,q}\end{array};z\right] = \frac{\prod\limits_{i=1}^{q}\Gamma(b_i)}{\prod\limits_{i=1}^{p}\Gamma(a_i)}\ {}_p\psi_q\left[\begin{array}{c}(a_i,1)_{1,p}\\(b_i,1)_{1,q}\end{array};z\right] = \sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{p}(a_i)_n\ z^n}{\prod\limits_{i=1}^{q}(b_i)_n\ n!}, $$

which is an entire function if $p<q+1$; and if $p=q+1$ it is analytic in Δ; and if $p=q+1, z\in\partial\Delta$, it is analytic if $\Re\{\sum_{i=1}^{q}b_i-\sum_{i=1}^{p}a_i\}>0$.

Thus, on taking $A_i=1,\ i=1,2,3,..,p;\ B_i=1,\ i=1,2,3,..,q$ and $C_i=1,\ i=1,2,3,..,r;\ D_i=1,\ i=1,2,3,..,s$ in the Main Results, we obtain following results:

Corollary 1 *Let for $f\in\mathcal{THS}$, the operator $\mathbf{J}$ be defined by (43) with $\Re(b_i)>0\ \ (i=1,2,..,p)$, $\Re(d_i)>0\ \ (i=1,2,..,q)$. Under the validity condition (in case $p=q+1,\ r=s+1$)*

$$ \sum_{i=1}^{q}\Re(b_i)-\sum_{i=1}^{p}|a_i|>1,\sum_{i=1}^{s}\Re(d_i)-\sum_{i=1}^{r}|c_i|>1, \tag{44} $$

if for $j=0,1$,

$$ {}_pF_q\left[\begin{array}{c}(|a_i|+j)_{1,p}\\(\Re(b_i)+j)_{1,q}\end{array};1\right]:=\ {}_pF_q^{j}\ ,\quad {}_rF_s\left[\begin{array}{c}(|c_i|+j)_{1,p}\\(\Re(d_i)+j)_{1,q}\end{array};1\right]:=\ {}_rF_s^{j} \tag{45} $$

satisfy for some $0\le\gamma<1, 0\le k<\infty$, gh inequality

$$ (k+1)\ \frac{\prod\limits_{i=1}^{p}|a_i|}{\prod\limits_{i=1}^{q}\Re(b_i)}\ {}_pF_q^1+(1-\gamma)\ {}_pF_q^0+(k+1)\ \frac{\prod\limits_{i=1}^{r}|c_i|}{\prod\limits_{i=1}^{s}\Re(d_i)}\ {}_rF_s^1+ $$
$$ (2k+1+\gamma)\quad {}_rF_s^0\le(2-\gamma), $$

then $\mathbf{J}\mathcal{THN}(\gamma)\subset\mathcal{HCV}(k;\gamma)$.

Corollary 2 *Under the same hypothesis and under the same validity condition (44) of Corollary 1, if gh inequality*

$$ \frac{\prod\limits_{i=1}^{p}|a_i|}{\prod\limits_{i=1}^{q}\Re(b_i)}\ {}_pF_q^1+\ {}_pF_q^0+\ \frac{\prod\limits_{i=1}^{r}|c_i|}{\prod\limits_{i=1}^{s}\Re(d_i)}\ \ {}_rF_s^1+\ {}_rF_s^0\le 2, \tag{46} $$

holds, then for $0 \leq \gamma < 1, 0 \leq k < \infty$, $\mathbf{J}\mathcal{THS}^*(k;\gamma) \subset \mathcal{HCV}(k;\gamma)$.

Corollary 3 *Under the same hypothesis of Corollary 1 and (in case* $p = q + 1$, $r = s + 1$*) under the validity conditions* $\sum_{i=1}^{q} \Re(b_i) - \sum_{i=1}^{p} |a_i| > 0$, $\sum_{i=1}^{s} \Re(d_i) - \sum_{i=1}^{r} |c_i| > 0$, *if* ${}_pF_q^0$, ${}_rF_s^0$ *defined by (45) (for* $j = 0$*), satisfy gh inequality*

$$ {}_pF_q^0 + {}_rF_s^0 \leq 2, \tag{47}$$

then for $0 \leq \gamma < 1, 0 \leq k < \infty$, $\mathbf{J}\mathcal{THCV}(k;\gamma) \subset \mathcal{HCV}(k;\gamma)$ *and* $\mathbf{J}\mathcal{THS}^*(k;\gamma) \subset \mathcal{HS}^*(k;\gamma)$.

Corollary 4 *Under the same hypothesis of Corollary 1, and (in case* $p = q + 1$, $r = s + 1$*) under the validity condition*

$$\sum_{i=1}^{q} \Re(b_i) - \sum_{i=1}^{p} |a_i| > 0, \sum_{i=1}^{s} \Re(d_i) - \sum_{i=1}^{r} |c_i| > 0, \tag{48}$$

if for $j = 0, 1$,

$$ {}_{p+1}F_{q+1}\left[\begin{matrix} (|a_i| + j)_{1,p}, 1 + j \\ (\Re(b_i) + j)_{1,q}, 2 + j \end{matrix}; 1\right] \ := \ {}_{p+1}F_{q+1}^j, \tag{49}$$

$$ {}_{r+1}F_{s+1}\left[\begin{matrix} (|c_i| + j)_{1,p}, 1 + j \\ (\Re(d_i) + j)_{1,q}, 2 + j \end{matrix}; 1\right] \ := \ {}_{r+1}F_{s+1}^j \tag{50}$$

satisfy for some $0 \leq \gamma < 1, 0 \leq k < \infty$, *gh inequality*

$$\begin{aligned} &(1+k)\frac{\prod_{i=1}^{p} |a_i|}{2\prod_{i=1}^{q} \Re(b_i)} \ {}_{p+1}F_{q+1}^1 + (1-\gamma) \ {}_{p+1}F_{q+1}^0 \\ &+(1+k) \ \frac{\prod_{i=1}^{r} |c_i|}{2\prod_{i=1}^{s} \Re(d_i)} \ {}_{r+1}F_{s+1}^1 + (2k+1+\gamma) \ {}_{r+1}F_{s+1}^0 \leq 2 - \gamma, \end{aligned}$$

then $\mathbf{I}\mathcal{THN}(\gamma) \subset \mathcal{HS}^*(k;\gamma)$ *and* $\mathbf{J}\mathcal{THR}(\gamma) \subset \mathcal{HCV}(k;\gamma)$.

Corollary 5 *Under the same hypothesis of Corollary 1, and if (in case* $p = q+1$, $r = s + 1$*) under the validity condition*

$$\sum_{i=1}^{q} \Re(b_i) > \max\left\{0, -1 + \sum_{i=1}^{p} |a_i|\right\}, \sum_{i=1}^{s} \Re(d_i) > \max\left\{0, -1 + \sum_{i=1}^{r} |c_i|\right\}, \tag{51}$$

${}_{p+1}F^j_{q+1},\ {}_{r+1}F^j_{s+1}$ *defined by (49) and (50) for* $j=0$, *satisfy gh inequality*

$$ {}_{p+1}F^0_{q+1} + {}_{r+1}F^0_{s+1} \leq 2, \tag{52}$$

then for $0 \leq \gamma < 1, 0 \leq k < \infty$, $\mathbf{J}\mathcal{THCV}(k;\gamma) \subset \mathcal{HS}^*(k;\gamma)$.

Corollary 6 *Under the same hypothesis of Corollary 1, and if (in case* $p = q+1$, $r = s+1$*) under the validity condition*

$$\sum_{i=1}^{q} \Re(b_i) > \max\left\{0, -1 + \sum_{i=1}^{p} |a_i|\right\}, \sum_{i=1}^{s} \Re(d_i) > \max\left\{0, -1 + \sum_{i=1}^{r} |c_i|\right\},$$

for $j = 0, 1$,

$$ {}_{p+2}F_{q+2}\left[\begin{array}{c} (|a_i|+j)_{1,p}, 1+j, 1+j \\ (\Re(b_i)+j)_{1,q}, 2+j, 2+j \end{array}; 1\right] \ :\ =\ {}_{p+2}F^j_{q+2},$$
$$ {}_{r+2}F_{s+2}\left[\begin{array}{c} (|c_i|+j)_{1,p}, 1+j, 1+j \\ (\Re(d_i)+j)_{1,q}, 2+j, 2+j \end{array}; 1\right] \ :\ =\ {}_{r+2}F^j_{s+2}$$

satisfy for some $0 \leq \gamma < 1, 0 \leq k < \infty$, *gh inequality*

$$(1+k)\frac{\prod_{i=1}^{p}|a_i|}{4\prod_{i=1}^{q}\Re(b_i)}\ {}_{p+2}F^1_{q+2} + (1-\gamma)\ {}_{p+2}F^0_{q+2}$$
$$+(1+k)\frac{\prod_{i=1}^{r}(|c_i|}{4\prod_{i=1}^{s}(\Re(d_i))}\ {}_{r+2}F^1_{s+2} + (2k+1+\gamma)\ {}_{r+2}F^0_{s+2} \leq 2-\gamma,$$

then $\mathbf{J}\mathcal{THR}(\gamma) \subset \mathcal{HS}^*(k;\gamma)$.

Remark 2 Similar to the earlier Remark 1, taking $k = 0$ in the above Corollaries, we can directly obtain sufficient gh inequalities for $\mathbf{J}f \in \mathcal{HK}(\gamma)$ $(\mathcal{HS}^*(\gamma))$ whenever $f \in \mathcal{THN}(\gamma)$ $(\mathcal{THR}(\gamma))$ $(\mathcal{THS}^*(\gamma))$ $(\mathcal{THK}(\gamma))$.

Taking $p = r = 2$ and $q = s = 1$, $\mathbf{J}$ reduces to Ω which involves Gauss's hypergeometric functions and for $f = h + \overline{g} \in \mathcal{HS}$ is defined by

$$\Omega f(z) = z\ {}_2F_1\left[\begin{array}{c} a_1, a_2 \\ b_1 \end{array}; z\right] * h(z) + \overline{z\ {}_2F_1\left[\begin{array}{c} c_1, c_2 \\ d_1 \end{array}; z\right]} * g(z) \in \mathcal{HS}. \tag{53}$$

Results, for the operator Ω can be obtained from Corollaries 1-6 , by adopting the similar way as it is used in [2], [3], [4] etc.

By choosing $p = q = r = s = 1$ and $a_1 = A_1 = c_1 = C_1 = 1$, the operator $\mathbf{I}$ reduces to the operator $\mathbf{E}$ and for $f = h + \overline{g} \in \mathcal{HS}$, is defined by

$$\mathbf{E}f(z) := z\,\Gamma(b_1)\;E^{1,1}_{B_1,b_1}\;[z] * h(z) + \overline{z\,\Gamma(d_1)\;E^{1,1}_{D_1,d_1}\;[z] * g(z)} \in \mathcal{HS},$$

which involve gneralized Mittag-Leffler functions:

$$E^{1,1}_{B_1,b_1}\;[z] = \;{}_1\psi_1\left[\begin{array}{c}(1,1)\\(b_1,B_1)\end{array};z\right] = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(b_1+nB_1)}$$

and

$$E^{1,1}_{D_1,d_1}\;[z] = \;{}_1\psi_1\left[\begin{array}{c}(1,1)\\(d_1,D_1)\end{array};z\right] = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(d_1+nD_1)}.$$

Further, involving Bessel-Maitland (Wright generalized Bessel) functions:

$$J^{\mu_1}_{\nu_1}[-z] = \;{}_0\psi_1\left[\begin{array}{c}-\\(1+\nu_1,\mu_1)\end{array};z\right] = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(1+\nu_1+n\mu_1)n!},$$

and

$$J^{\mu_2}_{\nu_2}[-z] = \;{}_0\psi_1\left[\begin{array}{c}-\\(1+\nu_2,\mu_2)\end{array};z\right] = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(1+\nu_2+n\mu_2)n!},$$

an operator $\mathbf{B}$ is defined by

$$\mathbf{B}f(z) := z\,\Gamma(1+\nu_1)\;J^{\mu_1}_{\nu_1}[-z] * h(z) + \overline{z\,\Gamma(1+\nu_2)\;J^{\mu_2}_{\nu_2}[-z] * g(z)} \in \mathcal{HS}.$$

As Mittag-Leffler functions and Bessel-Maitland functions are entire functions, results based on the *Main Results* for the operators $\mathbf{E}$ and $\mathbf{B}$ are quite obvious.

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Om P. Ahuja
Department of Mathematical Sciences,
Kent State University,
Burton, OH 44021-9500, USA
email: *oahuja@kent.edu*

Poonam Sharma
Department of Mathematics and Astronomy,
University of Lucknow, Lucknow 226007 INDIA
email: *sharma_poonam@lkouniv.ac.in*
† Corresponding author