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FROM CONFORMAL DEFORMATIONS OF THE JET BERWALD-MOR METRIC TO SOME NON-ISOTROPIC GEOMETRICAL EINSTEIN-LIKE EQUATIONS

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Abstract. In this paper we develop on the 1-jet space $J^1(\mathbb{R}, M^n)$ the Finsler-like geometry (in the sense of distinguished (d-) connection, d-torsions, d-curvatures and some gravitational-like and electromagnetic-like geometrical models) attached to the (t, x)-conformal deformation of the Berwald-Moór metric.

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1.Introduction

The geometric-physical Berwald-Moór structure ([5], [11], [10]) was intensively investigated by P.K. Rashevski [17] and further substantiated and developed by D.G. Pavlov, G.I. Garas'ko and S.S. Kokarev ([15], [6], [16]). At the same time, the physical studies of Asanov [1] or Garas'ko and Pavlov ([7], [14]) emphasize the importance of the Finsler geometry characterized by the total equality in rights of all non-isotropic directions in the theory of space-time structure, gravitation and electromagnetism. For such a reason, one underlines the important role played by the classical Berwald-Moór metric

$$F : TM \to \mathbb{R}, \qquad F(y) = \sqrt[n]{y^1 y^2 ... y^n}, \qquad n \geq 2,$$

whose tangent Finslerian geometry is studied by geometers as Matsumoto and Shimada [8] or Balan [3]. In such a perspective, according to the recent geometric-physical ideas proposed by Garas'ko ([6], [7]), we consider that a Finsler-like geometric-physical study for the conformal deformations of the jet Berwald-Moór structure is required. Consequently, we investigate on the 1-jet space $J^1(\mathbb{R}, M^n)$ the Finsler-like geometry (together with some gravitational-like and electromagnetic-like geometrical models) of the (t, x)-*conformal deformation of the Berwald-Moór metric*[1]

$$\overset{*}{F}(t, x, y) = e^{\sigma(x)} \sqrt{h^{11}(t)} \cdot \left[y_1^1 y_1^2 ... y_1^n \right]^{\frac{1}{n}}, \tag{1}$$

[1]We assume that we have $y_1^1 y_1^2 ... y_1^n > 0$. This is a domain of existence where we can y-differentiate the Finsler-like function $\overset{*}{F}(t, x, y)$.

where $\sigma(x)$ is a smooth non-constant function on M^n, $h^{11}(t)$ is the dual of a Riemannian metric $h_{11}(t)$ on $\mathbb{R}$, and

$$(t,x,y)=(t,x^1,x^2,...,x^n,y_1^1,y_1^2,...,y_1^n)$$

are the coordinates of the 1-jet space $J^1(\mathbb{R},M^n)$, which transform by the rules (the Einstein convention of summation is assumed everywhere):

$$\widetilde{t}=\widetilde{t}(t),\quad \widetilde{x}^i=\widetilde{x}^i(x^j),\quad \widetilde{y}_1^i=\frac{\partial\widetilde{x}^i}{\partial x^j}\frac{dt}{d\widetilde{t}}\cdot y_1^j, \tag{2}$$

where $i,j=\overline{1,n}$, rank $(\partial\widetilde{x}^i/\partial x^j)=n$ and $d\widetilde{t}/dt\neq 0$. Note that the particular jet Finsler-like geometries (together with their corresponding jet geometrical gravitational field-like theories) of the (t,x)-conformal deformations of the Berwald-Moór metrics of order three and four are now completely developed in the papers [12] and [13].

Based on the geometrical ideas promoted by Miron and Anastasiei in the classical Lagrangian geometry of tangent bundles [9], together with those used by Asanov in the geometry of 1-jet spaces [2], the differential geometry (in the sense of d-connections, d-torsions, d-curvatures, gravitational and electromagnetic geometrical theories) produced by an arbitrary jet rheonomic Lagrangian function $L:J^1(\mathbb{R},M^n)\to\mathbb{R}$ is now exposed in the monograph [4]. In what follows, we apply the general jet geometrical results from book [4] to the (t,x)-conformal deformed jet Berwald-Moór metric (1).

2.The canonical nonlinear connection

Let us rewrite the (t,x)-conformal deformed jet Berwald-Moór metric (1) in the form

$$\overset{*}{F}(t,x,y)=e^{\sigma(x)}\sqrt{h^{11}(t)}\cdot\left[G_{1[n]}(y)\right]^{1/n},$$

where $G_{1[n]}(y)=y_1^1y_1^2...y_1^n$. Hereinafter, the *fundamental metrical d-tensor* produced by the metric (1) is given by the formula[2] (see [4])

$$\overset{*}{g}_{ij}(t,x,y)\overset{def}{=}\frac{h_{11}(t)}{2}\frac{\partial^2\overset{*}{F}{}^2}{\partial y_1^i\partial y_1^j}\Rightarrow$$

$$\overset{*}{g}_{ij}(t,x,y):=\overset{*}{g}_{ij}(x,y)=\frac{e^{2\sigma(x)}}{n}\left(\frac{2}{n}-\delta_{ij}\right)\frac{G_{1[n]}^{2/n}}{y_1^iy_1^j}, \tag{3}$$

[2]Throughout this paper the Latin letters i, j, k, m, r,... take values in the set $\{1,2,...n\}$.

where we have no sum by i or j. Moreover, the matrix $\overset{*}{g} = (\overset{*}{g}_{ij})$ admits the inverse $\overset{*}{g}^{-1} = (\overset{*}{g}^{jk})$, whose entries are

$$\overset{*}{g}^{jk} = e^{-2\sigma(x)}(2 - n\delta^{jk})G_{1[n]}^{-2/n} y_1^j y_1^k \text{ (no sum by } j \text{ or } k). \tag{4}$$

Let us consider that the Christoffel symbol of the Riemannian metric $h_{11}(t)$ on $\mathbb{R}$ is

$$\varkappa_{11}^1 = \frac{h^{11}}{2}\frac{dh_{11}}{dt},$$

where $h^{11} = 1/h_{11} > 0$. Then, using a general formula from [4] and taking into account that we have

$$\frac{\partial G_{1[n]}}{\partial y_1^i} = \frac{G_{1[n]}}{y_1^i},$$

we find the following geometrical result:

Proposition 1. *For the (t, x)-conformal deformed Berwald-Moór metric (1), the energy action functional*

$$\overset{*}{\mathbf{E}}(t, x(t)) = \int_a^b \overset{*}{F}{}^2(t, x, y)\sqrt{h_{11}}dt = \int_a^b e^{2\sigma(x)}\left[y_1^1 y_1^2 ... y_1^n\right]^{2/n} \cdot h^{11}\sqrt{h_{11}}dt,$$

where $y = dx/dt$, produces on the 1-jet space $J^1(R, M^n)$ the canonical nonlinear connection

$$\overset{*}{\Gamma} = \left(M_{(1)1}^{(i)} = -\varkappa_{11}^1 y_1^i,\ N_{(1)j}^{(i)} = n\sigma_i y_1^i \delta_j^i\right), \tag{5}$$

where $\sigma_i = \partial\sigma/\partial x^i$.

Proof. For the energy action functional $\overset{*}{\mathbf{E}}$, the associated Euler-Lagrange equations can be written in the equivalent form (see [4])

$$\frac{d^2x^i}{dt^2} + 2H_{(1)1}^{(i)}\left(t, x^k, y_1^k\right) + 2G_{(1)1}^{(i)}\left(t, x^k, y_1^k\right) = 0, \tag{6}$$

where the local components

$$H_{(1)1}^{(i)} \overset{def}{=} -\frac{1}{2}\varkappa_{11}^1(t)y_1^i$$

and

$$G_{(1)1}^{(i)} \overset{def}{=} \frac{h_{11}\overset{*}{g}^{ip}}{4}\left[\frac{\partial^2 \overset{*}{F}{}^2}{\partial x^r \partial y_1^p}y_1^r - \frac{\partial \overset{*}{F}{}^2}{\partial x^p} + \frac{\partial^2 \overset{*}{F}{}^2}{\partial t \partial y_1^p} + \right.$$
$$\left. + \frac{\partial \overset{*}{F}{}^2}{\partial y_1^p}\varkappa_{11}^1(t) + 2h^{11}\varkappa_{11}^1 \overset{*}{g}_{pr} y_1^r\right] = \frac{n}{2}\sigma_i\left(y_1^i\right)^2$$

represent, from a geometrical point of view, a *spray* on the 1-jet space $J^1(\mathbb{R}, M^n)$.

Therefore, the *canonical nonlinear connection* associated to this spray has the local components (see [4])

$$M^{(i)}_{(1)1} \overset{def}{=} 2H^{(i)}_{(1)1} = -\text{K}^1_{11} y^i_1,$$

$$N^{(i)}_{(1)j} \overset{def}{=} \frac{\partial G^{(i)}_{(1)1}}{\partial y^j_1} = n\sigma_i y^i_1 \delta^i_j.$$

3. The Cartan $\overset{*}{\Gamma}$-linear connection. Its d-torsions and d-curvatures

The nonlinear connection (5) produces the dual *adapted bases* of d-vector fields (no sum by i)

$$\left\{ \frac{\delta}{\delta t} = \frac{\partial}{\partial t} + \text{K}^1_{11} y^p_1 \frac{\partial}{\partial y^p_1} \ ; \ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - n\sigma_i y^i_1 \frac{\partial}{\partial y^i_1} \ ; \ \frac{\partial}{\partial y^i_1} \right\} \subset \mathcal{X}(E) \tag{7}$$

and d-covector fields (no sum by i)

$$\left\{ dt \ ; \ dx^i \ ; \ \delta y^i_1 = dy^i_1 - \text{K}^1_{11} y^i_1 dt + n\sigma_i y^i_1 dx^i \right\} \subset \mathcal{X}^*(E), \tag{8}$$

where $E = J^1(\mathbb{R}, M^n)$. The naturalness of the geometrical adapted bases (7) and (8) is coming from the fact that, via a transformation of coordinates (2), their elements transform as the classical tensors. Therefore, the description of all subsequent geometrical objects on the 1-jet space $J^1(\mathbb{R}, M^n)$ (e.g., the Cartan canonical linear connection, its torsion and curvature etc.) will be given in local adapted components. Consequently, by direct computations, we obtain the following geometrical result:

Proposition 2. *The Cartan canonical $\overset{*}{\Gamma}$-linear connection, produced by the (t, x)-conformal deformed Berwald-Moór metric (1), has the following adapted local components (no sum by i, j or k):*

$$C\overset{*}{\Gamma} = \left(\text{K}^1_{11}, \ G^k_{j1} = 0, \ L^i_{jk} = n\delta^i_j \delta^i_k \sigma_i, \ C^{i(1)}_{j(k)} = \mathtt{C}^i_{jk} \cdot \frac{y^i_1}{y^j_1 y^k_1} \right), \tag{9}$$

where

$$\mathtt{C}^i_{jk} = -\frac{2}{n^2} + \frac{\delta^i_j + \delta^i_k + \delta_{jk}}{n} - \delta^i_j \delta^i_k.$$

Proof. The adapted components of the Cartan canonical connection are given by the formulas (see [4])

$$G_{j1}^{k} \overset{def}{=} \frac{\overset{*}{g}^{km}}{2}\frac{\delta \overset{*}{g}_{mj}}{\delta t} = 0,$$

$$L_{jk}^{i} \overset{def}{=} \frac{\overset{*}{g}^{im}}{2}\left(\frac{\delta \overset{*}{g}_{jm}}{\delta x^{k}} + \frac{\delta \overset{*}{g}_{km}}{\delta x^{j}} - \frac{\delta \overset{*}{g}_{jk}}{\delta x^{m}}\right) = n\delta_{j}^{i}\delta_{k}^{i}\sigma_{i},$$

$$C_{j(k)}^{i(1)} \overset{def}{=} \frac{\overset{*}{g}^{im}}{2}\left(\frac{\partial \overset{*}{g}_{jm}}{\partial y_{1}^{k}} + \frac{\partial \overset{*}{g}_{km}}{\partial y_{1}^{j}} - \frac{\partial \overset{*}{g}_{jk}}{\partial y_{1}^{m}}\right) = \frac{\overset{*}{g}^{im}}{2}\frac{\partial \overset{*}{g}_{jm}}{\partial y_{1}^{k}} = \mathtt{C}_{jk}^{i} \cdot \frac{y_{1}^{i}}{y_{1}^{j}y_{1}^{k}},$$

where we also used the equality

$$\frac{\delta \overset{*}{g}_{jm}}{\delta x^{k}} = n\delta_{jk}\overset{*}{g}_{jm}\sigma_{k} + n\delta_{mk}\overset{*}{g}_{jm}\sigma_{k}.$$

Remark 3. It is important to note that the vertical d-tensor $C_{j(k)}^{i(1)}$ also has the properties (see also [8], [12] and [13]):

$$C_{j(k)}^{i(1)} = C_{k(j)}^{i(1)}, \quad C_{j(m)}^{i(1)}y_{1}^{m} = 0, \quad C_{j(m)}^{m(1)} = 0, \quad C_{i(k)|m}^{m(1)} = 0, \tag{10}$$

with sum by m, where

$$C_{i(k)|j}^{l(1)} \overset{def}{=} \frac{\delta C_{i(k)}^{l(1)}}{\delta x^{j}} + C_{i(k)}^{r(1)}L_{rj}^{l} - C_{r(k)}^{l(1)}L_{ij}^{r} - C_{i(r)}^{l(1)}L_{kj}^{r}.$$

Proposition 4. *The Cartan canonical connection of the (t,x)-conformal deformed Berwald-Moór metric (1) has two effective local torsion d-tensors:*

$$R_{(1)ij}^{(r)} = n\left(\delta_{i}^{r}\sigma_{rj} - \delta_{j}^{r}\sigma_{ri}\right)y_{1}^{r},$$

$$P_{i(j)}^{r(1)} = \left(-\frac{2}{n^{2}} + \frac{\delta_{i}^{r} + \delta_{j}^{r} + \delta_{ij}}{n} - \delta_{i}^{r}\delta_{j}^{r}\right)\cdot\frac{y_{1}^{r}}{y_{1}^{i}y_{1}^{j}},$$

where $\sigma_{pq} := \left(\partial^{2}\sigma\right)/\left(\partial x^{p}\partial x^{q}\right).$

Proof. Generally, an *h-normal* Γ*-linear connection* on the 1-jet space $J^{1}(\mathbb{R}, M^{n})$ has *eight* effective local d-tensors of torsion (for more details, see [4]). For the Cartan canonical connection (9) these reduce only to *two* (the other six are zero):

$$R_{(1)ij}^{(r)} \overset{def}{=} \frac{\delta N_{(1)i}^{(r)}}{\delta x^{j}} - \frac{\delta N_{(1)j}^{(r)}}{\delta x^{i}},$$

$$P_{i(j)}^{r(1)} \stackrel{def}{=} C_{i(j)}^{r(1)}.$$

Proposition 5. *The Cartan canonical connection of the (t,x)-conformal deformed Berwald-Moór metric (1) has three effective local curvature d-tensors:*

$$R_{ijk}^{l} = \frac{\partial L_{ij}^{l}}{\partial x^{k}} - \frac{\partial L_{ik}^{l}}{\partial x^{j}} + L_{ij}^{r} L_{rk}^{l} - L_{ik}^{r} L_{rj}^{l} + C_{i(r)}^{l(1)} R_{(1)jk}^{(r)},$$
$$P_{ij(k)}^{l\ (1)} = -C_{i(k)|j}^{l(1)},$$
$$S_{i(j)(k)}^{l(1)(1)} \stackrel{def}{=} \frac{\partial C_{i(j)}^{l(1)}}{\partial y_{1}^{k}} - \frac{\partial C_{i(k)}^{l(1)}}{\partial y_{1}^{j}} + C_{i(j)}^{r(1)} C_{r(k)}^{l(1)} - C_{i(k)}^{r(1)} C_{r(j)}^{l(1)}.$$

Proof. Generally, an *h-normal* Γ*-linear connection* on the 1-jet space $J^1(\mathbb{R}, M^n)$ has *five* effective local d-tensors of curvature (for more details, see [4]). For the Cartan canonical connection (9) these reduce only to *three* (the other two are zero); these are

$$R_{ijk}^{l} \stackrel{def}{=} \frac{\delta L_{ij}^{l}}{\delta x^{k}} - \frac{\delta L_{ik}^{l}}{\delta x^{j}} + L_{ij}^{r} L_{rk}^{l} - L_{ik}^{r} L_{rj}^{l} + C_{i(r)}^{l(1)} R_{(1)jk}^{(r)},$$
$$P_{ij(k)}^{l\ (1)} \stackrel{def}{=} \frac{\partial L_{ij}^{l}}{\partial y_{1}^{k}} - C_{i(k)|j}^{l(1)} + C_{i(r)}^{l(1)} P_{(1)j(k)}^{(r)\ (1)} = -C_{i(k)|j}^{l(1)},$$
$$S_{i(j)(k)}^{l(1)(1)} \stackrel{def}{=} \frac{\partial C_{i(j)}^{l(1)}}{\partial y_{1}^{k}} - \frac{\partial C_{i(k)}^{l(1)}}{\partial y_{1}^{j}} + C_{i(j)}^{r(1)} C_{r(k)}^{l(1)} - C_{i(k)}^{r(1)} C_{r(j)}^{l(1)},$$

where we used the equality

$$P_{(1)j(k)}^{(r)\ (1)} \stackrel{def}{=} \frac{\partial N_{(1)j}^{(r)}}{\partial y_{1}^{k}} - L_{jk}^{r} = 0.$$

4. Gravitational-like geometrical model associated to the (t,x)-conformal deformation of the Berwald-Moór metric

The (t,x)-conformal deformed Berwald-Moór metric (1) produces on the 1-jet space $J^1(\mathbb{R}, M^n)$ the adapted metrical d-tensor (sum by i and j)

$$\mathbf{G} = h_{11} dt \otimes dt + \overset{*}{g}_{ij} dx^{i} \otimes dx^{j} + h^{11} \overset{*}{g}_{ij} \delta y_{1}^{i} \otimes \delta y_{1}^{j}, \tag{11}$$

where $\overset{*}{g}_{ij}$ is given by (3), and we have

$$\delta y_1^i = dy_1^i - \text{K}_{11}^1 y_1^i dt + n\sigma_i y_1^i dx^i \text{ (no sum by } i).$$

From an abstract physical point of view, the metrical d-tensor (11) may be regarded as a *"non-isotropic gravitational potential"*. In our geometric-physical approach, one postulates that the non-isotropic gravitational potential $\mathbf{G}$ is governed by the following *geometrical Einstein-like equations:*

$$\text{Ric }(C\overset{*}{\Gamma}) - \frac{\text{Sc }(C\overset{*}{\Gamma})}{2}\mathbf{G}=\mathcal{K}\mathcal{T}, \tag{12}$$

where

- Ric $(C\overset{*}{\Gamma})$ is the *Ricci d-tensor* associated to the Cartan canonical linear connection (9);
- Sc $(C\overset{*}{\Gamma})$ is the *scalar curvature*;
- $\mathcal{K}$ is the *Einstein constant* and $\mathcal{T}$ is the intrinsic *non-isotropic stress-energy d-tensor of matter.*

Therefore, using the adapted basis of vector fields (7), we can locally describe the global geometrical Einstein-like equations (12). Consequently, some direct computations lead to:

Lemma 6. *The Ricci tensor of the Cartan canonical connection $C\overset{*}{\Gamma}$ of the (t,x)-conformal deformed Berwald-Moór metric (1) has the following two effective local Ricci d-tensors (no sum by i, j, k or l):*

$$R_{ij} = \begin{cases} -\sigma_{ij} - \sum\limits_{\substack{m=1\\ m\neq j}}^{n} \sigma_{jm}\dfrac{y_1^m}{y_1^i}, & i \neq j \\ \\ 0, & i = j, \end{cases} \tag{13}$$

$$S_{(i)(j)}^{(1)(1)} = \left[\frac{2}{n^2} - \frac{1}{n} + \left(1 - \frac{2}{n}\right)\delta_{ij}\right] \cdot \frac{1}{y_1^i y_1^j}.$$

Proof. Generally, the Ricci tensor of the Cartan canonical connection $C\Gamma$ produced by an arbitrary jet Lagrangian function is determined by *six* effective local Ricci d-tensors (for more details, see [4]). For our particular Cartan canonical connection (9) these reduce only to the following *two* (the other four are zero):

$$R_{ij} \stackrel{def}{=} R^m_{ijm} = \frac{\partial L^m_{ij}}{\partial x^m} - \frac{\partial L^m_{im}}{\partial x^j} + L^r_{ij}L^m_{rm} - L^r_{im}L^m_{rj} + C^{m(1)}_{i(r)}R^{(r)}_{(1)jm},$$

$$S^{(1)(1)}_{(i)(j)} \stackrel{def}{=} S^{m(1)(1)}_{i(j)(m)} = \frac{\partial C^{m(1)}_{i(j)}}{\partial y^m_1} - \frac{\partial C^{m(1)}_{i(m)}}{\partial y^j_1} + C^{r(1)}_{i(j)}C^{m(1)}_{r(m)} - C^{r(1)}_{i(m)}C^{m(1)}_{r(j)} =$$

$$= \frac{\partial C^{m(1)}_{i(j)}}{\partial y^m_1} - C^{r(1)}_{i(m)}C^{m(1)}_{r(j)},$$

with sum by r and m.

Lemma 7. *The scalar curvature of the Cartan canonical connection* $C\overset{*}{\Gamma}$ *of the* (t, x)*-conformal deformed Berwald-Moór metric (1) has the value*

$$Sc\ (C\overset{*}{\Gamma}) = -e^{-2\sigma}G^{-2/n}_{1[n]}\left[4nY_{11} + \left(n^2 - 3n + 2\right)h_{11}\right],$$

where

$$Y_{11} = \sum_{\substack{p,q=1\\ p<q}}^{n} \sigma_{pq}y^p_1 y^q_1.$$

Proof. The scalar curvature of the Cartan canonical connection (9) is given by the formula (for more details, see [4]): Sc $(C\overset{*}{\Gamma}) = \overset{*}{g}{}^{pq}R_{pq} + h_{11}\overset{*}{g}{}^{pq}S^{(1)(1)}_{(p)(q)}$.

The local description in the adapted basis of vector fields (7) of the global geometrical Einstein-like equations (12) is given by (for more details, see [4]):

Proposition 8. *The geometrical Einstein-like equations of the* (t, x)*-conformal deformed Berwald-Moór metric (1) are locally described by*

$$\begin{cases} e^{-2\sigma}G^{-2/n}_{1[n]}\left[2nY_{11} + \dfrac{n^2-3n+2}{2}h_{11}\right]h_{11} = \mathcal{K}\mathcal{T}_{11} \\ R_{ij} + e^{-2\sigma}G^{-2/n}_{1[n]}\left[2nY_{11} + \dfrac{n^2-3n+2}{2}h_{11}\right]\overset{*}{g}_{ij} = \mathcal{K}\mathcal{T}_{ij} \\ S^{(1)(1)}_{(i)(j)} + e^{-2\sigma}G^{-2/n}_{1[n]}\left[2nY_{11} + \dfrac{n^2-3n+2}{2}h_{11}\right]h^{11}\overset{*}{g}_{ij} = \mathcal{K}\mathcal{T}^{(1)(1)}_{(i)(j)} \\ 0 = \mathcal{T}_{1i}, \quad 0 = \mathcal{T}_{i1}, \quad 0 = \mathcal{T}^{(1)}_{(i)1} \\ 0 = \mathcal{T}^{(1)}_{1(i)}, \quad 0 = \mathcal{T}^{(1)}_{i(j)}, \quad 0 = \mathcal{T}^{(1)}_{(i)j}. \end{cases} \tag{14}$$

Corollary 9. *The non-isotropic stress-energy d-tensor of matter T satisfies the following geometrical conservation laws (sum by m):*

$$\begin{cases} \mathcal{T}^1_{1/1} + \mathcal{T}^m_{1|m} + \mathcal{T}^{(m)}_{(1)1}|^{(1)}_{(m)} = 0 \\ \mathcal{T}^1_{i/1} + \mathcal{T}^m_{i|m} + \mathcal{T}^{(m)}_{(1)i}|^{(1)}_{(m)} = E^m_{i|m} \\ \mathcal{T}^{1(1)}_{\ \ (i)/1} + \mathcal{T}^{m(1)}_{\ \ (i)|m} + \mathcal{T}^{(m)(1)}_{(1)(i)}|^{(1)}_{(m)} = \dfrac{2e^{-2\sigma}G^{-2/n}_{1[n]}}{\mathcal{K}} \cdot \left[n\dfrac{\partial Y_{11}}{\partial y^i_1} - 2\dfrac{Y_{11}}{y^i_1}\right], \end{cases}$$

where (sum by r):

$$\mathcal{T}^1_1 \stackrel{def}{=} h^{11}\mathcal{T}_{11} = \mathcal{K}^{-1}e^{-2\sigma}G^{-2/n}_{1[n]}\left[2nY_{11} + \frac{n^2-3n+2}{2}h_{11}\right],$$

$$\mathcal{T}^m_1 \stackrel{def}{=} \overset{*}{g}{}^{mr}\mathcal{T}_{r1} = 0, \quad \mathcal{T}^{(m)}_{(1)1} \stackrel{def}{=} h_{11}\overset{*}{g}{}^{mr}\mathcal{T}^{(1)}_{(r)1} = 0, \quad \mathcal{T}^1_i \stackrel{def}{=} h^{11}\mathcal{T}_{1i} = 0,$$

$$\mathcal{T}^m_i \stackrel{def}{=} \overset{*}{g}{}^{mr}\mathcal{T}_{ri} := E^m_i = \mathcal{K}^{-1}\left[\overset{*}{g}{}^{mr}R_{ri} + \right.$$

$$\left. + e^{-2\sigma}G^{-2/n}_{1[n]}\left(2nY_{11} + \frac{n^2-3n+2}{2}h_{11}\right)\delta^m_i\right],$$

$$\mathcal{T}^{(m)}_{(1)i} \stackrel{def}{=} h_{11}\overset{*}{g}{}^{mr}\mathcal{T}^{(1)}_{(r)i} = 0, \quad \mathcal{T}^{1(1)}_{\ \ (i)} \stackrel{def}{=} h^{11}\mathcal{T}^{\ \ (1)}_{1(i)} = 0, \quad \mathcal{T}^{m(1)}_{\ \ (i)} \stackrel{def}{=} \overset{*}{g}{}^{mr}\mathcal{T}^{\ \ (1)}_{r(i)} = 0,$$

$$\mathcal{T}^{(m)(1)}_{(1)(i)} \stackrel{def}{=} h_{11}\overset{*}{g}{}^{mr}\mathcal{T}^{(1)(1)}_{(r)(i)} = \frac{e^{-2\sigma}G^{-2/n}_{1[n]}}{\mathcal{K}} \cdot \left[\frac{n-2}{n}h_{11}\frac{y^m_1}{y^i_1} + \right.$$

$$\left. + \left(2nY_{11} + \frac{n^2-5n+6}{2}h_{11}\right)\delta^m_i\right],$$

and we also have (summation by m and r, but no sum by i)

$$\mathcal{T}^1_{1/1} = \frac{\delta\mathcal{T}^1_1}{\delta t}, \quad \mathcal{T}^m_{1|m} \stackrel{def}{=} \frac{\delta\mathcal{T}^m_1}{\delta x^m} + \mathcal{T}^r_1 L^m_{rm},$$

$$\mathcal{T}^{(m)}_{(1)1}|^{(1)}_{(m)} \stackrel{def}{=} \frac{\partial\mathcal{T}^{(m)}_{(1)1}}{\partial y^m_1} + \mathcal{T}^{(r)}_{(1)1}C^{m(1)}_{r(m)} = \frac{\partial\mathcal{T}^{(m)}_{(1)1}}{\partial y^m_1},$$

$$\mathcal{T}^1_{i/1} \stackrel{def}{=} \frac{\delta\mathcal{T}^1_i}{\delta t} + \mathcal{T}^1_i\mathrm{K}^1_{11} - \mathcal{T}^1_r G^r_{i1} = \frac{\delta\mathcal{T}^1_i}{\delta t} + \mathcal{T}^1_i\mathrm{K}^1_{11},$$

$$\mathcal{T}^m_{i|m} \overset{def}{=} \frac{\delta \mathcal{T}^m_i}{\delta x^m} + \mathcal{T}^r_i L^m_{rm} - \mathcal{T}^m_r L^r_{im} = E^m_{i|m} := \frac{\delta E^m_i}{\delta x^m} + nE^m_i \sigma_m - nE^i_i \sigma_i,$$

$$\mathcal{T}^{(m)}_{(1)i}|^{(1)}_{(m)} \overset{def}{=} \frac{\partial \mathcal{T}^{(m)}_{(1)i}}{\partial y^m_1} + \mathcal{T}^{(r)}_{(1)i} C^{m(1)}_{r(m)} - \mathcal{T}^{(m)}_{(1)r} C^{r(1)}_{i(m)} = \frac{\partial \mathcal{T}^{(m)}_{(1)i}}{\partial y^m_1} - \mathcal{T}^{(m)}_{(1)r} C^{r(1)}_{i(m)},$$

$$\mathcal{T}^{1(1)}_{(i)/1} \overset{def}{=} \frac{\delta \mathcal{T}^{1(1)}_{(i)}}{\delta t} + 2\mathcal{T}^{1(1)}_{(i)} \mathrm{K}^1_{11}, \quad \mathcal{T}^{m(1)}_{(i)|m} \overset{def}{=} \frac{\delta \mathcal{T}^{m(1)}_{(i)}}{\delta x^m} + \mathcal{T}^{r(1)}_{(i)} L^m_{rm} - \mathcal{T}^{m(1)}_{(r)} L^r_{im},$$

$$\mathcal{T}^{(m)(1)}_{(1)(i)}|^{(1)}_{(m)} \overset{def}{=} \frac{\partial \mathcal{T}^{(m)(1)}_{(1)(i)}}{\partial y^m_1} + \mathcal{T}^{(r)(1)}_{(1)(i)} C^{m(1)}_{r(m)} - \mathcal{T}^{(m)(1)}_{(1)(r)} C^{r(1)}_{i(m)} = \frac{\partial \mathcal{T}^{(m)(1)}_{(1)(i)}}{\partial y^m_1}.$$

Proof. The local Einstein-like equations (14), together with some direct computations, lead us to what we were looking for. Also note that we have (summation by m and r)

$$\mathcal{T}^{(m)(1)}_{(1)(r)} C^{r(1)}_{i(m)} = 0.$$

5. Electromagnetic-like geometrical model associated to the (t,x)-conformal deformation of the Berwald-Moór metric

In book [4], a geometrical theory for electromagnetism was also created, using only a given Lagrangian function L on the 1-jet space $J^1(\mathbb{R}, M^n)$. In the background of the jet single-time (one-parameter) Lagrange geometry from [4], one works with the *non-isotropic electromagnetic distinguished* 2*-form* (sum by i and j)

$$\mathbf{F} = F^{(1)}_{(i)j} \delta y^i_1 \wedge dx^j,$$

where (sum by m and r)

$$F^{(1)}_{(i)j} = \frac{h^{11}}{2} \left[\overset{*}{g}_{jm} N^{(m)}_{(1)i} - \overset{*}{g}_{im} N^{(m)}_{(1)j} + \left(\overset{*}{g}_{ir} L^r_{jm} - \overset{*}{g}_{jr} L^r_{im} \right) y^m_1 \right].$$

This is characterized by some natural *geometrical Maxwell-like equations* (for more details, see [9] and [4]).

Remark 10. The Lagrangian function that governs the movement law of a particle of mass $m \neq 0$ and electric charge e, which is displaced concomitantly into an environment endowed both with a gravitational field and an electromagnetic one, is given by

$$L(t, x^k, y^k_1) = mch^{11}(t)\, \varphi_{ij}(x^k)\, y^i_1 y^j_1 + \frac{2e}{m} A^{(1)}_{(i)}(t, x^k)\, y^i_1, \tag{15}$$

where

- the semi-Riemannian metric $\varphi_{ij}(x)$ represents the *isotropic gravitational potential*;
- $A_{(i)}^{(1)}(t,x)$ are the components of a d-tensor on the 1-jet space $J^1(\mathbb{R}, M^n)$ representing the *electromagnetic potential.*

Note that the jet Lagrangian function (15) is a natural extension of the Lagrangian (defined on the tangent bundle) used in electrodynamics by Miron and Anastasiei [9]. In our jet geometrical formalism applied to Lagrangian (15), the *electromagnetic-like components* become classical ones (see [4]):

$$F_{(i)j}^{(1)} = -\frac{e}{2m}\left(\frac{\partial A_{(i)}^{(1)}}{\partial x^j} - \frac{\partial A_{(j)}^{(1)}}{\partial x^i}\right).$$

Moreover, the second set of *geometrical Maxwell-like equations* reduce to the classical ones too (for more details, see [9], [4]):

$$\sum_{\{i,j,k\}} F_{(i)j|k}^{(1)} = 0,$$

where

$$F_{(i)j|k}^{(1)} = \frac{\partial F_{(i)j}^{(1)}}{\partial x^k} - F_{(m)j}^{(1)}\gamma_{ik}^m - F_{(i)m}^{(1)}\gamma_{jk}^m.$$

Also, the *geometrical Einstein-like equations* attached to the Lagrangian (15) (see [9], [4]) are the same with the famous classical ones (associated to the semi-Riemannian metric $\varphi_{ij}(x)$). In author's opinion, these facts suggest some kind of naturalness for the present abstract Lagrangian non-isotropic electromagnetic and gravitational geometrical theories.

Via some direct calculations, we easily deduce that the (t,x)-conformal deformed Berwald-Moór metric (1) produces null non-isotropic electromagnetic components:

$$F_{(i)j}^{(1)} = 0.$$

It follows that our conformal deformed Berwald-Moór geometrical electromagnetic-like theory is trivial. This fact probably suggests that the (t,x)-conformal deformed Berwald-Moór geometrical structure (1) has rather gravitational connotations than electromagnetic ones.

As a conclusion, it is possible for the recent Voicu-Siparov approach of the electromagnetism in spaces with anisotropic metrics (that electromagnetic approach is different from the electromagnetic theory exposed above, and it is developed in the paper [18]) to give other interesting electromagnetic-geometrical results for spaces endowed with the Berwald-Moór geometrical structure.

Open problem. The author of this paper believes the finding of some possible real physical interpretations for the present non-isotropic Berwald-Moór geometrical approach of gravity and electromagnetism may be an open problem for physicists.

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