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GAPS OF A CLASS OF PSEUDO SYMMETRIC NUMERICAL SEMIGROUPS

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Abstract. In this study, we give some results about the gaps, fundamental and special gaps of a pseudo symmetric numerical semigroup in the form of $S=<3,3+s,3+2s>$ for $s\in\mathbb{Z}^{+}$ and $3\nmid s$.

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1. Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and nonnegative integers. respectively. A numerical semigroup is a subset S of $\mathbb{N}$ that is closed under addition where $0\in S$ and $\mathbb{N}\backslash S$ is finite. It is well known that every numerical semigroup is finitely generated [1], that is to say, there exist $s_1,s_2,...,s_p\in\mathbb{N}$ such that $s_1<s_2<...<s_p$ and

$$S=<s_1,s_2,...,s_p>=\{s_1k_1+s_2k_2+...+s_pk_p : k_i\in\mathbb{N}, 1\le i\le p\}.$$

Moreover, every numerical semigroup has a unique minimal system of generators.

Following the notation used in [2,3], if S is a numerical semigroup then the greatest integer in $\mathbb{Z}\backslash S$ is the *Frobenius number* of S , denoted by $g(S)$. The elements of $\mathbb{N}\backslash S$, denoted by $H(S)$ are called *gaps* of S. If $x\in H(S)$ and $\{2x,3x\}\subset S$ then x is called the *fundamental gap*. We denote by $FH(S)$ the set of fundamental gaps of S.

S is *symmetric* if for every $x\in\mathbb{Z}\backslash S$, the integer $g(S)-x\in S$. Similarly, S is *pseudo symmetric* if $g(S)$ is even and there exists an integer $x\in\mathbb{Z}\backslash S$ such that $x=\frac{g(S)}{2}$ and $g(S)-x\notin S$. For more background on symmetric and pseudo symmetric numerical semigroups, the reader is encouraged to see [2,3,4,7,9].

Let S be a numerical semigroup and $m\in S\backslash\{0\}$. The Apery set of S with respect to m is defined by $Ap(S,\ m)=\{s\in S:\ s-m\notin S\}$. Hence, $Ap(S,\ m)=\{w(0)=0,w(1),w(2),...,w(m-1)\}$ and $g(S)=\max(Ap(S,m))-m$, where $w(i)$ is the least element in S that is congruent with i modulo m. For instance see [6] and [10].

The following can be found in [7]: Let S be a numerical semigroup. We say that $x \in \mathbb{Z} \backslash S$ is a *pseudo Frobenius number* of S if $x + s \in S$ for all $s \in S \backslash \{0\}$. We denote by $Pg(S)$ the set of pseudo Frobenius numbers of S. The cardinal of $Pg(S)$ is called the type of S and denoted by $type(S)$. Notice that $g(S)$ is always an element of $Pg(S)$. In [11], it is proved that a numerical semigroup is symmetric if and only if $Pg(S) = \{g(S)\}$ i.e. $type(S) = 1$. Furthermore, we define in S the following partial order:

$$a \leq_S b \text{ if } b - a \in S.$$

For $m \in S \backslash \{0\}$, it is proved that$Pg(S) = \{w(i) - m : \ w(i) maximals \leq_S Ap(S, m)\}$ in [7].

An element $x \in Pg(S)$ is a *special gap* of S if $2x \in S$. We denote by $SH(S)$ the set of special gaps of S. That is, $SH(S) = \{x \in Pg(S) : \ 2x \in S\}$. The following proposition is proved in [8]:

$$x \in Pg(S) \text{ if and only if } S \cup \{x\} \text{ is a numerical semigroup.}$$

The main goal of this paper is to prove Theorem 2 and Theorem 3 which gives the sets $H(S)$ and $FH(S)$ with respect to s. We also find the cardinality $\sharp(FH(S))$ and give the relations between $\sharp(H(S))$ and $\sharp(FH(S))$ in Corollary 4 and Corollary 5.

In this paper, S is defined as $S =< 3, \ 3 + s, \ 3 + 2s >$ for $s \in \mathbb{Z}^+$ and $3 \nmid s$.

2. Results

In this section, we will give some results related to the gaps, fundamental and special gaps of a pseudo symmetric numerical semigroup in the form $S =< 3, \ 3 + s, \ 3 + 2s >$ for $s \in \mathbb{Z}^+$ and $3 \nmid s$.

Firstly we give following theorem:

Theorem 1. *$S =< 3, 3+s, 3+2s >$ is a pseudo symmetric numerical semigroup, for $s \in \mathbb{Z}^+$ and $3 \nmid s$. [see 5,9].*

Notation: We can write the following cases for S:

(i) If $s = 6k + 1$ or $s = 6k + 4$ then

$$S =< 3, \ 3 + s, \ 3 + 2s >= \{0, 3, ..., s - 1, s + 2, s + 3, s + 5, ..., 2s + 1, \rightarrow ...\}$$

(ii) If $s = 6k + 2$ or $s = 6k + 5$ then

$$S =< 3, \ 3 + s, \ 3 + 2s >= \{0, 3, ..., s - 2, s + 1, s + 3, s + 4, ..., 2s + 1, \rightarrow ...\}$$

where $k \in \mathbb{N}$.

Teorem 2. *The set of gaps of S is as follows:*

(i) if $s = 6k+1$ *or* $s = 6k+4$, *then*

$$H(S) = \{1, 2, 4, 5, ..., s, s+1, s+4, ..., 2s\}.$$

(ii) if $s = 6k+2$ *or* $s = 6k+5$, *then*

$$H(S) = \{1, 2, 4, 5, ..., s, s+2, s+5, s+8, ..., 2s\}$$

where $k \in \mathbb{N}$.

Proof. By definition, every non-positive integer k with $k \leq s$, $3 \nmid k$ is in $H(S)$. That is, $\{1, 2, 4, 5, ..., s\} \subseteq H(S)$. In addition, for the different states of s:

(i) If $s = 6k+1$ ($k \in \mathbb{N}$) then $3 \nmid (s+1)$, so $s+1 \in H(S)$. However, $s+2,\ s+3 \in S$. In this case, $s+1+3t \leq 2s$ ($t \in \mathbb{N}$). Otherwise, let $s+1+3t \notin H(S)$ for $s+1+3t \leq 2s$, then $s+1+3t \in S$. Thus, $3 \mid (s+1)$ since $3 \mid (s+1+3t)$ that is $3 \mid (6k+2)$. This is a contradiction. Therefore, $H(S) = \{1, 2, 4, 5, ..., s, s+1, s+4, ..., 2s\}$.

If $s = 6k+4$, then $3 \nmid s$, but $s+2,\ s+3 \in S$. That is $s+1 \in H(S)$. On the contrary, let $s+1 \notin H(S)$. Then $3 \mid s+1$ and $3 \mid 6k+5$ which is a contradiction. Thus, $s+1+3t \in H(S)$ is obtained for $s+1+3t \leq 2s$. Consequently, $H(S) = \{1, 2, 4, 5, ..., s, s+1, s+4, ..., 2s\}$.

(ii) If $s = 6k+2$, then $3 \mid s+1$ and $s+1, s+3, s+4 \in S$; but $s+2 \notin S$. In order words, $s+2 \in H(S)$. We assume that $s+2 \notin H(S)$. Then, $3 \mid s+2$, that is $3 \mid 6k+4$. Hence, $3 \mid 4$ which gives a contradiction. Thus, we have that $H(S) = \{1, 2, 4, 5, ..., s, s+2, s+5, s+8, ..., 2s\}$.

If $s = 6k+5$ then $s+1, s+3 \in S$. But $s+2 \notin S$, i.e. $s+2 \in H(S)$. Conversely, $s+2 \notin H(S)$. Then $3 \mid s+2$ and $3 \mid 6k+7$ which is a contradiction. Hence, $s+2+3t \in H(S)$ for $s+1+3t \leq 2s$. Thus, $H(S) = \{1, 2, 4, 5, ..., s, s+2, s+5, s+8, ..., 2s\}$ is obtained.

Theorem 3. *The set of fundamental gaps of S is given as follows:*

(a) *if* $s = 6k+1$ *or* $s = 6k+5$, *then* $FH(S) = \left\{\frac{3+s}{2}, \frac{3+s}{2}+3, ..., 2s\right\}$

(b) *if* $s = 6k+2$ *or* $s = 6k+4$, *then* $FH(S) = \left\{\frac{6+s}{2}, \frac{6+s}{2}+3, ..., 2s\right\}$

where $k \in \mathbb{N}$.

Proof. (a) We must firstly show that $T=\left\{\frac{3+s}{2},\frac{3+s}{2}+3,...,2s\right\}\neq\varnothing$ and $T\subseteq H(S)$: Thus it suffices to prove $\frac{3+s}{2}\notin S$ (since $n=\frac{3+s}{2}\in H(S)$ for $\frac{3+s}{2}\notin S$ and $n+3t\leq 2s$ $(t\in\mathbb{N})$, $n+3t\in H(S)$). Conversely, assume that $\frac{3+s}{2}\in S$. In this case, $\frac{3+s}{2}=3n_1+(3+s)n_2+(3+2s)n_3$ $(n_1,n_2,n_3\in\mathbb{N})$. Thus, we write $s=3(2n_1-1)+(3+s)2n_2+(3+2s)2n_3\in S$. But this yields $s\in S$ which contradicts with the definition of S. Now let us show that $T=FH(S)$:

$$\begin{aligned}
x &\in T\Longrightarrow x=\frac{3+s}{2}+3t,\ (t\in\mathbb{N})\\
&\Longrightarrow 2x=2(\frac{3+s}{2}+3t)\ \textit{and}\ 3x=3(\frac{3+s}{2}+3t)\\
&\Longrightarrow 2x=3+s+6t\ \textit{and}\ [3x=3(\frac{3+6k+1}{2}+3t)\ \textit{or}\ 3x=3(\frac{3+6k+5}{2}+3t)]\\
&\Longrightarrow 2x\in S\ \textit{and}\ [3x=6+9k+9t\ \textit{or}\ 3x=12+9k+9t]\\
&\Longrightarrow 2x\in S\ \textit{and}\ 3x\in S\\
&\Longrightarrow x\in FH(S).
\end{aligned}$$

For the other implication, let us show that $FH(S)\subseteq T$. Conversely, assume that $FH(S)\nsubseteq T$. Then, $\exists y\in FH(S)\ \ni y\notin T$, i.e., $y\notin H(S)$, which gives $y\in S$. This is a contradiction. As a result $FH(S)=T$.

(b) $A=\left\{\frac{6+s}{2},\frac{6+s}{2}+3,...,2s\right\}$ is a subset of $H(S)$: For this, it suffices to prove $\frac{6+s}{2}\notin S$ (since $v=\frac{6+s}{2}\in H(S)$ for $\frac{6+s}{2}\notin S$, and $v+3t\leq 2s$ $(t\in\mathbb{N})$, $v+3t\in H(S)$). Conversely, assume that $\frac{6+s}{2}\in S$. In this case, $\frac{6+s}{2}=3u_1+(3+s)u_2+(3+2s)u_3$ $(u_1,u_2,u_3\in\mathbb{N})$. Thus, we write $s=3(2u_1-2)+(3+s)2u_2+(3+2s)2u_3\in S$. This contradicts with the definition of S. Furthermore, $T=FH(S)$:

$$\begin{aligned}
x &\in T\Longrightarrow x=\frac{6+s}{2}+3t,\ (t\in\mathbb{N})\\
&\Longrightarrow 2x=2(\frac{6+s}{2}+3t)\ \textit{and}\ 3x=3(\frac{6+s}{2}+3t)\\
&\Longrightarrow 2x=6+s+6t\ \textit{and}\ [3x=3(\frac{6+6k+2}{2}+3t)\ \textit{or}\ 3x=3(\frac{6+6k+4}{2}+3t)]\\
&\Longrightarrow 2x\in S\ \textit{and}\ [3x=12+9k+9t)\ \textit{or}\ 3x=15+9k+9t)]\\
&\Longrightarrow 2x\in S\ \textit{and}\ 3x\in S\\
&\Longrightarrow x\in FH(S).
\end{aligned}$$

On the other hand, $FH(S)\subseteq T$ can be shown as in (a).

Corollary 4.

(i)If s is odd, then $\sharp(FH(S))=\frac{s+1}{2}$.
(ii)If s is even, then $\sharp(FH(S))=\frac{s}{2}$.

Proof. By Theorem 3, we have that $FH(S) = \left\{\frac{3+s}{2}, \frac{3+s}{2}+3, ..., 2s\right\}$ and $FH(S) = \left\{\frac{6+s}{2}, \frac{6+s}{2}+3, ..., 2s\right\}$ are obtained where s is odd and even, respectively. Thus, if s is odd, then $\sharp(FH(S)) = \frac{2s-\frac{3+s}{2}}{3}+1 = \frac{3s-3}{6}+1 = \frac{s+1}{2}$. If s is even, then $\sharp(FH(S)) = \frac{2s-\frac{6+s}{2}}{3}+1 = \frac{3s-6}{6}+1 = \frac{s}{2}$.

Corollary 5. *The following corollary a result of Corollary 4*
(i) If s is odd, then $\sharp(H(S)) = 2\sharp(FH(S))$.
(ii) If s is even, then $\sharp(H(S)) = 2\sharp(FH(S))+1$.

Proposition 6. The set of special gaps of S is $\{2s\}$, that is, $SH(S) = \{2s\}$.

Proof. We can write that $Ap(S,3) = \{0, 3+s, 2s+3\}$ and

$$Maximals \leq_S (Ap(S,3)) = \left\{\frac{2s}{2}+3, 2s+3\right\}$$

from [5] and [9], respectively. Thus, we write that

$$SH(S) = \{x \in Pg(S) : 2x \in S\}$$

since $Pg(S) = \{s, 2s\}$.

Corollary 7. $SH(S) \subset FH(S) \subset H(S)$.

Example 8. Let $S =< 3, 7, 11 >= \{0, 3, 6, 7, 9, 10, 11, \rightarrow ...\}$ be a pseudo symmetric numerical semigroup for $s = 4$. Since $s = 4 = 6.0 + 4$; $g(S) = 8$, $Ap(S,3) = \{0, 3+4, 2.4+3\} = \{0, 7, 11\}$, and $H(S) = \{1, 2, 4, 5, 8\}$, $FH(S) = \left\{\frac{6+4}{2}, \frac{6+4}{2}+3\right\} = \{5, 8\}$, $SH(S) = \{8\}$.

Thus, $\sharp(H(S)) = 4+1 = 5 = 2\sharp(FH(S))+1$ and $\{8\} \subset \{5,8\} \subset \{1,2,4,5,8\}$.

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