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ITERATED ORDER OF MEROMORPHIC SOLUTIONS OF CERTAIN HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS OF FINITE ITERATED ORDER

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Abstract. In this paper, we investigate the iterated order of meromorphic solutions of homogeneous and nonhomogeneous linear differential equations where the coefficients are meromorphic functions satisfying certain growth conditions. And some estimates of iterated convergence exponent are also given.

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1. Introduction and main results

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions(see [6, 11, 15, 16]). Let us define inductively, for $r \in [0,\infty), \exp_1 r = e^r$ and $\exp_{i+1} r = \exp(\exp_i r), i \in N$. For all sufficiently large r, we define $\log_1 r = \log r$ and $\log_{i+1} r = \log(\log_i r), i \in N$. We also denote $\exp_0 r = r = \log r, \log_{-1} r = \exp_1 r$, and $\exp_{-1} r = \log_1 r$. In order to express the rate of growth of meromorphic functions of infinite order more precisely, we recall the following definitions (see [2, 10, 12]).

Definition 1.1. The iterated p-order $\sigma_p(f)$ of a meromorphic function $f(z)$ is defined by

$$\sigma_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log r}, \quad p \in N.$$

Remark 1.1. 1) If $p = 1$, the classical growth of order of an entire function $f(z)$ is defined by (see [6, 11])

$$\sigma(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_2 M(r,f)}{\log r}.$$

2) If $p = 2$, the hyper-order of an entire function $f(z)$ is defined by (see [15])

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log_2 T(r,f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_3 M(r,f)}{\log r}.$$

3) If $f(z)$ is an entire function, then the iterated p-order of $f(z)$ is defined by

$$\sigma_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r,f)}{\log r}, \quad p \in N.$$

Definition 1.2. The iterated p-lower order $\mu_p(f)$ of a meromorphic function f is defined by

$$\mu_p(f) = \underline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log r}, \quad p \in N.$$

Definition 1.3. The finiteness degree of the order of a meromorphic function $f(z)$ is defined by

$$i(f) = \begin{cases} 0, & \text{if } f \text{ is rational;} \\ \min\{p \in N : \sigma_p(f) < \infty\}, & \text{if } f \text{ is transcendental} \\ & \text{and } \sigma_p(f) < \infty \text{ for some } p \in N; \\ \infty, & \text{if } \sigma_p(f) = \infty \text{ for all } p \in N. \end{cases}$$

Remark 1.2. Similarly, we can define the finiteness degree of the lower order $\underline{i}(f)$ of a meromorphic function $f(z)$.

Definition 1.4. Let $n(r,a)$ be the unintegrated counting function for the sequence of a-points of a meromorphic function $f(z)$. The iterated convergence exponent of the sequence of a-points is defined by

$$\lambda_p(f-a) = \lambda_p(f,a) = \overline{\lim_{r\to\infty}} \frac{\log_p n(r,a)}{\log r}, \quad p \in N,$$

where $n(r,a) = n(r,a,f) = n(r, \frac{1}{f-a})$.

Remark 1.3. We also use the notation $\overline{\lambda}_p(f,a)$ to denote the iterated convergence exponent of the sequence of distinct a-points. In the definition of the iterated convergence exponent, we may replace $n(r,a)$ with the integrated counting function $N(r,a)$, and we have

$$\lambda_p(f,a) = \overline{\lim_{r\to\infty}} \frac{\log_p n(r,a)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_p N(r,\frac{1}{f-a})}{\log r}, \quad p \in N,$$

where $N(r,a) = N(r,a,f) = N(r, \frac{1}{f-a})$. If $a = 0$, the iterated convergence exponent of the zeros is defined by

$$\lambda_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p n(r,\frac{1}{f})}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_p N(r,\frac{1}{f})}{\log r}, \quad p \in N.$$

If $a = \infty$, the iterated convergence exponent of the poles is defined by

$$\lambda_p(\frac{1}{f}) = \overline{\lim_{r\to\infty}} \frac{\log_p n(r,f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_p N(r,f)}{\log r}, \quad p \in N.$$

Definition 1.5. The finiteness degree of the convergence exponent is defined by

$$i_\lambda(f,a) = \begin{cases} 0, & \text{if } n(r,a) = O(\log r)\ ; \\ \min\{p \in N : \lambda_p(f,a) < \infty\}, & \text{if } \lambda_p(f,a) < \infty \text{ for some } p \in N; \\ \infty, & \text{if } \lambda_p(f,a) = \infty \text{ for all } p \in N. \end{cases}$$

Remark 1.4. If $a = 0$, then we set $i_\lambda(f,a) = i_\lambda(f)$. If $a = \infty$, then we set $i_\lambda(f,a) = i_\lambda(\frac{1}{f})$. Similarly, we can define the finiteness degree $i_{\overline{\lambda}}(f,a)$ of $\overline{\lambda}_p(f,a)$.

Moreover, we define the linear measure of a set $H \subset [0,\infty)$ by $m(H) = \int_H dt$ and the logarithmic measure of a set $F \subset [1,+\infty)$ by $m_l(F) = \int_F \frac{dt}{t}$ (see [7]). The upper and the lower densities of H are defined by (see [9])

$$\overline{dens}H = \overline{\lim_{r\to\infty}} \frac{m(H\cap[0,r])}{r}, \qquad \underline{dens}H = \underline{\lim_{r\to\infty}} \frac{m(H\cap[0,r])}{r}.$$

For almost four decades, the Nevanlinna's value distribution theory has been a useful tool in investigating the complex oscillation of differential equations. Recently, the concepts of hyper-order (see [4, 15]) and iterated order (see [10, 11]) were used to further investigate the growth of infinite order meromorphic solutions of complex differential equations. The following results have been obtained.

Theorem A (see [9]) Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$, and let $A(z), B(z)$ be entire functions such that for some constants $\alpha, \beta > 0$,

$$|A(z)| \le \exp\{o(1)|z|^\beta\} \quad \text{and} \quad |B(z)| \ge \exp\{(1+o(1))\alpha|z|^\beta\}$$

as $z \to \infty$ for $z \in H$. Then every solution of $f \not\equiv 0$ of the equation

$$f'' + A(z)f' + B(z)f = 0 \tag{1.1}$$

satisfies $\sigma(f) = +\infty$ and $\sigma_2(f) \ge \beta$.

Theorem B (see [4]) Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$, and let $A(z), B(z)$ be entire functions, with $\sigma(A) \le \sigma(B) = \sigma < +\infty$ such that for some real constant $C(> 0)$ and for any given $\varepsilon > 0$,

$$|A(z)| \le \exp\{o(1)|z|^{\sigma-\varepsilon}\} \quad \text{and} \quad |B(z)| \ge \exp\{(1+o(1))C|z|^{\sigma-\varepsilon}\}$$

as $z \to \infty$ for $z \in H$. Then every solution $f \not\equiv 0$ of the equation (1.1) satisfies $\sigma(f) = +\infty$ and $\sigma_2(f) = \sigma$.

Theorem C (see [1]) Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$, and let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be entire functions such that for some constants $0 \le \beta < \alpha$ and $\mu > 0$, we have

$$|A_0(z)| \ge \exp\{\alpha|z|^{\mu}\} \quad \text{and} \quad |A_j(z)| \le \exp\{\beta|z|^{\mu}\}, j = 1, \ldots, k-1$$

as $z \to \infty$ for $z \in H$. Then every solution $f \not\equiv 0$ of the equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = 0 \tag{1.2}$$

satisfies $\sigma(f) = +\infty$ and $\sigma_2(f) \ge \mu$.

Theorem D (see [1]) Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$, and let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be entire functions with $\max\{\sigma(A_j) : j = 1, \ldots, k-1\} \le \sigma(A_0) = \sigma < +\infty$ such that for some real constants $0 \le \beta < \alpha$, we have

$$|A_0(z)| \ge \exp\{\alpha|z|^{\sigma-\varepsilon}\} \quad \text{and} \quad |A_j(z)| \le \exp\{(\beta|z|^{\sigma-\varepsilon}\}, j = 1, \ldots, k-1$$

as $z \to \infty$ for $z \in H$. Then every solution $f \not\equiv 0$ of the equation (1.2) satisfies $\sigma(f) = +\infty$ and $\sigma_2(f) = \sigma$.

Let $k \ge 2$ be an integer, $A_0(z), \ldots, A_{k-1}(z), A_k(z)$ with $A_0(z) \not\equiv 0$ and $A_k(z) \not\equiv 0$ be entire functions. It is well known that if $A_k(z) \equiv 1$, then all solutions of the linear differential equation

$$A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = 0 \tag{1.3}$$

are entire functions. We also know that if some of coefficients $A_0(z), \ldots, A_{k-1}(z)$ are transcendental and $A_k(z) \equiv 1$, then the equation (1.3) has at least one solution of infinite order. But when $A_k(z)$ is a nonconstant meromorphic function, the equation (1.3) can possess meromorphic solutions. For example, the equation $\frac{z^4}{e^z(z^3-3z^2+6z-6)}f''' - \frac{z^3}{e^z(z^2-2z+2)}f'' + f' + \frac{1-z}{z}f = 0$ has a meromorphic solution $f(z) = \frac{e^z}{z}$. Thus the natural question is: what conditions on $A_0(z), \ldots, A_k(z)$ will guarantee that every meromorphic solution $f \not\equiv 0$ of (1.3) has an infinite order? Recently, Chen [3] improved their results to the second order linear differential equations with meromorphic coefficients. Our main purpose of this paper is to improve and generalize the results of Theorems C and D and Chen [3]. We obtain some results of iterated order of meromorphic solutions of the higher order linear differential equations (1.2)-(1.5) and give some estimates of iterated convergence exponent.

Theorem 1.1. Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$ and let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be meromorphic functions of finite iterated orders such that for some constants $\alpha_2 > \alpha_1 \geq 0, \mu > 0$, we have

$$|A_0(z)| \geq \exp_p\{\alpha_2|z|^\mu\} \quad \text{and} \quad |A_j(z)| \leq \exp_p\{\alpha_1|z|^\mu\} \ (j = 1, \ldots, k-1)$$

as $z \to \infty$ for $z \in H$. If the equation (1.2) have meromorphic solutions, then every meromorphic solution $f(\not\equiv 0)$ satisfies $\sigma_{p+1}(f) \geq \mu$. Furthermore, if $\max\{|A_j(z)|, j = 0, 1, \ldots, k-1\} \leq \exp_p\{\beta|z|^\mu\}$ as $z \to \infty$, where $\beta(> 0)$ is a constant , then every meromorphic solution $f(\not\equiv 0)$ with $\lambda_p(\frac{1}{f}) < \mu_p(f)$ satisfies $i(f) = p+1$ and $\sigma_{p+1}(f) = \mu$.

Theorem 1.2. Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$ and let $A_0(z), A_1(z), \ldots, A_k(z)$ be meromorphic functions of finite iterated orders satisfying $\max\{\sigma_p(A_j), j = 0, 1, \ldots, k\} = \sigma < \infty$, such that for some constants $\alpha_2 > \alpha_1 \geq 0$ and for any given $\varepsilon > 0$, we have

$$|A_0(z)| \geq \exp_p\{\alpha_2|z|^{\sigma-\varepsilon}\} \quad \text{and} \quad |A_j(z)| \leq \exp_p\{\alpha_1|z|^{\sigma-\varepsilon}\} \ (j = 1, \ldots, k)$$

as $z \to \infty$ for $z \in H$. If the equation (1.3) have meromorphic solutions, then every meromorphic solution $f(\not\equiv 0)$ satisfies $\sigma_{p+1}(f) \geq \sigma$. Furthermore, if $\lambda_p(\frac{1}{f}) < \mu_p(f)$, then $i(f) = p + 1$ and $\sigma_{p+1}(f) = \sigma$.

Corollary 1.1. Let $A_0(z), \ldots, A_k(z), H$ satisfy all of the hypothesis of Theorem 1.2, and let $g(z)(\not\equiv 0)$ be a meromorphic function satisfying $i(g) < p+1$ or $\sigma_{p+1}(g) < \sigma$. Then every meromorphic solution $f(z)(\not\equiv 0)$ with $\lambda_p(\frac{1}{f}) < \mu_p(f)$ of the equation (1.3) satisfies $i_{\overline{\lambda}}(f - g) = p + 1$ and $\overline{\lambda}_{p+1}(f - g) = \lambda_{p+1}(f - g) = \sigma_{p+1}(f - g) = \sigma$.

Theorem 1.3. Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$ and $F(z)(\not\equiv 0)$ be a meromorphic function with $|F(z)| \leq \exp_q\{\alpha|z|^\mu\}$ as $z \to \infty$ or $\sigma_q(F) \leq \mu$ $(0 < q \leq p < \infty)$. Let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be meromorphic functions of finite iterated orders satisfying the following conditions:
(i)for some constants $\alpha_2 > \alpha_1 \geq 0, \mu > 0$,

$$|A_0(z)| \geq \exp_p\{\alpha_2|z|^\mu\} \quad \text{and} \quad |A_j(z)| \leq \exp_p\{\alpha_1|z|^\mu\} \ (j = 1, \ldots, k-1)$$

as $z \to \infty$ for $z \in H$;
(ii)$\max\{|A_j(z)|, j = 0, 1, \ldots, k-1\} \leq \exp_p\{\beta|z|^\mu\}$ as $z \to \infty$, where $\beta(> 0)$ is a constant.
If the equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = F(z) \tag{1.4}$$

have meromorphic solutions, then every meromorphic solution $f(z)(\not\equiv 0)$ with $\lambda_p(\frac{1}{f}) < \mu_p(f)$ satisfies $i(f) = p+1$ and $\sigma_{p+1}(f) = \mu$, with at most one exceptional solution $f_0(z)$ with $i(f) < p+1$ or $\sigma_{p+1}(f_0) < \mu$.

Theorem 1.4. Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$ and $F(z)(\not\equiv 0)$ be a meromorphic function with $\sigma_q(F) \leq \sigma$ $(0 < q \leq p < \infty)$. And let $A_0(z), A_1(z), \ldots, A_k(z)$ be meromorphic functions of finite iterated orders satisfying the following conditions:
(i)for some constants $\alpha_2 > \alpha_1 \geq 0$, and for any given $\varepsilon > 0$,

$$|A_0(z)| \geq \exp_p\{\alpha_2 |z|^{\sigma-\varepsilon}\} \quad \text{and} \quad |A_j(z)| \leq \exp_p\{\alpha_1 |z|^{\sigma-\varepsilon}\}\ (j = 1, \ldots, k)$$

as $z \to \infty$ for $z \in H$;
(ii)$\max\{\sigma_p(A_j), j = 0, 1, \ldots, k\} = \sigma < \infty$.
If the equation

$$A_k(z)f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = F(z) \tag{1.5}$$

have meromorphic solutions, then every meromorphic solution $f(z)$ with $\lambda_p(\frac{1}{f}) < \mu_p(f)$ satisfies $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma$, with at most one exceptional solution.

Remark 1.5. If $p < q$ in Theorems 1.3 or 1.4, then we can obtain $\sigma_{q+1}(f) = \mu$ or σ respectively.

Corollary 1.2. Let $A_0(z), \ldots, A_k(z), F(z), H$ satisfy all of the hypothesis of Theorem 1.4, and let $g(z)(\not\equiv 0)$ be a meromorphic function satisfying $i(g) < p+1$ or $\sigma_{p+1}(g) < \sigma$, and $F - [A_k(z)g^{(k)} + A_{k-1}(z)g^{(k-1)} + \ldots + A_0 g] \not\equiv 0$. Then every meromorphic solution $f(z)(\not\equiv 0)$ with $i_{\overline{\lambda}}(f) = i(f) = p+1$ and $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma$ of the equation (1.5) satisfies $i_{\overline{\lambda}}(f-g) = p+1$ and $\overline{\lambda}_{p+1}(f-g) = \lambda_{p+1}(f-g) = \sigma_{p+1}(f-g) = \sigma$.

2. Preliminary Lemmas

Lemma 2.1 (see [5]) Let $f(z)$ be a meromorphic function, and let $\alpha > 1$ and $\varepsilon > 0$ be given real constants. Then there exist a constant $C > 0$ and a set $E_1 \subset [0, \infty)$ having finite linear measure such that for all z satisfying $|z| = r \notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq C[T(\alpha r, f)r^{\varepsilon} \log T(\alpha r, f)]^j \ \ (j \in N).$$

Lemma 2.2 (see [13]) Let $f(z) = \frac{g(z)}{d(z)}$, where $g(z), d(z)$ are entire functions of finite iterated order satisfying $\mu_p(g) = \mu_p(f) \leq \sigma_p(g) = \sigma_p(f) < \infty, 0 < p <$

$\infty, i(d) < p$ or $\sigma_p(d) = \beta < \mu_p(f)$. Let z be a point with $|z| = r$ at which $|g(z)| = M(r, g)$ and $\nu_g(r)$ denote the central index of g, then the estimation

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^j (1 + o(1)) \quad (j \in N)$$

holds for all $|z| = r$ outside a set E_2 of r of finite logarithmic measure.

Lemma 2.3 (see [8,13]) Let $g(z)$ be an entire function of finite iterated order satisfying $i(g) = p + 1, \sigma_{p+1}(g) = \sigma, \mu_{q+1}(g) = \mu, 0 < q \le p < \infty$, and let $\nu_g(r)$ be the central index of g, then we have

$$\overline{\lim_{r\to\infty}} \frac{\log_{p+1} \nu_g(r)}{\log r} = \sigma, \quad \underline{\lim_{r\to\infty}} \frac{\log_{q+1} \nu_g(r)}{\log r} = \mu.$$

Lemma 2.4 (see [11]) Let $g : (0, +\infty) \to R, h : (0, +\infty) \to R$ be monotone increasing functions such that (i) $g(r) \le h(r)$n.e or (ii) $g(r) \le h(r)$ outside of an exceptional set E_3 of finite logarithmic measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \le h(\alpha r)$ for all $r > r_0$.

Lemma 2.5 (see [14]) Let $f(z)$ be a meromorphic function of finite iterated order with $i(f) = p, p \in N$. Then exist entire functions $\pi_1(z), \pi_2(z)$ and $D(z)$ such that

$$f(z) = \frac{\pi_1(z)e^{D(z)}}{\pi_2(z)}, \text{ and } \sigma_p(f) = \max\{\sigma_p(\pi_1), \sigma_p(\pi_2), \sigma_p(e^{D(z)})\}.$$

Moreover, for any given $\varepsilon > 0$, we have

$$\exp\{-\exp_{p-1}\{r^{\sigma_p(f)+\varepsilon}\}\} \le |f(z)| \le \exp_p\{r^{\sigma_p(f)+\varepsilon}\}, \quad (r \notin E_4),$$

where E_4 is a set of r of finite linear measure.

Lemma 2.6 (see [13]) Let $A_0(z), A_1(z), \ldots, A_{k-1}(z), F(z)(\not\equiv 0)$ be meromorphic functions and let $f(z)$ be a meromorphic solution of (1.3) satisfying one of the following conditions:

$(i)\max\{i(F) = q, i(A_j)(j = 0, \ldots, k-1)\} < i(f) = p + 1$ $(0 < p < \infty)$,

$(ii) b = \max\{\sigma_{p+1}(F), \sigma_{p+1}(A_j)(j = 0, \ldots, k-1)\} < \sigma_{p+1}(f) = \sigma$, then $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma$.

Lemma 2.7 Let $A_0(z), A_1(z), \ldots, A_k(z), F(z)(\not\equiv 0)$ be meromorphic functions and let $f(z)$ be a meromorphic solution of (1.3) satisfying one of the following conditions:

$(i)\max\{i(F) = q, i(A_j)(j = 0, \ldots, k)\} < i(f) = p + 1$ $(0 < p < \infty)$,

$(ii) b = \max\{\sigma_{p+1}(F), \sigma_{p+1}(A_j)(j = 0, \ldots, k)\} < \sigma_{p+1}(f) = \sigma$, then $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma$.

Proof. By using the same method of the proof of Lemma 2.6, we obtain the conclusion of Lemma 2.7.

3. Proof of Theorem 1.1-1.4

Proof of Theorem 1.1. Suppose that $f(z)(\not\equiv 0)$ ia a meromorphic solution of (1.2). It follows by (1.2) that

$$|A_0(z)| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + |A_{k-1}(z)| \left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \ldots + |A_1(z)| \left|\frac{f'(z)}{f(z)}\right|. \tag{3.1}$$

Then by Lemma 2.1, there exists a set $E_1 \subset [0, \infty)$ with finite linear measure such that for all z satisfying $|z| = r \notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq Cr[T(2r, f)]^{j+1} \quad (j = 1, \ldots, k), \tag{3.2}$$

where C is a constant. By the hypothesis of Theorem 1.1, there exists a set H with $\overline{dens}\{|z| : z \in H\} > 0$ such that for all $z \to \infty$ for $z \in H$, we have

$$|A_0(z)| \geq \exp_p\{\alpha_2|z|^\mu\} \;\text{ and }\; |A_j(z)| \leq \exp_p\{\alpha_1|z|^\mu\} \;\; (j = 1, \ldots, k-1). \tag{3.3}$$

By substituting (3.2) and (3.3) into (3.1), for all z satisfying $z \to \infty$ for $z \in H$ and $|z| = r \notin E_1$, we have

$$\exp_p\{\alpha_2|z|^\mu\} \leq kCr[T(2r, f)]^{k+1} \exp_p\{\alpha_1|z|^\mu\}. \tag{3.4}$$

Hence, there exists a set $E \subset (0, \infty)$ with positive upper density such that

$$(1 - o(1)) \exp_p\{\alpha_2|z|^\mu\} \leq kCr[T(2r, f)]^{k+1} \tag{3.5}$$

as $r \to \infty$ in E. By (3.5), we have $\sigma_{p+1}(f) \geq \mu$.

Furthermore, by the hypothesis of Theorem 1.1, for sufficiently large r, we have

$$\max\{|A_j(z)|, j = 0, 1, \ldots, k-1\} \leq \exp_p\{\beta|z|^\mu\}, \tag{3.6}$$

where $\beta > 0$ is a constant. By Hadamard factorization theorem, we can write $f(z)$ as $f(z) = \frac{g(z)}{d(z)}$, where $g(z), d(z)$ are entire functions of finite iterated order satisfying $\mu_p(g) = \mu_p(f) \leq \sigma_p(g) = \sigma_p(f), i(d) < p$ or $\sigma_p(d) = \lambda_p(d) = \lambda_p(\frac{1}{f}) < \mu_p(f)$. By Lemma 2.2, there exists a set E_2 having finite logarithmic measure such that for all z satisfying $|z| = r \notin E_2$ and $|g(z)| = M(r, g)$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| = \left(\frac{\nu_g(r)}{z}\right)^j (1 + o(1)) \;\; (j = 1, \ldots, k). \tag{3.7}$$

It follows by (1.2) that

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq |A_{k-1}(z)| \left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \ldots + |A_1(z)| \left|\frac{f'(z)}{f(z)}\right| + |A_0(z)|. \tag{3.8}$$

By substituting (3.6) and (3.7) into (3.8), we obtain

$$\nu_g(r)|1+o(1)| \leq kr^k \exp_p\{\beta|z|^\mu\}|1+o(1)|, \tag{3.9}$$

where z satisfies $|z| = r \notin [0,1] \cup E_2, r \to \infty$ and $|g(z)| = M(r,g)$. By (3.9), Lemmas 2.3 and 2.4, we obtain $\sigma_{p+1}(f) = \sigma_{p+1}(g) \leq \mu$. Hence, every meromorphic solution $f(\not\equiv 0)$ with $\lambda_p(\frac{1}{f}) < \mu_p(f)$ satisfies $i(f) = p+1$ and $\sigma_{p+1}(f) = \mu$. Theorem 1.1 is thus proved.

Proof of Theorem 1.2. Assume that $f(\not\equiv 0)$ is a meromorphic solution of equation (1.3). By using the same arguments as in Theorem 1.1, we get $\sigma_{p+1}(f) \geq \sigma - \varepsilon$. And since ε is arbitrary, we get $\sigma_{p+1}(f) \geq \sigma$.

On the other hand, by the hypothesis of Theorem 1.2 and Lemma 2.5, for any given $\varepsilon > 0$, there exists a set E_4 with a finite linear measure, for all z satisfying $|z| = r \notin E_4$ such that

$$|A_j(z)| \leq \exp_p\{r^{\sigma+\varepsilon}\} \ \ (j = 0, 1, \ldots, k-1) \ \text{ and } \ |A_k(z)| \geq \exp\{-\exp_{p-1}\{r^{\sigma+\varepsilon}\}\}. \tag{3.10}$$

It follows by (1.3) that

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq \frac{1}{|A_k(z)|}\left(|A_{k-1}(z)| \left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \ldots + |A_1(z)| \left|\frac{f'(z)}{f(z)}\right| + |A_0(z)|\right). \tag{3.11}$$

By Hadamard factorization theorem, we can write $f(z)$ as $f(z) = \frac{g(z)}{d(z)}$, where $g(z), d(z)$ are entire functions of finite iterated order satisfying $\mu_p(g) = \mu_p(f) \leq \sigma_p(g) = \sigma_p(f), i(d) < p$ or $\sigma_p(d) = \lambda_p(d) = \lambda_p(\frac{1}{f}) < \mu_p(f)$. By Lemma 2.2, there exists a set E_2 having finite logarithmic measure such that for all z satisfying $|z| = r \notin E_2$ and $|g(z)| = M(r,g)$, we have (3.7). By substituting (3.7) and (3.10) into (3.11), we obtain

$$\nu_g(r)|1+o(1)| \leq kr^k \exp_p\{r^{\sigma+2\varepsilon}\}|1+o(1)|, \tag{3.12}$$

where z satisfying $|z| = r \notin [0,1] \cup E_2 \cup E_4$ and $|g(z)| = M(r,f), r \to \infty$. By (3.12), Lemmas 2.3 and 2.4, we obtain $\sigma_{p+1}(f) = \sigma_{p+1}(g) \leq \sigma + \varepsilon$. Since ε is arbitrary, we have $\sigma_{p+1}(f) \leq \sigma$. Hence, we obtain $i(f) = p+1$ and $\sigma_{p+1}(f) = \sigma$. Theorem 1.2 is thus proved.

Proof of Theorem 1.3. Case 1. We assume that $|F(z)| \leq \exp_q\{\alpha|z|^{\mu}\}$ as $z \to \infty$.It follows by (1.4) that

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq |A_{k-1}(z)| \left|\frac{f^{(k)}(z)}{f(z)}\right| + \ldots + |A_1(z)| \left|\frac{f'(z)}{f(z)}\right| + |A_0(z)| + \left|\frac{F(z)}{f(z)}\right|. \quad (3.13)$$

By Hadamard factorization theorem, we can write $f(z)$ as $f(z) = \frac{g(z)}{d(z)}$, where $g(z), d(z)$ are entire functions of finite iterated order such that $\mu_p(g) = \mu_p(f) \leq \sigma_p(g) = \sigma_p(f), \sigma_p(d) = \lambda_p(d) = \lambda_p(\frac{1}{f}) < \mu_p(f)$, By Lemma 2.2, there exists a set E_2 having finite logarithmic measure such that for all z satisfying $|z| = r \notin E_2$ and $|g(z)| = M(r, g)$, we have (3.7). By Lemma 2.5, for any given $\varepsilon > 0$, there exists a set E_4 with finite linear measure such that for all z satisfying $|z| = r \notin E_4, r \to \infty$ and $|g(z)| = M(r, g)$, we have

$$\left|\frac{F(z)}{f(z)}\right| = \frac{|F(z)||d(z)|}{|g(z)|} = \frac{|F(z)||d(z)|}{M(r,g)} \leq \frac{\exp_q\{\alpha|z|^{\mu}\} \exp_p\{r^{\sigma_p(d)+\varepsilon}\}}{\exp_p\{r^{\mu_p(g)-\varepsilon}\}}$$

$$\leq \exp_q\{\alpha|z|^{\mu}\}. \quad (3.14)$$

By the condition (ii) of hypothesis of Theorem 1.3, for sufficiently large r, we have (3.6), where $\beta(> 0)$ is a constant. By substituting (3.6),(3.7) and (3.14) into (3.13), for z satisfying $|z| = r \notin [0, 1] \cup E_2 \cup E_4, r \to \infty$ and $|g(z)| = M(r, g)$, we have

$$\nu_g(r)|1 + o(1)| \leq (k + 1)r^k \exp_p\{\max\{\alpha, \beta\}|z|^{\mu}\}. \quad (3.15)$$

Hence, by (3.15), Lemmas 2.3 and 2.4, we obtain $\sigma_{p+1}(f) \leq \mu$.

We assume that f_0 is a meromorphic solution of the equation (1.4) and satisfies $i(f_0) < p + 1$ or $\sigma_{p+1}(f_0) < \mu$. If there exists another meromorphic solution f_1 with $i(f_1) < p + 1$ or $\sigma_{p+1}(f_1) < \mu$, then $\sigma_{p+1}(f_1 - f_0) < \mu$. However, $f_1 - f_0$ is a solution of the corresponding homogeneous equation (1.2) and by the first section of hypothesis of Theorem 1.1, we can obtain $\sigma_{p+1}(f_1 - f_0) \geq \mu$. This is a contradiction with $\sigma_{p+1}(f_1 - f_0) < \mu$. Hence, every meromorphic solution $f(z)$ with $\lambda_p(\frac{1}{f}) < \mu_p(f)$ satisfies $i(f) = p+1$ and $\sigma_{p+1}(f) = \mu$, with at most one exceptional solution f_0 with $i(f) < p + 1$ and $\sigma_{p+1}(f_0) < \mu$.

Case 2. We assume that $\sigma_q(F) \leq \mu$. By Lemma 2.5, for any given $\varepsilon > 0$, there is a set E_4 with finite linear measure such that for z satisfying $|z| = r \notin E_4, r \to \infty$, we have

$$|F(z)| \leq \exp_p\{r^{\sigma_q(F)+\varepsilon}\} \leq \exp_p\{r^{\mu+\varepsilon}\}. \quad (3.16)$$

By using the same reasoning as in Case 1, we obtain $i(f) < p + 1$ and $\sigma_{p+1}(f) = \mu$, with at most one exceptional solution f_0 having $i(f_0) < p + 1$ and $\sigma_{p+1}(f) < \mu$. Theorem 1.3 is thus proved.

Proof of Theorem 1.4 Assume that $f(\not\equiv 0)$ is a meromorphic solution of equation (1.5). It follows by (1.5) that

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq \frac{|A_{k-1}(z)|}{|A_k(z)|}\left|\frac{f^{(k-1)}(z)}{f(z)}\right|+\ldots+\frac{|A_1(z)|}{|A_k(z)|}\left|\frac{f'(z)}{f(z)}\right|+\frac{|A_0(z)|}{|A_k(z)|}+\frac{1}{|A_k(z)|}\left|\frac{F(z)}{f(z)}\right|. \tag{3.17}$$

By Hadamard factorization theorem, we can write $f(z)$ as $f(z) = \frac{g(z)}{d(z)}$, where $g(z), d(z)$ are entire functions of finite iterated order such that $\mu_p(g) = \mu_p(f) \leq \sigma_p(g) = \sigma_p(f), \sigma_p(d) = \lambda_p(d) = \lambda_p(\frac{1}{f}) < \mu_p(f)$, by Lemma 2.2, there exists a set E_2 having finite logarithmic measure such that for all z satisfying $|z| = r \notin E_2$ and $|g(z)| = M(r, g)$, we have (3.7). By Lemma 2.5 and the condition (ii) of hypothesis of Theorem 1.4, for any given $\varepsilon > 0$, there exists a set E_4 with finite linear measure such that for all z satisfying $|z| = r \notin E_4$ and $|g(z)| = M(r, g)$, for sufficiently large r, we have (3.10) and

$$\frac{1}{|A_k(z)|}\left|\frac{F(z)}{f(z)}\right| = \frac{|F(z)||d(z)|}{|A_k(z)||g(z)|} = \frac{|F(z)||d(z)|}{|A_k(z)|M(r,g)} \leq \frac{\exp_p\{r^{\sigma+\varepsilon}\}\exp_p\{r^{\sigma_p(d)+\varepsilon}\}}{\exp\{-\exp_{p-1}\{r^{\sigma+\varepsilon}\}\}\exp_p\{r^{\mu_p(g)-\varepsilon}\}}$$

$$\leq \exp_p\{r^{\sigma+2\varepsilon}\}. \tag{3.18}$$

By substituting (3.7), (3.10) and (3.18) into (3.17), for z satisfying $|z| = r \notin [0,1] \cup E_2 \cup E_4, r \to \infty$ and $|g(z)| = M(r, g)$, we have

$$\nu_g(r)|1+o(1)| \leq (k+1)r^k \exp_p\{r^{\sigma+2\varepsilon}\}|1+o(1)|. \tag{3.19}$$

Hence, by (3.19), Lemmas 2.3 and 2.4, we obtain $\sigma_{p+1}(f) \leq \sigma$.

By using the same arguments as in Theorem 1.3, we get $\sigma_{p+1}(f) = \sigma$, with at most one exceptional solution. Since $\sigma_q(F) \leq \sigma, \max\{\sigma_p(A_j), j = 0, \ldots, k\} = \sigma$, and by Lemma 2.7, we obtain $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma$, with at most one exceptional solution. Theorem 1.4 is thus proved.

4. Proofs of Corollaries 1.1 and 1.2

Proof of Corollary 1.1 Setting $h = f - g$. Since $i(g) < p+1$ or $\sigma_{p+1}(g) < \sigma$, and by Theorem 1.2, we have $\sigma_{p+1}(h) = \sigma_{p+1}(f) = \sigma$ and $\overline{\lambda}_{p+1}(h) = \overline{\lambda}_{p+1}(f-g)$. By substituting $f = h + g$ into (1.2), we get

$$A_k(z)h^{(k)}+A_{k-1}(z)h^{(k-1)}+\ldots+A_0(z)h = -[A_k(z)g^{(k)}+A_{k-1}(z)g^{(k-1)}+\ldots+A_0(z)g].$$

Set $F(z) = A_k(z)g^{(k)} + A_{k-1}(z)g^{(k-1)} + \ldots + A_1(z)g' + A_0(z)g$. If $F(z) \equiv 0$, By the first part of Theorem 1.2, we can get $\sigma_{p+1}(g) \geq \sigma$. This is a contradiction with

$\sigma_{p+1}(g) < \sigma$. So we have $F(z) \not\equiv 0$. Since $F(z) \not\equiv 0$ and $\sigma_{p+1}(F) < \sigma = \sigma_{p+1}(f)$, and by Lemma 2.7, we obtain $i_{\overline{\lambda}}(f - g) = p + 1$ and $\overline{\lambda}_{p+1}(f - g) = \lambda_{p+1}(f - g) = \sigma_{p+1}(f - g) = \sigma$.

Proof Corollary 1.2 By using the similar proof of Corollary 1.1, we can get the results of Corollary 1.2.

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