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DECOMPOSITIONS OF CONTINUITY VIA $\zeta-$OPEN SETS

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Abstract. Al-Hawary [1] introduced and explored the class of $\zeta-$open sets which is strictly weaker than open and proved that the collection of all $\zeta-$open subsets of a space forms a topology that is finer than the original one. In this paper, we introduce what we call $\zeta-$continuity and ζ_X-continuity and we give several characterizations and two decompositions of $\zeta-$continuity. Finally, new decompositions of continuity are provided.

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INTRODUCTION

Let $(X, \mathfrak{T})$ be a topological space (or simply, a space). If $A \subseteq X$, then the closure of A and the interior of A will be denoted by $Cl_{\mathfrak{T}}A)$ and $Int_{\mathfrak{T}}(A)$; respectively. If no ambiguity appears, we use $\overline{A}$ and A^o instead. By X, Y and Z we mean topological spaces with no separation axioms assumed. $\mathfrak{T}_{s\tan dard}$, $\mathfrak{T}_{indiscrete}$, $\mathfrak{T}_{leftray}$ and $\mathfrak{T}_{cocountable}$ stand for the standard, indiscrete, left ray and the cocountable topologies, respectively. A space $(X, \mathfrak{T})$ is anti locally countable if all non-empty open subsets are uncountable.

In [1], the relatively new notion of $\zeta-$open subset which is strictly weaker than open was introduced. It was proved that the collection of all $\zeta-$open subsets of a space forms a topology that is finer than the original one. Several characterizations and properties of this class were also given as well as connections to other well-known "generalized open" subsets. Analogous to [3, 6, 7], in Section 2 we introduce the relatively new notion of $\zeta-$-continuity, which is closely related to continuity. Moreover, we show that a space $(X, \mathfrak{T})$ is Lindelof if and only if $(X, \mathfrak{T}_\zeta)$ is Lindelof, where $\mathfrak{T}_\zeta$ is the collection of all ζ-open subsets of X. Sections 3 is devoted for studying three weaker notions of ζ-continuity by which we provide two decompositions of ζ-continuity. Finally, in Section 4 we give several decompositions of continuity which seem to be new.

Next, we recall several necessary definitions and results from [1].

Definition 1 A subset A of a space $(X, \mathfrak{T})$ is called ζ-open if for every $x \in A$, there exists an open subset $U \subseteq X$ containing x and such that $U \backslash sInt(A)$ is countable. The complement of an $\zeta-$open subset is called $\zeta-$closed.

Clearly every open set is ζ-open, but the converse needs not be true.

Example 1 Consider the real line $\mathbb{R}$ with the topology $\mathfrak{T} = \{U \subseteq \mathbb{R} : \mathbb{R}\backslash U$ is finite or $0 \in \mathbb{R}\backslash U\}$ and set $A = \mathbb{R}\backslash U \cup \{0\}$. Then A is not open while it is ζ-open.

Theorem 1 If $(X, \mathfrak{T})$ is a space, then $(X, \mathfrak{T}_\zeta)$ is a space such that $\mathfrak{T} \subseteq \mathfrak{T}_\zeta$.

Corollary 1 If $(X, \mathfrak{T})$ is a p-space, then $\mathfrak{T} = \mathfrak{T}_\zeta$.

Next we show that $\zeta-$open notion is independent of both PO and SO notions.

Example 2 Consider $\mathbb{R}$ with the standard topology. Then $\mathbb{Q}$ is PO but not ζ-open. Also $[0, 1]$ is SO but not ζ-open.

Example 3 Let $X = \{a, b\}$ and $\mathfrak{T} =\{\varnothing, X, \{a\}\}$ Then $\{b\}$ is ζ-open but neither PO nor open.

Next we characterize $\mathfrak{T}_\zeta$ when X is a countable space.

Corollary 2 If $(X, \mathfrak{T})$ is a countable space, then $\mathfrak{T}_\zeta$ is the discrete topology.

The following result, in which a new characterization of ζ-open subsets is given, will be a basic tool throughout the rest of the paper.

Lemma 1 A subset A of a space X is ζ-open if and only if for every $x \in A$, there exists an open subset U containing x and a countable subset C such that $U - C \subseteq sInt(A)$.

Lemma 2 A subset A of a space X is ζ-closed if and only if $Cl_\zeta(A) = A$.

Theorem 2 If A is ζ-open subset of X, then $\mathfrak{T}_{\zeta|A} \subseteq (\mathfrak{T}_A)_\zeta$. If A is open subset of X, then $\mathfrak{T}_{\zeta|A} \subseteq (\mathfrak{T}_A)_\zeta$.

Lemma 3 If X is a Lindelof space, then $A - sInt(A)$ is countable for every closed subset $A \in \mathfrak{T}_\zeta$:

Corollary 3 If X is a second countable space, then $A - sInt(A)$ is countable for every closed subset $A \in \mathfrak{T}_\zeta$.

Theorem 3 Let $(X, \mathfrak{T})$ be a space and $C \subseteq X$ is ζ-closed. Then $Cl_\mathfrak{T}(C) \subseteq K \cup B$ for some closed subset K and a countable subset B.

2.ζ-CONTINUOUS MAPPINGS

We begin this section by introducing the notion of ζ-continuous mappings. Several characterizations of this class of mappings are also provided.

Definition 2 A map $f : X \to Y$ is ζ-*continuous at* $x \in X$ if for every open subset V in Y containing $f(x)$, there exists an ζ-open subset U in X containing x such that $f(U) \subseteq V$. f is ζ-*continuous* if it is ζ-continuous at every $x \in X$.

As every open set is ζ-open , every continuous map is ζ-continuous. The converse needs not be true.

Example 4 Let $X = \{a, b\}$, $\mathfrak{T}_1 = \{\varnothing, X, \{a\}\}$ and $\mathfrak{T}_2 = \{\varnothing, X, \{b\}\}$. Then the identity map $id : (X, \mathfrak{T}_1) \to (X, \mathfrak{T}_2)$ is ζ-continuous but not continuous.

The proofs of the following three results are easy ones and thus omitted.

Lemma 4 Let X, Y and Z be spaces. Then

(1) If $f : X \to Y$ is ζ-continuous surjection and $g : Y \to Z$ is continuous surjection, then $g \circ f$ is ζ-continuous.

(2) If $f : X \to Y$ is ζ-continuous surjection and $A \subseteq X$, then $f|_A$ is ζ-continuous.

(3) If $f : X \to Y$ is a map such that $X = X_1 \cup X_2$ where X_1 and X_2 are closed and both $f|_{X_1}$ and $f|_{X_2}$ are ζ-continuous, then f is ζ-continuous.

(4) If $f_1 : X \to X_1$ and $f_2 : X \to X_2$ are maps and $g : X \to X_1 \times X_2$ is the map defined by $g(x) = (f_1(x), f_2(x))$ for all $x \in X$, then g is ζ-continuous if and only if f_1 and f_2 are ζ-continuous.

Lemma 5 For a map $f : X \to Y$, the following are equivalent:

(1) f is ζ-continuous.

(2) The inverse image of every open subset of Y is ζ-open in X.

(3) The inverse image of every closed subset of Y is ζ-closed in X.

(4) The inverse image of every basic open subset of Y is ζ-open in X.

(5) The inverse image of every subbasic open subset of Y is ζ-open in X.

Lemma 6 A space $(X, \mathfrak{T}_X)$ is Lindelof if and only if $(X, \mathfrak{T}_\zeta)$ is Lindelof.

Next we show that being Lindelof is preserved under ζ-continuity.

Theorem 4 If $f : (X, \mathfrak{T}_X) \to (Y, \mathfrak{T}_Y)$ is ζ-continuous and X is Lindelof, then Y is Lindelof.

Proof. Let $\mathfrak{B} = \{V_\alpha : \alpha \in \nabla\}$ be an open cover of Y. Since f is ζ-continuous, $\mathfrak{A} = \{f^{-1}(V_\alpha) : \alpha \in \nabla\}$ is a cover of X by ζ-open subsets and as X is Lindelof, by Lemma , $\mathfrak{A}$ has a countable subcover $\{f^{-1}(V_{\alpha_n}) : n \in \mathbb{N}\}$. Now $Y = f(X) = f(\cup\{f^{-1}(V_{\alpha_n}) : n \in \mathbb{N})\} \subseteq \cup\{V_{\alpha_n} : n \in \mathbb{N}\}$. Therefore Y is Lindelof. $\square$

If X is a countable space, then every subset of X is ζ-open and hence every map $f : X \to Y$ is ζ-continuous. Next, we show that if X is uncountable such that every ζ-continuous map $f : X \to Y$ is a constant map, then X has to be connected.

Theorem 5 If X is uncountable space such that every ζ-continuous map $f : X \to Y$ is a constant map, then X is connected.

Proof. If X is disconnected, then there exists a non-empty proper subset A of X which is both open and closed. Let $Y = \{a, b\}$ and $\mathfrak{T}_Y = \{\varnothing, Y, \{b\}\}$ and $f : X \to Y$ defined by $f(A) = \{a\}$ and $f(X \backslash A) = \{b\}$. Then f is a non-constant ζ-continuous map. $\square$

The converse of the preceding result need not be true even when X is uncountable.

Example 5 The identity map $id : (\mathbb{R}, \mathfrak{T}_{leftray}) \to (\mathbb{R}, \mathfrak{T}_{indiscrete})$ is a non-constant ζ-continuous.

3. DECOMPOSITION OF ζ-CONTINUITY

We begin by recalling the following well-known two definitions.

Definition 3 A map $f : X \to Y$ is *weakly continuous at* $x \in X$ if for every open subset V in Y containing $f(x)$, there exists an open subset U in X containing x such that $f(U) \subseteq \overline{V}$. f is *weakly continuous* if it is weakly continuous at every $x \in X$.

Definition 4 A map $f : X \to Y$ is *W^*-continuous* if for every open subset V in Y, $f^{-1}(Fr(V))$ is closed in X, where $Fr(V) = \overline{V}\backslash \overset{o}{V}$.

Weakly continuity and W*-continuity are independent notions that are weaker than continuity and the two together characterize continuity (see for example [5]). Next we give two relatively new such definitions.

Definition 5 A map $f : X \to Y$ is *weakly ζ-continuous at $x \in X$* if for every open subset V in Y containing $f(x)$, there exists an ζ-open subset U in X containing x such that $f(U) \subseteq \overline{V}$. f is *weakly ζ-continuous* if it is weakly ζ-continuous at every $x \in X$.

Clearly, every ζ-continuous and every weakly continuous map is weakly ζ-continuous. Non of the converses need be true as shown next.

Example 6 Let $Y = \{a, b, c\}$ and $\mathfrak{T} = \{\varnothing, Y, \{a\}, \{c\}, \{a, c\}\}$. Then the map $f : (\mathbb{R}, \mathfrak{T}_{cocountable}) \to (Y, \mathfrak{T})$ defined by $f(x) = a$ for all $x \in \mathbb{R}$. Then f is weakly ζ-continuous but not ζ-continuous.

Example 7 Let $Y = \{a, b, c\}$ and $\mathfrak{T} = \{\varnothing, Y, \{a\}, \{c\}, \{a, c\}\}$. Then the map $f : (\mathbb{R}, \mathfrak{T}_{cocountable}) \to (Y, \mathfrak{T})$ defined by $f(x) = \begin{cases} a & x \in \mathbb{Q} \\ b & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$ for all $x \in \mathbb{R}$. Then f is weakly continuous and hence weakly ζ-continuous but not ζ-continuous.

Definition 6 A map $f : X \to Y$ is *coweakly ζ-continuous* if for every open subset V in Y, $f^{-1}(Fr(V))$ is ζ-closed in X, where $Fr(V) = \overline{V}\backslash \overset{o}{V}$.

Clearly, every ζ-continuous is coweakly ζ-continuous. The converse need not be true.

Example 8 Let $X = Y = \{a, b\}$, $\mathfrak{T}_X = \{\varnothing, X\}$ and $\mathfrak{T}_Y = \{\varnothing, Y, \{a\}, \{b\}\}$Then the identity map $id : X \to Y$ is coweakly ζ-continuous but not ζ-continuous.

Our first characterization of ζ-continuity in terms of the preceding two notions of continuity is given next.

Theorem 6 The following are equivalent for a map $f : (X, \mathfrak{T}_X) \to (Y, \mathfrak{T}_Y)$:

(1) f is ζ-continuous.

(2) $f : (X, \mathfrak{T}_\zeta) \to (Y, \mathfrak{T}_Y)$ is continuous.

(3) $f : (X, \mathfrak{T}_\zeta) \to (Y, \mathfrak{T}_Y)$ is weakly continuous and W*-continuous.

Proof. $(1) \Rightarrow (2)$: Obvious.

$(2) \Rightarrow (3)$: See for example [5].

$(3) \Rightarrow (1)$: Since $f : (X, \mathfrak{T}_\zeta) \to (Y, \mathfrak{T}_Y)$ is W*-continuous, it is coweakly ζ-continuous and as it is weakly-continuous, it is weakly ζ-continuous. Thus $f : (X, \mathfrak{T}_X) \to (Y, \mathfrak{T}_Y)$ is ζ-continuous. $\square$

We show that weakly ζ-continuity and coweakly ζ-continuity are independent notions, but together they characterize ζ-continuity. This will be our first decomposition of ζ-continuity.

Example 9 The map id in Example is coweakly ζ-continuous but not weakly ζ-continuous.

Example 10 Let $Y = \{a, b\}$ and $\mathfrak{T} = \{\varnothing, Y, \{a\}\}$. Then the map $f : (\mathbb{R}, \mathfrak{T}_{cocountable}) \to (Y, \mathfrak{T})$ defined by $f(x) = \begin{cases} a & x \in \mathbb{Q} \\ b & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$ for all $x \in \mathbb{R}$. Then f is weakly ζ-continuous but not coweakly ζ-continuous.

Theorem 7 A map $f : X \to Y$ is ζ-continuous if and only if f is both weakly and coweakly ζ-continuous.

Proof. ζ-continuity implies weakly and coweakly ζ-continuity is obvious. Conversely, suppose $f : X \to Y$ is both weakly and coweakly ζ-continuous and let $x \in X$ and V be an open subset of Y such that $f(x) \in V$. Then as f is weakly ζ-continuous, there exists an ζ-open subset U of X containing x such that $f(U) \subseteq \overline{V}$. Now $Fr(V) = \overline{V}\backslash V$ and hence $f(x) \notin Fr(V)$. So $x \in U\backslash f^{-1}(Fr(V))$ which is ζ-open in X since f is coweakly ζ-continuous. For every $y \in f(U\backslash f^{-1}(Fr(V)))$, $y = f(a)$ for some $a \in U\backslash f^{-1}(Fr(V))$ and hence $f(a) = y \in f(U) \subseteq \overline{V}$ and $y \notin Fr(V)$. Thus $f(a) = y \notin Fr(V)$ and thus $f(a) \in V$. Therefore, $f(U\backslash f^{-1}(Fr(V))) \subseteq V$ and hence f is ζ-continuous.

4. DECOMPOSITION OF CONTINUITY

We begin this section by introducing the notion of an ω_X^o-set. We then introduce the notion of ω_X^o-continuity which gives an immediate decomposition of continuity.

Definition 7 For a space $(X, \mathfrak{T})$, let $\zeta_X =: \{A \subseteq X : Int_{\mathfrak{T}_\zeta}(A) = Int_{\mathfrak{T}}(A)\}$. A is an ζ_X-set if $A \in \zeta_X$.

The proof of the following result is trivial.

Corollary 4 If $(X, \mathfrak{T})$ is anti locally countable, then ζ_X contains all ζ-closed subsets of X.

We remark that, in general, an ζ-closed set need not be an ζ_X-set as shown in the next example.

Example 11 Let $X = \{a, b\}$ and $\mathfrak{T} = \{\varnothing, X, \{a\}\}$. Set $A = \{b\}$. Then A is ζ-closed but not an ζ_X-set.

As every open set is ζ-open, every open set is an ζ_X-set but the converse need not be true.

Example 12 Consider $\mathbb{R}$ with the standard topology $\mathfrak{T}_{standard}$. Then $\mathbb{Q}$ is an ζ_X-set which is not open.

Next, we show that the notions of ζ_X-set and ζ-open are independent, but together they characterize open sets.

Example 13 In Example , A is ζ-open but not an ζ_X-set.

Example 14 In Example , $\mathbb{Q}$ is an ζ_X-set which is not ζ-open.

Theorem 8 A subset A of a space X is open if and only if A is ζ-open and an ζ_X-set.

Proof. Trivially every open set is ζ-open and an ζ_X-set.. Conversely, let A be an ζ-open set that is ζ_X-set. Then $A = Int_{\Im_\zeta}(A) = Int_{\Im}(A)$ and therefore A is open.$\square$

Definition 8 A map $f : X \to Y$ is ζ_X-continuous if the inverse image of every open subset of Y is an ζ_X-set.

Clearly every continuous map is ζ_X-continuous, but the converse need not be true as not every ζ_X-set is open. An immediate consequence of Theorems and are the following decompositions of continuity, which seem to be new.

Theorem 9 For a map $f : X \to Y$, the following are equivalent:

(1) f is continuous.

(2) f is ζ-continuous and ζ_X-continuous.

(3) f is both weakly ζ-continuous, coweakly ζ-continuous and ζ_X-continuous.

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