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THERMAL CONDUCTION IN GRIDWORKS (CYLINDRICAL DOMAINS)

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ABSTRACT. In this paper we study a stationary thermal problem on gridworks, characterized by two small parameters: ε- period and δ- thickness distributed along the structure layers.

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1. Introducere

We now consider a particular case of three-dimensional lattice structures, the gridworks that consist in regular array of thin wires.

The specifics of these structures consists in just following two-way periodicity ox_1 and ox_2. The two small parameters on which the structure are ε- during which distribute the reference cell and δ- small thickness of the material distributed along the structure cross section. There is also a third parameter e, which is the thickness of the structure in longitudinal section. In our case e and ε have the same order $e = k\varepsilon$.

The novelty of our problem consists in reticulated structure plate geometry: the period that we're on the covers is made up of horizontal bars, vertical and oblique. Due to this the limit problem obtained after the homogenization by two parameters ε and δ is new and at the same time simple: we started from a thermal problem on a heterogeneous domain which depends on ε and δ, and we have a two-dimensional problem with partial differential second order with constant and elliptical coefficients.

Homogenization reduces initial problem to two simple problems: one on the cell of periodicity and another one on a fixed domain without holes.

In the first stage we use a result obtained in [1]. Here, applying the variational method of Tartar, homogenized the initial equation after $\varepsilon \to 0$.

In the second stage we got our new result, using the method of dilation introduced in [2]. Dilatation technique we use is to changes the appropriate variables that transforms bars $H_\delta, V_\delta, O_\delta^1$ and O_δ^2 in the entire cell reference Y.

In our result, homogenized coefficients obtained by $\delta \to 0$ are simple algebraic combinations between the characteristics of the material. The problem obtained in theorem 2, can be solved explicit. We should note the following aspect: the crossing of the boundary after $\varepsilon \to 0$, then $\delta \to 0$ we get homogenized coefficients that depends strictly reticulated structure, namely the periodicity cell of structure.

2.The geometry of the structure

Let $\omega = (0, L_1) \times (0, L_2) \subset \mathbf{R}^2$ and $\Omega^e = \omega \times \left(-\frac{e}{2}, \frac{e}{2}\right)$, and ω is covered periodically with the reference cell $Y = (0,1) \times (0,1)$. We have $\frac{L_1}{L_2} \in \mathbf{Q}$ and we choose ε such that , $N_\varepsilon^1 = \frac{L_1}{\varepsilon}, N_\varepsilon^2 = \frac{L_2}{\varepsilon}$ to be integer numbers. ε is called the period which is distributed Y in ω. In Figure 1 represent the periodicity cell Y_δ defined by:

$$Y_\delta = \left\{ y \in Y \,\middle|\, dist\,(y, \partial Y) < \frac{\delta}{2} \right\} \cup O_\delta^1 \cup O_\delta^2$$

where O_δ^1, O_δ^2 are rectangular slash of length $\sqrt{2}$ and thickness δ.

We define the hole $T_\delta = Y \backslash \bar{Y}_\delta$.

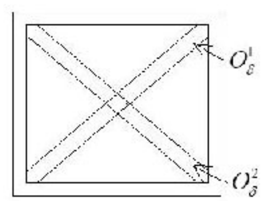

Figure 1: The cell Y_δ.

Y_δ is occupied by the material from the cell Y. We consider $\omega_{\varepsilon\delta}$ the perforated area from ω or occupied by the material from ω after distribution of the periodicity cell Y_δ with period ε by two directions ox_1 and ox_2. The domain $\omega_{\varepsilon\delta}$ has $N_\varepsilon^1 \times N_\varepsilon^2$ holes which do not intersects the border of ω.

Consider three-dimensional perforated domain $\Omega_{\varepsilon\delta}^e = \omega_{\varepsilon\delta} \times \left(-\frac{e}{2}, \frac{e}{2}\right)$ which is a gridworks type plates which depends on three small parameters: ε the period, e the plate thickness and δ thickness of the bars (oblique, horizontal and vertical) which forms covers. The periodicity cell of the structure is $\mathbf{Y}_\delta = Y_\delta \times \left(-\frac{1}{2}, \frac{1}{2}\right)$.

Because the correctors $w_\alpha^{\delta k}, w_3^{\delta k}$ that appear in Theorem 1 are Y-periodic, choose - for ease of calculations that appear in the method of dilation of theorem 2 of this article - the next cell reference $Y = \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$
and the periodicity cell Y_δ shown in Figure 2, is defined by

$$Y_\delta = H_\delta \cup V_\delta \cup O_\delta^1 \cup O_\delta^2,$$

where the bars O_δ^1 and O_δ^2 are the same as Figure 1, and

$$H_\delta = \left\{ y \in Y \,\middle|\, |y_1| \le \tfrac{1}{2}, |y_2| \le \tfrac{\delta}{2} \right\},$$
$$V_\delta = \left\{ y \in Y \,\middle|\, |y_1| \le \tfrac{\delta}{2}, |y_2| \le \tfrac{1}{2} \right\}.$$

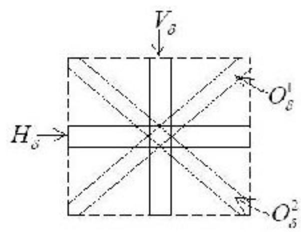

Figure 2: The period Y_δ.

3.Statement of the problem

Let the stationary temperature problem on $\Omega^e_{\varepsilon\delta}$:

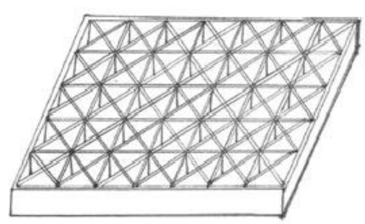

Figure 3: The three-dimensional perforated domain $\Omega^e_{\varepsilon\delta}$.

$$\begin{cases} -\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u^{\varepsilon e\delta}}{\partial x_j}\right) = f_e^\varepsilon \text{ in } \Omega^e_{\varepsilon\delta} \\ a_{3j}\frac{\partial u^{\varepsilon e\delta}}{\partial x_j} n_3 = g_e^{\varepsilon\pm} \text{ in } \Gamma^{e\pm}_{\varepsilon\delta} \\ a_{\alpha j}\frac{\partial u^{\varepsilon e\delta}}{\partial x_j} n_\alpha = 0 \text{ on } \delta T^e_{\varepsilon\delta} \\ u^{\varepsilon e\delta} = 0 \text{ on } \Gamma_0^e \end{cases} \tag{1}$$

where:

$\Gamma^{e\pm}_{\varepsilon\delta} = \omega_{\varepsilon\delta} \times \{\pm\frac{e}{2}\}$, are the two covers on the structure
$T_{\varepsilon\delta} = \omega \backslash \omega_{\varepsilon\delta}$, the set of holes
$T^{e}_{\varepsilon\delta} = T_{\varepsilon\delta} \times (-\frac{e}{2}, \frac{e}{2})$,
$\Gamma^{e}_{0} = \delta\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, external border of the structure
and we make the assumptions:

1. $f^{\varepsilon}_{e} \in C^{1}(\mathbf{R}^{3}) \cap L^{2}(\Omega)$ and $g^{\varepsilon\pm}_{e} \in C^{1}(\mathbf{R}^{2}) \cap L^{2}(\omega)$.
2. There is a constant $A > 0$ such that: $a_{ij}\xi_i\xi_j \geq A\xi_i\xi_j,\ \forall \xi \in \mathbf{R}^{3}$.

Consider the case $e = k\varepsilon$, so when the period and the plate thickness are the same power.

We are making changes of variables and functions:
$z_1 = x_1,\ \ z_2 = x_2,\ \ z_3 = \frac{x_3}{k\varepsilon}$;
$u^{\varepsilon e\delta}(x_1, x_2, x_3) = u^{\varepsilon e\delta}(z_1, z_2, k\varepsilon z_3) = u^{\varepsilon\delta}_{k}(z_1, z_2, z_3)$;
$f^{\varepsilon}_{e}(z_1, z_2, k\varepsilon z_3) = f^{\varepsilon}_{k}(z_1, z_2, z_3)$; $g^{\varepsilon\pm}_{e}(z_1, z_2, k\varepsilon z_3) = g^{\varepsilon\pm}_{k}(z_1, z_2, z_3)$.
$\Omega^{e}_{\varepsilon\delta}$ passes into $\Omega_{\varepsilon\delta} = \omega_{\varepsilon\delta} \times \left(-\frac{1}{2}, \frac{1}{2}\right)$; $\Gamma^{e\pm}_{\varepsilon\delta}$ passes into $\Gamma^{\pm}_{\varepsilon\delta} = \omega_{\varepsilon\delta} \times \left\{\pm\frac{1}{2}\right\}$; Γ^{e}_{0} passes into $\Gamma_0 = \delta\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, and Ω^{e} in $\Omega = \omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)$.

After the change the variable and function, problem (1) is written variational:

$$\int_{\Omega_{\varepsilon\delta}} \left[a_{\alpha\beta}\frac{\partial u^{\varepsilon\delta}_{k}}{\partial z_\beta}\frac{\partial v}{\partial z_\alpha} + (k\varepsilon)^{-1}\left(a_{\alpha 3}\frac{\partial u^{\varepsilon\delta}_{k}}{\partial z_3}\frac{\partial v}{\partial z_\alpha} + a_{3\beta}\frac{\partial u^{\varepsilon\delta}_{k}}{\partial z_\beta}\frac{\partial v}{\partial z_3}\right) + (k\varepsilon)^{-2}\left(a_{33}\frac{\partial u^{\varepsilon\delta}_{k}}{\partial z_3}\frac{\partial v}{\partial z_3}\right)\right] dz =$$
$$= \int_{\Omega_{\varepsilon\delta}} f^{\varepsilon}_{k} v\,dz + (k\varepsilon)^{-1}\int_{\Gamma^{+}_{\varepsilon\delta}} g^{\varepsilon+}_{k} v\,dz_1 dz_2 + (k\varepsilon)^{-1}\int_{\Gamma^{-}_{\varepsilon\delta}} g^{\varepsilon-}_{k} v\,dz_1 dz_2, \tag{2}$$

for all
$v \in V_{\varepsilon\delta} = \left\{v \in H^{1}(\Omega_{\varepsilon\delta}) : v = 0 \text{ on } \partial\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right\}$
and
$\|v\|_{V_{\varepsilon\delta}} = \left[\sum_{i=1}^{3}\int_{\Omega_{\varepsilon\delta}}\left(\frac{\partial v}{\partial x_i}\right)^2 dx\right]^{1/2}$.

4. The homogenization of the problem

First we do $\varepsilon \to 0$ and δ consider fixed.

After applying Tartar's variational method [3], we find:

Theorem 1. *Consider the following assumptions:*

$$\begin{aligned} & f^{\varepsilon}_{k}\chi_{\Omega_{\varepsilon\delta}} \overset{\varepsilon\to 0}{\longrightarrow} \frac{\text{meas } \mathrm{Y}_\delta}{\text{meas } \mathrm{Y}} \cdot f^{*}_{k} \text{ weak in } L^{2}(\Omega) \\ & (k\varepsilon)^{-1} g^{\varepsilon\pm}_{k}\chi_{\omega_{\varepsilon\delta}} \overset{\varepsilon\to 0}{\longrightarrow} \frac{\text{meas } \mathrm{Y}_\delta}{\text{meas } \mathrm{Y}} \cdot g^{*\pm}_{k} \text{ weak in } L^{2}(\omega). \end{aligned} \tag{3}$$

Then there is an extension operator $P^{\varepsilon\delta} \in \mathbf{L}\left(V_{\varepsilon\delta}; H^{1}_{0}(\Omega)\right)$ such that

$P^{\varepsilon\delta}u_k^{\varepsilon\delta} \to [\varepsilon \to 0]u_k^{\delta}$ *weak in* $H^1(\Omega)$ *where* $u_k^{\delta} = u_k^{\delta}(z_1, z_2)$, $u_k^{\delta} \in H_0^1(\omega)$ *satisfies the problem*

$$\begin{cases} -q_{\alpha\beta}^{\delta k}\frac{\partial^2 u_k^{\delta}}{\partial z_\alpha \partial z_\beta} = \frac{\text{meas } Y_\delta}{\text{meas } Y} \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} f_k^*(z_1, z_2, z_3)\, dz_3 + \frac{\text{meas } Y_\delta}{\text{meas } Y}\left(g_k^{*+} + g_k^{*-}\right) & \text{in } \omega \\ u_k^{\delta} = 0 & \text{on } \partial\omega \end{cases} \quad (4)$$

where the homogenized coefficients are:

$$q_{\alpha\beta}^{\delta k} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{Y_\delta} \left(a_{\gamma\beta}\frac{\partial w_\alpha^{\delta k}}{\partial y_\gamma} + k^{-1}a_{3\beta}\frac{\partial w_\alpha^{\delta k}}{\partial y_3}\right) dy \quad (5)$$

where the correction functions $w_\alpha^{\delta k}$ *satisfies the problem:*

$$\begin{cases} -\frac{\partial}{\partial y_\beta}\left(a_{\gamma\beta}\frac{\partial w_\alpha^{\delta k}}{\partial y_\gamma}\right) - k^{-1}\frac{\partial}{\partial y_\beta}\left(a_{3\beta}\frac{\partial w_\alpha^{\delta k}}{\partial y_3}\right) - k^{-1}\frac{\partial}{\partial y_3}\left(a_{\gamma 3}\frac{\partial w_\alpha^{\delta k}}{\partial y_\gamma}\right) - \\ -k^{-2}\frac{\partial}{\partial y_3}\left(a_{33}\frac{\partial w_\alpha^{\delta k}}{\partial y_3}\right) = 0 \text{ in } Y_\delta \times \left(-\frac{1}{2}, \frac{1}{2}\right) \\ \left(a_{\gamma j}\frac{\partial w_\alpha^{\delta k}}{\partial y_\gamma} + k^{-1}a_{3j}\frac{\partial w_\alpha^{\delta k}}{\partial y_3}\right) n_j = 0 \text{ on } \left[\partial T_\delta \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right] \cup \left[Y_\delta \times \left\{\pm\frac{1}{2}\right\}\right] \\ \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{Y_\delta} w_\alpha^{\delta k} dy = 0. \end{cases} \quad (6)$$

$w_\alpha^{\delta k} - y_\alpha$ *is periodic in* y_1 *and* y_2.

Further, we do $\delta \to 0$, and using the dilatation method, find:

Theorem 2. *We have* $u_k^{\delta} \to [\delta \to 0]u_k^*$ *weak in* $H_0^1(\omega)$, *where:*

$$\begin{cases} -q_{\alpha\beta}^*\frac{\partial^2 u_k^*}{\partial z_\alpha \partial z_\beta} = 2\left(1+\sqrt{2}\right)\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} f_k^*(z_1, z_2, z_3)\, dz_3 + \left(g_k^{*+} + g_k^{*-}\right)\right] & \text{in } \omega \\ u_k^* = 0 & \text{on } \partial\omega \end{cases} \quad (7)$$

where the coefficients $q_{\alpha\beta}^*$ *are elliptical and are given by:*

$$\begin{cases} q_{11}^* = D\left[\frac{1}{A_{22}} + \frac{\sqrt{2}}{a_{11}-a_{13}-a_{31}+a_{33}} + \frac{\sqrt{2}}{a_{11}+a_{22}+a_{33}+a_{13}+a_{22}+a_{31}}\right] \\ q_{22}^* = D\left[\frac{1}{A_{11}} + \frac{\sqrt{2}}{a_{11}-a_{13}-a_{31}+a_{33}} + \frac{\sqrt{2}}{a_{11}+a_{22}+a_{33}+a_{13}+a_{22}+a_{31}}\right] \\ q_{12}^* = q_{21}^* = \sqrt{2}D\left[\frac{1}{a_{11}-a_{13}-a_{31}+a_{33}} - \frac{1}{a_{11}+2a_{22}+a_{33}+a_{13}+a_{31}}\right] \end{cases} \quad (8)$$

where:

$D = \det A$, and A_{11}, A_{22} *are algebraic complements.*

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