Acta Universitatis Apulensis No. 34/2013
ISSN: 1582-5329 pp. 355-369

A SUBCLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH THE HURWITZ - LERCH ZETA FUNCTION

Shahram Najafzadeh and Elnaz Pezeshki

Abstract. Making use of a convolution operator involving the Hurwitz-Lerch Zeta function,we introduce a new class of analytic functions $PT(\lambda,\alpha,\beta)$ defined in the open unit disc,and investigate its various characteristics.Further we obtained distortion bounds, extreme points and radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $PT(\lambda,\alpha,\beta)$.

2000 *Mathematics Subject Classification*: 30C45, 30C50.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic and univalent in the open disc $\mathbb{U} = \{z : z \in \mathbb{C}; |z| < 1\}$. For functions $f \in A$ given by (1) and $g \in A$ given by $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \qquad z \in \mathbb{U} \tag{2}$$

We now recall a general Hurwitz- Lerch Zeta function $\Phi(z,s,a)$ (cf.,e.g., [18]) defined by

$$\Phi(z,s,a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s} \qquad (a \in \mathbb{C}\backslash\{\mathbb{Z}_0^-\}; s \in \mathbb{C}, \Re(s) > 1 \ \ and \ \ |z| = 1) \tag{3}$$

where, as usual,

$$\mathbb{Z}_0^- := \mathbb{Z}\backslash\{\mathbb{N}\}, \quad (\mathbb{Z} := \{\pm 1, \pm 2, \pm 3, ...\}); \mathbb{N} := \{1, 2, 3, ...\} \ .$$

Several interesting properties and characteristics of the Hurwitz - Lerch Zeta function $\Phi(z,s,a)$ can be found in the recent investigations by Choi and Srivastava [4], Ferreira and Lopez [5], Garg et al. [7], Lin and Srivastava [11], Lin et al. [12], and others. In 2007, Srivastava and Attiya [17] (see also Riaducanu and Srivastava [14], Prajapat and Goyal [13]) introduced and investigated the linear operator:

$$\jmath_{\mu,b} : A \to A$$

defined, in terms of the Hadamard product (or convolution), by

$$\jmath_{\mu,b}f(z) = g_{\mu,b} * f(z), \tag{4}$$

$(z \in \mathbb{U}; b \in \mathbb{C}\backslash\{\mathbb{Z}_0^-\}; \mu \in \mathbb{C}; f \in A)$, where, for convenience,

$$g_{\mu,b}(z) := (1+b)^{\mu}[\Phi(z,\mu,b) - b^{-\mu}] \quad (z \in \mathbb{U}) \; . \tag{5}$$

We recall here the following relationships (given earlier in [13, 14]) which follow easily by using (1), (4) and (5)

$$\jmath_{\mu,b}f(z) = z + \sum_{k=2}^{\infty} C_k(b,\mu) a_k z^k, \tag{6}$$

where

$$C_k(b,\mu) = (\frac{1+b}{k+b})^{\mu}, \tag{7}$$

and (throughout this paper unless otherwise mentioned) the parameters μ and b are constrained as $b \in \mathbb{C}\backslash\{\mathbb{Z}_0^-\}$ and $\mu \in \mathbb{C}$.
(1) For $\mu = 0$,

$$\jmath_{0,b}f(z) := f(z). \tag{8}$$

(2) For $\mu = 1, b = 0$,

$$\jmath_{1,0}f(z) := \int_0^z \frac{f(t)}{t}dt := L_b f(z). \tag{9}$$

(3) For $\mu = 1$ and $b = \nu \quad (\nu > -1)$,

$$\jmath_{1,\nu}f(z) := \frac{1+\nu}{z^{\nu}} \int_0^z t^{\nu-1} f(t)dt = z + \sum_{k=2}^{\infty} (\frac{1+\nu}{k+\nu}) a_k z^k := F_{\nu} f(z). \tag{10}$$

(4) For $\mu = \sigma(\sigma > 0)$ and $b = 1$

$$\jmath_{\sigma,1}f(z) := z + \sum_{k=2}^{\infty} (\frac{2}{k+1})^{\sigma} a_k z^k := J^{\sigma} f(z), \tag{11}$$

where $L_b(f)$ and F_ν are the integral operators introduced by Alexander [1] and Bernardi [3], respectively, and $\jmath^\sigma(f)$ is the Jung-Kim-Srivastava integral operator [9] closely related to some multiplier transformations studied by Flett [6]. Making use of the operator$\jmath_{\mu,b}$ we introduce a new subclass of analytic functions with negative coefficients, and discuss some standard properties of geometric function theory in relation to this generalized class. For$\lambda \geq 0, 0 \leq \alpha < 1$ and $0 < \beta \leq 1$, we let $P(\lambda, \alpha, \beta)$ be the subclass of A consisting of functions of the form (1) and satisfying the inequality

$$\left| \frac{\jmath_{\mu_{b,\lambda}} f(z) - 1}{2\gamma(\jmath_\mu^{b,\lambda} f(z) - \alpha) - (\jmath_\mu^{b,\lambda} f(z) - 1)} \right| < \beta, \tag{12}$$

where

$$\jmath_\mu^{b,\lambda} f(z) = (1-\lambda)\frac{\jmath_{\mu,b} f(z)}{z} + \lambda(\jmath_{\mu,b} f(z))', \tag{13}$$

$0 < \gamma \leq 1$, and$\jmath_\mu^b f(z)$ is given by (6). We further let

$$PT(\lambda, \alpha, \beta) = P(\lambda, \alpha, \beta) \cap T,$$

where

$$T := \{f \in A : f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \quad (z \in \mathbb{U})\} \tag{14}$$

is a subclass of A introduced and studied by Silverman [16]. Furthermore, we note that by suitably specializing the values of α, β, γ and λ the class $PT(\lambda, \alpha, \beta)$ and the above subclasses reduce to the various subclasses introduced and studied in the literature, for example see [2,9].

In the following section we obtain coefficient estimates and extreme points for the class $PT(\lambda, \alpha, \beta)$.

2.Coefficient bounds

Theorem 1. Let the function f be defined by (14). Then $f \in PT(\lambda, \alpha, \beta)$ if and only if

$$\sum_{k=2}^{\infty}(1+\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|\,a_k \leq 2\beta\gamma(1-\alpha)\ . \tag{15}$$

The result is sharp for the function

$$f(z) = z - \frac{2\beta\gamma(1-\alpha)}{(1+\lambda(k-1))}[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|\,z^k, \quad k \geq 2\ , \tag{16}$$

where $C_k(b,\mu)$ is defined by (7).

Proof. Suppose f satisfies (15). Then for $|z|<1$ we have,

$$\begin{aligned}
&\left|\jmath_\mu^{b,\lambda} f(z)-1\right|-\beta\left|2\gamma(\jmath_\mu^{b,\lambda} f(z)-\alpha)-(\jmath_\mu^{b,\lambda} f(z)-1)\right|\\
&=\left|-\sum_{k=2}^{\infty}(1+\lambda(k-1))C_k(b,\mu)a_k z^{k-1}\right|\\
&-\beta\left|2\gamma(1-\alpha)-\sum_{k=2}^{\infty}(1+\lambda(k-1))(2\gamma-1)C_k(b,\mu)a_k z^{k-1}\right|\\
&\leq\sum_{k=2}^{\infty}(1+\lambda(k-1))\left|C_k(b,\mu)\right|a_k-2\beta\gamma(1-\alpha)\\
&+\sum_{k=2}^{\infty}(1+\lambda(k-1))\beta(2\gamma-1)\left|C_k(b,\mu)\right|a_k\\
&=\sum_{k=2}^{\infty}(1+\lambda(k-1))[1+\beta(2\gamma-1)]\left|C_k(b,\mu)\right|a_k-2\beta\gamma(1-\alpha)\\
&\leq 0,
\end{aligned}$$

by (15). Hence, by the maximum modulus Theorem and (12), $f\in PT(\lambda,\alpha,\beta)$. Conversely, assume that

$$\begin{aligned}
&\left|\frac{\jmath_\mu^{b,\lambda} f(z)-1}{2\gamma(\jmath_\mu^{b,\lambda} f(z)-\alpha)-(\jmath_\mu^{b,\lambda} f(z))-1)}\right|\\
&=\left|\frac{-\sum_{k=2}^{\infty}(1+\lambda(k-1))C_k(b,\mu)a_k z^{k-1}}{2\gamma(1-\alpha)-\sum_{k=2}^{\infty}(1+\lambda(k-1))(2\gamma-1)C_k(b,\mu)a_k z^{k-1}}\right|\\
&\leq\beta,\quad z\in\mathbb{U}.
\end{aligned}$$

Or, equivalently,

$$Re\{\frac{\sum_{k=2}^{\infty}(1+\lambda(k-1))\left|C_k(b,\mu)\right|a_k z^{k-1}}{2\gamma(1-\alpha)-\sum_{k=2}^{\infty}(1+\lambda(k-1))(2\gamma-1)C_k(b,\mu)a_k z^{k-1}}\}<\beta. \qquad (17)$$

Since $Re(z)\leq|z|$ for all z, choose values of z on the real axis so that $\jmath_\mu^{b,\lambda} f(z)$ is real. Upon clearing the denominator in (17) and letting $z\to 1$ through real values, we obtain the desired inequality (15). □

Corollary 1. If $f(z)$ of the form (14) is in $PT(\lambda,\alpha,\beta)$ then

$$a_k \le \frac{2\beta\gamma(1-\alpha)}{(1+\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|}, \qquad k\ge 2, \tag{18}$$

with equality only for functions of the form (16).

Theorem 2. Let

$$f_1(z)=z$$

and

$$f_k(z)=z-\frac{2\beta\gamma(1-\alpha)}{1+\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|}z^k, \qquad k\ge 2, \tag{19}$$

for $0\le\alpha<1, 0<\beta\le 1, \lambda\ge 0$ and $0<\gamma\le 1$. Then $f(z)$ is in the class $PT(\lambda,\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=2}^{\infty}\omega_k f_k(z), \tag{20}$$

where $\omega_k\ge 0$ and $\sum_{k=1}^{\infty}\omega_k=1$.

Proof. Suppose $f(z)$ can be written as in (20). Then

$$f(z)=z-\sum_{k=2}^{\infty}\omega_k\frac{2\beta\gamma(1-\alpha)}{(1+\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|}z^k\ .$$

Now,

$$\sum_{k=2}^{\infty}\frac{(1+\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)}\omega_k\frac{2\beta\gamma(1-\alpha)}{(1+\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|}$$

$$=\sum_{k=2}^{\infty}\omega_k=1-\omega_1\le 1\ .$$

Thus $f\in PT(\lambda,\alpha,\beta)$. Conversely, let $f\in PT(\lambda,\alpha,\beta)$. Then by using (18), we set

$$\omega_k=\frac{(1+(\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)}a_k, \qquad k\ge 2$$

and $\omega_1=1-\sum_{k=2}^{\infty}\omega_k$. Then we have $f(z)=\sum_{k=1}^{\infty}\omega_k f_k(z)$, and hence this completes the proof of Theorem 2. □

3.Distortion bounds

In this section we obtain distortion bounds for the class $PT(\lambda,\alpha,\beta)$.
Theorem 3. If $f\in PT(\lambda,\alpha,\beta)$, then

$$r-\frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\,|C_2(b,\mu)|}r^2\leq|f(z)| \tag{21}$$

$$\leq r+\frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\,|C_2(b,\mu)|}r^2$$

holds if the sequence $\{\sigma_k(\lambda,\beta,\gamma)\}_{k=2}^{\infty}$ is non-decreasing, and

$$1-\frac{4\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\,|C_2(b,\mu)|}r\leq\left|f'(z)\right| \tag{22}$$

$$\leq 1+\frac{4\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\,|C_2(b,\mu)|}r$$

holds if the sequence $\{\sigma_k(\lambda,\beta,\gamma)/k\}_{k=2}^{\infty}$ is non-decreasing, where

$$\sigma_k(\lambda,\beta,\gamma)=(1+\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|\ .$$

The bounds in (21) and (22) are sharp, since the equalities are attained by the function

$$f(z)=z-\frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\,|C_2(b,\mu)|}z^2,\quad z=\pm r\ . \tag{23}$$

Proof. In view of Theorem 1, we have

$$\sum_{k=2}^{\infty}a_k\leq\frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\,|C_2(b,\mu)|}. \tag{24}$$

Using (14) and (24), we obtain

$$|z|-|z|^2\sum_{k=2}^{\infty}a_k\leq|f(z)|$$

$$\leq|z|+|z|^2\sum_{k=2}^{\infty}a_k.$$

So,

$$r - r^2 \frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\,|C_2(b,\mu)|} \leq |f(z)| \tag{25}$$

$$\leq r + r^2 \frac{2\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\,|C_2(b,\mu)|} \ .$$

Hence (21) follows from (25). Further,

$$\sum_{k=2}^{\infty} k a_k \leq \frac{4\beta\gamma(1-\alpha)}{(1+\lambda)[1+\beta(2\gamma-1)]\,|C_2(b,\mu)|} \ .$$

Hence (22) follows from

$$1 - r\sum_{k=2}^{\infty} k a_k \leq \left|f'(z)\right| \leq 1 + r\sum_{k=2}^{\infty} k a_k \ .$$

□

4.Radius of starlikeness and convexity

The radii of close-to-convexity, starlikeness and convexity for the class $PT(\lambda,\alpha,\beta)$ are given in this section.

Theorem 4. Let the function $f(z)$ defined by (14) belong to the class $PT(\lambda,\alpha,\beta)$, Then $f(z)$ is close-to-convex of order δ, $(0 \leq \delta < 1)$ in the disc $|z| < R_1$, where

$$R_1 := \inf_{k\geq 2}\left[\frac{(1-\delta)(1+\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|}{2k\beta\gamma(1-\alpha)}\right]^{\frac{1}{k-1}} \tag{26}$$

The result is sharp, with extremal function $f(z)$ given by (19).

Proof. Given $f \in T$ and f is close-to-convex of order δ, we have

$$\left|f'(z) - 1\right| < 1 - \delta \ . \tag{27}$$

For the left hand side of (27) we have

$$\left|f'(z) - 1\right| \leq \sum_{k=2}^{\infty} k a_k \, |z|^{k-1} \ .$$

The last expression is less than $1-\delta$ if

$$\sum_{k=2}^{\infty}\frac{k}{1-\delta}a_k\,|z|^{k-1}<1\ .$$

Using the fact that $f\in PT(\lambda,\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty}\frac{(1+\lambda(k-1))[1+\beta(2\gamma-1)]a_k\,|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)}\leq 1,$$

So (27) is true if

$$\frac{k}{1-\delta}\,|z|^{k-1}\leq\frac{1+\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)}\ .$$

Or, equivalently,

$$|z|^{k-1}\leq[\frac{(1-\delta)(1+\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|}{2k\beta\gamma(1-\alpha)}],$$

which completes the proof. □

Theorem 5. Let $f\in PT(\lambda,\alpha,\beta)$. Then
(1) f is starlike of order δ, $(0\leq\delta<1)$, in the disc $|z|<R_2$, where

$$R_2=\inf_{k\geq 2}\{\frac{(1-\delta)(1+\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)(k-\delta)}\}^{\frac{1}{k-1}}$$

(2) f is convex of order δ , $(0\leq\delta<1)$, in the disc $|z|<R_3$, that is where

$$R_3=\inf_{k\geq 2}\{\frac{(1-\delta)(1+\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)k(k-\delta)}\}^{\frac{1}{k-1}}\ .$$

Each of these results is sharp for the extremal function $f(z)$ given by (19).

Proof. (1) Given $f\in T$ and f starlike of order δ , we have

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1-\delta\ . \tag{28}$$

For the left hand side of (28) we have

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq\frac{\sum_{k=2}^{\infty}(k-1)a_k\,|z|^{k-1}}{1-\sum_{k=2}^{\infty}a_k\,|z|^{k-1}}\ .$$

The last expression is less than $1-\delta$ if

$$\sum_{k=2}^{\infty}\frac{k-\delta}{1-\delta}a_k\,|z|^{k-1}<1\ .$$

Using the fact that $f\in PT(\lambda,\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty}\frac{(1+\lambda(k-1))[1+\beta(2\gamma-1)]a_k\,|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)}<1,$$

we can say (28) is true if

$$\frac{k-\delta}{1-\delta}\,|z|^{k-1}<\frac{(1+\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)}\ .$$

Or, equivalently,

$$|z|^{k-1}<\frac{(1-\delta)(1+\lambda(k-1))[1+\beta(2\gamma-1)]\,|C_k(b,\mu)|}{2\beta\gamma(1-\alpha)(k-\delta)}$$

which yields the starlikeness of the family.
(2) Using the fact that f is convex if and only if zf' is starlike, we can prove (2) on lines similar to the proof of (1). □

5.Neighborhood property

In this section we study neighborhood property for functions in the class $PT(\lambda,\alpha,\beta)$.

Definition. For functions f belong to $P(\lambda,\alpha,\beta)$ of the form (1) and $\gamma\geq 0$, we define $\eta-\gamma$-neighborhood of f by

$$N_{\gamma}^{\eta}(f)=\{g(z)\in P(\lambda,\alpha,\beta):g(z)=z+\sum_{k=2}^{\infty}b_kz^k,\quad \sum_{k=2}^{\infty}k^{\eta+1}\,|a_k-b_k|\leq\gamma\},$$

where η is a fixed positive integer.

By using the following lemmas we will investigate the $\eta-\gamma$-neighborhood of functions in $PT(\lambda,\alpha,\beta)$.

Lemma 1. Let $p\geq 0$ and $-1\leq\theta<1$. if $g(z)=z+\sum_{k=2}^{\infty}b_kz^k$ satisfies

$$\sum_{k=2}^{\infty}k_{\rho+1}\,|b_k|\leq\frac{2\theta\gamma(1-\alpha)}{1+\theta(2\gamma-1)},$$

then $g(z) \in PT(\lambda, \alpha, \beta)$.

Proof. By using Theorem 1, it is sufficient to show that

$$\frac{(1+\lambda(k-1))[1+\theta(2\gamma-1)]}{2\theta\gamma(1-\alpha)}(\frac{\rho+1}{\rho+k})^{\mu} = \frac{k^{\rho+1}}{2\theta\gamma(1-\alpha)}(1+\theta(2\gamma-1)) \ .$$

But

$$\frac{[1+\theta(2\gamma-1)]}{2\theta\gamma(1-\alpha)}(\frac{\rho+1}{\rho+k})^{\mu} \le \frac{k^{\rho+1}}{2\theta\gamma(1-\alpha)}[1+\theta(2\gamma-1)] \ .$$

Therefore it is enough to prove that

$$Q(k,\rho) = \frac{(\frac{\rho+1}{\rho+k})^{\mu}}{k^{\rho+1}} \le 1 \ .$$

The result follows because the last inequality holds for all $k \ge 2$. □

Lemma 2. Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in T$, $-1 \le \alpha < 1$, $0 < \beta \le 1$, $\lambda \ge 0$ and $\epsilon \ge 0$. If $\frac{f(z)+\epsilon z}{1+\epsilon} \in PT(\lambda, \alpha, \beta)$ then

$$\sum_{k=2}^{\infty} k^{\rho+1} a_k \le \frac{2^{\rho+1}[2\beta\gamma(1-\alpha)(1+\epsilon)]}{(1+\lambda)[1+\beta(2\gamma-1)]}(\frac{b+2}{b+1})^{\mu}$$

where either $\rho = 0$ and $b \ge 0$ or $\rho = 1$ and $1 \le b \le 2$.The result is sharp with the extremal function

$$f(z) = z - \frac{2\beta\gamma(1-\alpha)(1+\epsilon)}{(1+\lambda)[1+\beta(2\gamma-1)]}(\frac{b+2}{b+1})^{\mu} z^2, \quad (z \in \mathbb{U}) \ .$$

Proof. Letting $g(z) = \frac{f(z)+\epsilon z}{1+\epsilon}$ we have

$$g(z) = z - \sum_{k=2}^{\infty} \frac{a_k}{1+\epsilon} z^k, \quad (z \in \mathbb{U}) \ .$$

In view of Theorem 2, $g(z) = \sum_{k=1}^{\infty} \omega_k g_k(z)$ where $\omega_k \ge 0$, $\sum_{k=1}^{\infty} \omega_k = 1$,

$$g_1(z) = z$$

and

$$g_k(z) = z - \frac{2\beta\gamma(1-\alpha)(1+\epsilon)}{(1+\lambda(k-1))[1+\beta(2\gamma-1)]}(\frac{b+k}{b+1})^{\mu} z^k \quad (k \geq 2) .$$

So we obtain

$$g(z) = \omega_1 g_1(z) + \sum_{k=2}^{\infty} \omega_k [z - \frac{2\beta\gamma(1-\alpha)(1+\epsilon)}{(1+\lambda(k-1))[1+\beta(2\gamma-1)]}(\frac{b+k}{b+1})^{\mu} z^k]$$

$$= z - \sum_{k=2}^{\infty} \omega_k [\frac{2\beta\gamma(1-\alpha)(1+\epsilon)}{(1+\lambda(k-1))[1+\beta(2\gamma-1)]}(\frac{b+k}{b+1})^{\mu}] z^k .$$

Since $\omega_k \geq 0$ and $\sum_{k=2}^{\infty} \omega_k \leq 1$, it follows that

$$\sum_{k=2}^{\infty} k^{\rho+1} a_k \leq 2^{\rho+1}[\frac{2\beta\gamma(1-\alpha)(1+\epsilon)}{(1+\lambda(k-1))[1+\beta(2\gamma-1)]}(\frac{b+k}{b+1})^{\mu}] .$$

Since whenever $\rho = 0$ and $b \geq 0$ or $\rho = 1$ and $1 \leq b \leq 2$ we conclude

$$W(k,\rho,\alpha,\beta,\epsilon,b,\mu) = k^{\rho+1}[\frac{2\beta\gamma(1-\alpha)(1+\epsilon)}{(1+\lambda(k-1))[1+\beta(2\gamma-1)]}(\frac{b+k}{b+1})^{\mu}],$$

is a decreasing function of k , the result will follow. So the proof is complete. □

Theorem 6. Let either $\rho = 0$ and $b \geq 0$ or $\rho = 1$ and $1 \leq b \leq 2$. Suppose $-1 \leq \beta < 1$, and

$$-1 \leq \theta < \frac{[1+\beta(2\gamma-1)](1+\lambda)(b+1)^{\mu} - 2^{\eta+1}[2\beta\gamma(1-\alpha)(1+\epsilon)(b+2)^{\mu}]}{(1+\lambda)[1+\beta(2\gamma-1)](b+1)^{\mu}},$$

$f(z) \in T$ and $\frac{f(z)+\epsilon z}{1+\epsilon} \in PT(\lambda,\alpha,\beta)$. Then the $\eta-\gamma$-neighborhood of f is the subset of $PT(\lambda,\alpha,\beta)$, where

$$\gamma = \frac{[1+\beta(2\gamma-1)]2\theta\gamma(1-\alpha)(1+\lambda)(b+1)^{\mu} - 2^{\eta+1}[2\beta\gamma(1-\alpha)(1+\epsilon)(b+2)^{\mu}(1+\theta(2\gamma-1)]}{(1+\theta(2\gamma-1))(1+\lambda)[1+\beta(2\gamma-1)](b+1)^{\mu}} .$$

The result is sharp.

Proof. For $f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k$, let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ be in $N_{\gamma}^{\eta}(f)$. So by Lemma 2, we have

$$\sum_{k=2}^{\infty} k^{\eta+1} |b_k| = \sum_{k=2}^{\infty} k^{\eta+1} |a_k - b_k - a_k|$$

$$\leq \gamma + 2^{\eta+1}[\frac{2\beta\gamma(1-\alpha)(1+\epsilon)}{(1+\lambda)[1+(2\gamma-1)]}(\frac{b+2}{b+1})^{\mu}]$$

By using Lemma 2, $g(z) \in PT(\lambda, \alpha, \beta)$ if

$$\gamma + 2^{\eta+1}[\frac{2\beta\gamma(1-\alpha)(1+\epsilon)}{(1+\lambda)[1+\beta(2\gamma-1)](\frac{b+2}{b+1})^{\mu}}] \leq \frac{2\theta\gamma(1-\alpha)}{1+\theta(2\gamma-1)} \ .$$

That is, $\gamma \leq$

$$\frac{1+\beta(2\gamma-1)]2\theta\gamma(1-\alpha)(1+\lambda)(b+1)^{\mu} - 2^{k+1}[2\beta\gamma(1-\alpha)(1+\epsilon)(b+2)^{\mu}(1+\theta(2\gamma-1))]}{(1+\lambda)[1+\beta(2\gamma-1)](b+1)^{\mu}(1+\theta(2\gamma-1))}$$

and the proof is complete. □

6. Partial sums

In last section we verify some properties of partial sums of functions in the class $PT(\lambda, \alpha, \beta)$.

Theorem 7. Let $f(z) \in PT(\lambda, \alpha, \beta)$ and define the partial sums $f_1(z)$ and $f_n(z)$ by

$$f_1(z) = z$$

and

$$f_n(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad (n \in \mathbb{N}, n > 1) \tag{29}$$

If

$$\sum_{k=2}^{\infty} c_k |a_k| \leq 1, \tag{30}$$

where

$$c_k = \frac{[1+\lambda(k-1)][1+\beta(2\gamma-1)]}{2\beta\gamma(1-\alpha)}(\frac{b+1}{b+k})^{\mu}. \tag{31}$$

Then $f_k(z) \in PT(\lambda, \alpha, \beta)$. Moreover

$$Re\{\frac{f(z)}{f_n(z)}\} > 1 - \frac{1}{c_{n+1}}, \qquad (z \in \mathbb{U}, n \in \mathbb{N}) \tag{32}$$

$$Re\{\frac{f_n(z)}{f(z)}\} > \frac{c_{n+1}}{1+c_{n+1}} \tag{33}$$

Proof. It is easy to show that $f_1(z) = z \in PT(\lambda, \alpha, \beta)$. So by TLemma 2, and condition (30), we have $N_1^{\eta}(z) \subset PT(\lambda, \alpha, \beta)$, so $f_k \in PT(\lambda, \alpha, \beta)$. Next, for the coefficient c_k it is easy to show that

$$c_{k+1} > c_k > 1 \ .$$

Therefore by using (30) we obtain

$$\sum_{k=2}^{n} |a_k| + c_{n+1} \sum_{k=n+1}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} c_k \, |a_k| \leq 1. \tag{34}$$

By putting

$$h_1(z) = c_{n+1}\{\frac{f(z)}{f_n(z)} - (1 - \frac{1}{c_{n+1}})\} = 1 + c_{n+1}(\frac{f(z)}{f_n(z)} - 1)$$

$$= 1 + c_{n+1}(\frac{z + \sum_{k=2}^{\infty} a_k z^k}{z + \sum_{k=2}^{n} a_k z^k} - 1) = 1 + c_{n+1}(\frac{1 + \sum_{k=2}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{n} a_k z^{k-1}} - 1)$$

$$= 1 + c_{n+1}[\frac{1 + \sum_{k=2}^{\infty} a_k z^{k-1} - 1 - \sum_{k=2}^{n} a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} a_k z^{k-1}}]$$

$$= 1 + \frac{c_{n+1} \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{n} a_k z^{k-1}},$$

and using (34), for all $z \in \mathbb{U}$ we have

$$\left| \frac{h_1(z) - 1}{h_1(z) + 1} \right| = \left| \frac{c_{n+1} \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{n} a_k z^{k-1}} + \frac{c_{n+1} \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{n} a_k z^{k-1}} \right|$$

$$\leq \frac{c_{n+1} \sum_{k=2}^{\infty} |a_k|}{2 - 2\sum_{k=2}^{n} |a_k| - c_{n+1} \sum_{k=n+1}^{\infty} |a_k|} \leq 1,$$

which proves (32). Similarly, if we put

$$h_2(z) = \{\frac{f_n(z)}{f(z)} - \frac{c_{n+1}}{1 + c_{n+1}}\}(1 + c_{n+1})$$

$$= 1 - \frac{(1 + c_{n+1} \sum_{k=n+1}^{\infty} a_k z^{k-1})}{1 + \sum_{k=2}^{\infty} a_k z^{k-1}},$$

and using (34) we obtain

$$\left|\frac{h_2(z)-1}{h_2(z)+1}\right| \leq 1, \quad (z \in \mathbb{U}),$$

which yields the condition (33). So the proof is complete. □

Acknowledgment

This paper is obtained from the project with the same title which was approved by the support of Payame Noor Univeresity.

References

[1] Alexander J.W ,*Functions which map the interior of the unit circle upon simple regions*, Ann. of Math. 17, 1915, 12-22.

[2] Altintas O., *A subclass of analytic functions with negative coefficients*, Hacettepe Univ. Bull. Nat. Sciences & Engineering 19, 1990, 15- 24.

[3] Bernardi, S.D., *Convex and starlike univalent functions*, Trans. Amer. Math. Soc. 135, 1969, 429-446.

[4] Choi, J. ans Srivastava, H. M., *Certain families of series associated with the Hurwitz - Lerch Zeta function*, Appl. Math. Comput. 170, 2005, 399 - 409.

[5] Ferreira, C. and Lopez, J. L., *Asymptotic expansions of the Hurwitz - Lerch Zeta function*, J. Math. Anal. Appl. 298, 2004, 210 - 224.

[6] Flett, T. M., *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl. 38, 1972, 746 - 765.

[7] Garg, M., Jain, K. and Srivastava, H. M., *Some relationships between the generalized Apostol - Bernoulli polynomials and Hurwitz - Lerch Zeta function*, Integral Transform. Spec. Funct. 17, 2006, 803 - 815.

[8] G. Murugusundaramoorthy, *A subclass of analytic functions associated with the hurwitz- lerch zeta function*, Volume 39(2), 2010, 265- 272.

[9] Jung, I. B., Kim, Y. C. and Srivastava, H. M., *The Hardy space of analytic functions associated with certain one - parameter families of integral operators*, J. Math. Anal. Appl. 176, 1993, 138 - 147.

[10] Owa, S. and Lee, S. K., *Certain generalized class of analytic functions with negative coefficient*, Bull. Cal. Math. Soc. 82, 1990, 284 - 289.

[11] Lin, S. -D. and Srivastava, H. M., *Some families of the Hurwitz - Lerch Zeta functions and associated fractional derivative and other integral representations*, Appl. Math. Comput. 154, 2004, 725 - 733.

[12] Lin, S. D., Srivastava , H. M. and Wang, P. - Y., *Some expansion formulas for a class of generalized Hurwitz - Lerch Zeta functions*, Integral Transform. Spec. Funct. 17, 2006, 817 - 827.

[13] Prajapat, J. K. and Goyal, S. P., *Applications of Srivastava - Attiya operator to the classes of strongly starlike and strongly convex functions*, J. Math. Inequal. 3, 2009, 129 - 137.

[14] Riaducanu, D. and Srivastava, H. M., *A new class of analytic functions defined by means of a convolution operator involving the Hurwitz - Lerch Zeta function*, Integral Transform. Spec. Funct. 18, 2007, 933 - 943.

[15] Sh.Najafzadeh and Ali Ebadian, *Neighborhood and Partial sum property for univalent holomorphic functions in terms of komatu operators* No, 19, 2009.

[16] Silverman, H., *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. 51, 1975, 109 - 116.

[17] Srivastava, H. M. and Attiya, A. A., *An integral operator associated with the Hurwitz - Lerch Zeta function and differential subordination*, Integral Transform. Spec. Funct. 18, 2007, 207 - 216.

[18] Srivastava, H. M. and choi, J., *Series associated with the Zeta and related functions.* Kluwer Academic Publishers, Dordrecht, Boston, London, 2001.

Shahram Najafzadeh and Elnaz Pezeshki
Department of Mathematics
Payam Noor University
Iran
email: *najafzadeh1234@yahoo.ie, elnaz_pezeshki@yahoo.com*