DIFFERENTIAL SUBORDINATIONS OBTAINED BY USING GENERALIZED SĂLĂGEAN AND RUSCHEWEYH OPERATORS

Georgia Irina Oros, Gheorghe Oros

Abstract In this paper we consider the operator $D^\alpha_\lambda f$ in terms of the generalized Sălăgean and Ruschweyh operators, and we study several differential subordinations generated by $D^\alpha_\lambda f$.

2000 Mathematics Subject Classification: 30C45, 30A20, 34A40.

Keywords and phrases: differential subordination, convex function, best dominant, differential operator.

1. Introduction and preliminaries

Let U be the unit disc in the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U.
Also let

$$A_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \dots,\ z \in U\}$$

with $A_1 = A$.
For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, we denote by

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U),\ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots,\ z \in U\}.$$

Let

$$K = \left\{f \in A,\ \mathrm{Re}\,\frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U\right\},$$

denote the class of normalized convex functions in U.

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination

(i) $\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z),\ z \in U,$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (i).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (i) is said to be the best dominant of (i). (Note that the best dominant is unique up to a rotation of U).

To prove our main results, we need the following lemmas:

Lemma A. (Hallenbeck and Ruscheweyh [2, Th. 3.1.6, p.71]) *Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with* $\operatorname{Re} \gamma \geq 0$. *If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z), \quad z \in U$$

then

$$p(z) \prec g(z) \prec h(z), \quad z \in U$$

where

$$g(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1}dt, \quad z \in U.$$

Lemma B. (Miller and Mocanu [2]) *Let g be a convex function in U and let*

$$h(z) = g(z) + n\alpha zg'(z), \quad z \in U,$$

where $\alpha > 0$ and n is a positive integer.

If

$$p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \dots, \quad z \in U$$

is holomorphic in U *and*

$$p(z) + \alpha z p'(z) \prec h(z), \quad z \in U$$

then

$$p(z) \prec g(z)$$

and this result is sharp.

Definition 1. (Gr. Şt. Sălăgean [4]) *For* $f \in A$, $n \in \mathbb{N}^* \cup \{0\}$, *let* S^n *be the operator given by* $S^n : A \to A$

$$\begin{aligned} &S^0 f(z) = f(z) \\ &S^1 f(z) = z f'(z) \\ &\dots \\ &S^{n+1} f(z) = z[S^n f(z)]', \ z \in U. \end{aligned}$$

Remark 1. If $f \in A$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j,$$

then

$$S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j, \quad z \in U.$$

If n is replaced by a positive number, we obtain:

Definition 2. *For* $f \in A$, $\alpha \geq 0$, *let* S^α *be the operator given by* $S^\alpha : A \to A$

$$\begin{aligned} &S^0 f(z) = f(z) \\ &\dots \\ &S^\alpha f(z) = z[S^{\alpha-1} f(z)]', \ z \in U. \end{aligned}$$

Remark 2. If $f \in A$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j,$$

then

$$S^\alpha f(z) = z + \sum_{j=2}^{\infty} j^\alpha a_j z^j, \quad z \in U.$$

Definition 3. (Şt. Ruscheweyh [3]) *For $f \in A$, $n \in \mathbb{N}^* \cup \{0\}$, let R^n be the operator given by $R^n : A \to A$*

$$\begin{aligned}
&R^0 f(z) = f(z) \\
&R^1 f(z) = z f'(z) \\
&\ldots \\
&(n+1) R^{n+1} f(z) = z[R^n f(z)]' + n R^n f(z), \ z \in U.
\end{aligned}$$

Remark 3. If $f \in A$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j,$$

then

$$R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_j z^j, \quad z \in U.$$

If n is replaced by a positive real number, we obtain:

Definition 4. *For $f \in A$, $\alpha \geq 0$, let R^α be the operator given by $R^\alpha : A \to A$*

$$\begin{aligned}
&R^0 f(z) = f(z) \\
&\ldots \\
&(\alpha+1) R^{\alpha+1} f(z) = z[R^\alpha f(z)]' + \alpha R^\alpha f(z), \ z \in U.
\end{aligned}$$

Remark 4. If $f \in A$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j,$$

then

$$R^{\alpha}f(z)=z+\sum_{j=2}^{\infty}C_{\alpha+j-1}^{\alpha}a_jz^j,\quad z\in U.$$

DEFINITION 5. [1] *Let $n\in\mathbb{N}$ and $\lambda\geq 0$. Also let D_{λ}^n denote the operator defined by $D_{\lambda}^n:A\to A$*

$$\begin{aligned}
&D_{\lambda}^0f(z)=f(z),\\
&D_{\lambda}^1f(z)=(1-\lambda)f(z)+\lambda zf'(z)=D_{\lambda}f(z)\\
&\ldots\\
&D_{\lambda}^nf(z)=(1-\lambda)D_{\lambda}^{n-1}f(z)+\lambda z(D_{\lambda}^{n-1})'=D_{\lambda}[D_{\lambda}^{n-1}f(z)].
\end{aligned}$$

REMARK 5. [1] We observe that D_{λ}^n is a linear operator and for

$$f(z)=z+\sum_{j=2}^{\infty}a_jz^j$$

we have

$$D_{\lambda}^nf(z)=z+\sum_{j=2}^{\infty}[1+(j-1)\lambda]^na_jz^j.$$

Further, it is not difficult to deduce that if $\lambda=1$ in the above definition, then we obtain the Sălăgean differential operator.

2. MAIN RESULTS

DEFINITION 6. *Let $\alpha\geq 0$, $\lambda\geq 0$. Also let D_{λ}^{α} denote the operator given by*

$$D_{\lambda}^{\alpha}:A\to A,$$

$$D_{\lambda}^{\alpha}f(z)=(1-\lambda)S^{\alpha}f(z)+\lambda R^{\alpha}f(z),\quad z\in U.$$

Here S^{α} and R^{α} are the operators given by Definition 2 and Definition 4.

REMARK 6. We observe that D_{λ}^{α} is a linear operator and for

$$f(z)=z+\sum_{j=2}^{\infty}a_jz^j$$

we have

$$D_\lambda^\alpha f(z) = z + \sum_{j=2}^{\infty} [(1-\lambda) j^\alpha + \lambda C_{\alpha+j-1}^\alpha] a_j z^j, \quad z \in U.$$

Remark 7. For $\lambda = 0$, $D_0^\alpha f(z) = S^\alpha f(z)$, $\lambda = 1$, $D_1^\alpha f(z) = R^\alpha f(z)$, $z \in U$.

Remark 8. For $\alpha = 0$,

$$D_\lambda^0 f(z) = (1-\lambda) S^0 f(z) + \lambda R^0 f(z) = f(z) = R^0 f(z) = S^0 f(z), \quad z \in U$$

and for $\alpha = 1$,

$$D_\lambda^1 f(z) = (1-\lambda) S^1 f(z) + \lambda R^1 f(z) = z f'(z) = R^1 f(z) = S^1 f(z), \quad z \in U.$$

Theorem 1. *Let g be a convex function such that $g(0) = 1$, and let h be the function*

$$h(z) = g(z) + z g'(z), \quad z \in U.$$

If $\alpha \geq 0$, $\lambda \geq 0$, $f \in A$ and the differential subordination

$$[D_\lambda^{\alpha+1} f(z)]' + \frac{\lambda \alpha z (R^\alpha f(z))''}{\alpha + 1} \prec h(z), \quad z \in U \tag{1}$$

holds, then

$$[D_\lambda^\alpha f(z)]' \prec g(z), \quad z \in U.$$

This result is sharp.

Proof. By using the properties of operator D_λ^α, we obtain

$$D_\lambda^{\alpha+1} f(z) = (1-\lambda) S^{\alpha+1} f(z) + \lambda R^{\alpha+1} f(z), \quad z \in U. \tag{2}$$

Then (1) becomes

$$[(1-\lambda) S^{\alpha+1} f(z) + \lambda R^{\alpha+1} f(z)]' + \frac{\lambda \alpha z [R^\alpha f(z)]''}{\alpha + 1} \prec h(z), \quad z \in U. \tag{3}$$

After a short calculation, we obtain

$$(1-\lambda)[S^{\alpha+1}f(z)]' + \lambda[R^{\alpha+1}f(z)]' + \frac{\lambda\alpha z[R^{\alpha}f(z)]''}{\alpha+1} \prec h(z),\ z\in U. \quad (4)$$

Taking into account the properties of operators S^α and R^α, we deduce in view of (4) that

$$(1-\lambda)[z(S^{\alpha}f(z))']' + \lambda\frac{[z(R^{\alpha}f(z))' + \alpha R^{\alpha}f(z)]'}{\alpha+1} \quad (5)$$

$$+\frac{\lambda\alpha z[R^{\alpha}f(z)]''}{\alpha+1} \prec h(z), \quad z\in U.$$

Making an elementary computation into above, we obtain from (5) that

$$(1-\lambda)[S^{\alpha}f(z)]' + (1-\lambda)z[S^{\alpha}f(z)]'' \quad (6)$$

$$+\lambda\frac{[R^{\alpha}f(z)]' + \alpha[R^{\alpha}f(z)]' + z(R^{\alpha}f''(z))}{\alpha+1} + \frac{\lambda\alpha z[R^{\alpha}f(z)]''}{\alpha+1} \prec h(z),\ z\in U.$$

The above relation is equivalent to

$$(1-\lambda)[S^{\alpha}f(z)]' + \lambda[R^{\alpha}f(z)]' \quad (7)$$

$$+z[(1-\lambda)(S^{\alpha}f(z))'' + \lambda(R^{\alpha}f(z))''] \prec h(z), \quad z\in U.$$

Let

$$p(z) = (1-\lambda)[S^{\alpha}f(z)]' + \lambda[R^{\alpha}f(z)]' = [D^{\alpha}_{\lambda}f(z)]' \quad (8)$$

$$= (1-\lambda)\left[z+\sum_{j=2}^{\infty} j^{\alpha}a_j z^j\right]' + \lambda\left[z+\sum_{j=2}^{\infty} C^{\alpha}_{\alpha+j-1}a_j z^j\right]$$

$$= (1-\lambda)\left[1+\sum_{j=2}^{\infty} j^{\alpha+1}a_j z^{j-1}\right] + \lambda\left[1+\sum_{j=2}^{\infty} jC^{\alpha}_{\alpha+j-1}a_j z^{j-1}\right]$$

$$= 1+\sum[(1-\lambda)j^{\alpha+1} + \lambda jC^{\alpha}_{\alpha+j-1}]a_j z^{j-1} = 1+p_1z+p_2z^2+\dots$$

In view of (8), we deduce that $p\in\mathcal{H}[1,1]$.
Using the notation in (8), the differential subordination (7) becomes

$$p(z)+zp'(z) \prec h(z) = g(z)+zg'(z).$$

By using Lemma B, we have

$$p(z) \prec g(z), \quad z \in U$$

i.e.

$$[D_\lambda^\alpha f(z)]' \prec g(z), \quad z \in U$$

and this result is sharp.

EXAMPLE 1. If $\lambda = 1$, $\alpha = 1$, $f \in A$ we deduce that

$$zf'(z) + zf''(z) + \frac{z^2 f''(z)}{2} + \frac{z^2 f'''(z)}{2} \prec 1 + 2z,$$

which yields that

$$f'(z) + zf''(z) \prec 1 + z, \quad z \in U.$$

THEOREM 2. *Let g be a convex function, $g(0) = 1$ and let h be the function*

$$h(z) = g(z) + zg'(z), \quad z \in U.$$

If $f \in A$, $\alpha \geq 0$, $\lambda \geq 0$ and satisfies the differential subordination

$$[D_\lambda^\alpha f(z)]' \prec h(z), \quad z \in U \tag{9}$$

then

$$\frac{D_\lambda^\alpha f(z)}{z} \prec g(z), \quad z \in U$$

and this result is sharp.

Proof. Taking into account the properties of operator D_λ^α, we have

$$D_\lambda^\alpha f(z) = z + \sum_{j=2}^{\infty} [(1-\lambda) j^\alpha + \lambda C_{\alpha+j-1}^\alpha] a_j z^j.$$

Let

$$p(z) = \frac{D_\lambda^\alpha f(z)}{z} = \frac{z + \sum_{j=2}^{\infty} [(1-\lambda) j^\alpha + \lambda C_{\alpha+j-1}^\alpha] a_j z^j}{z} \tag{10}$$

$$= 1 + p_1 z + p_2 z^2 + \dots, \quad z \in U.$$

From (10) we have $p \in \mathcal{H}[1,1]$.

Let

$$D_\lambda^\alpha f(z) = zp(z), \quad z \in U. \tag{11}$$

Differentiating (11), we obtain

$$[D_\lambda^\alpha f(z)]' = p(z) + zp'(z), \quad z \in U. \tag{12}$$

Then (9) becomes

$$p(z) + zp'(z) \prec h(z), \quad z \in U. \tag{13}$$

By using Lemma B, we have

$$p(z) \prec g(z), \quad z \in U,$$

i.e.

$$\frac{D_\lambda^\alpha f(z)}{z} \prec g(z), \quad z \in U.$$

Example 2. For $\alpha = 1$, $\lambda = 1$, $f \in A$, we deduce that

$$f'(z) + zf''(z) \prec \frac{1 + 2z - z^2}{(1-z)^2}, \quad z \in U$$

implies

$$f'(z) \prec \frac{1+z}{1-z}, \quad z \in U.$$

Theorem 3. *Let*

$$h(z) = \frac{1 + (2\beta - 1)z}{1+z},$$

be convex in U, with $h(0) = 1$, $0 \le \beta < 1$.

Assume $\alpha \ge 0$, $\lambda \ge 0$, and $f \in A$ satisfies the differential subordination

$$[D_\lambda^{\alpha+1} f(z)]' + \frac{\lambda \alpha z [R^\alpha f(z)]''}{\alpha + 1} \prec h(z), \quad z \in U. \tag{14}$$

Then

$$[D_\lambda^\alpha f(z)]' \prec q(z), \quad z \in U,$$

where q is given by

$$q(z) = 2\beta - 1 + 2(1-\beta)\frac{\ln(1+z)}{z}, \quad z \in U. \tag{15}$$

The function q is convex and is the best dominant.

Proof. By following similar steps to those in the proof of Theorem 1 and using (8), the differential subordination (14) becomes:

$$p(z) + zp'(z) \prec h(z) = \frac{1+(2\beta-1)z}{1+z}, \quad z \in U.$$

In view of Lemma A, we have $p(z) \prec q(z)$, i.e.,

$$[D_\lambda^\alpha f(z)]' \prec q(z) = \frac{1}{z}\int_0^z h(t)t^{1-1}dt$$

$$= \frac{1}{z}\int_0^z \frac{1+(2\beta-1)t}{1+t}dt = \frac{1}{z}\int_0^z \left(2\beta - 1 + \frac{2(1-\beta)}{1+t}\right)dt$$

$$= 2\beta - 1 + 2(1-\beta)\frac{1}{z}\ln(z+1), \quad z \in U.$$

THEOREM 4. *Let $h \in \mathcal{H}(U)$ such that $h(0) = 1$ and*

$$\mathrm{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \quad z \in U.$$

If $f \in A$ satisfies the differential subordination

$$[D_\lambda^{\alpha+1} f(z)]' + \frac{\lambda\alpha z[R^\alpha f(z)]''}{\alpha+1} \prec h(z), \quad z \in U \tag{16}$$

then $[D_\lambda^\alpha f(z)]' \prec q(z), \quad z \in U$ where q is given by $q(z) = \frac{1}{z}\int_0^z h(t)dt$.
The function q is convex and is the best dominant.

Proof. A simple application of the differential subordination technique [2, Corollary 2.6.g.2, p.66] yields that the function g is convex.
By using the properties of operator D_λ^α, in (8), we obtain after a simple computation that

$$[D_\lambda^{\alpha+1} f(z)]' + \frac{\lambda\alpha z[R^\alpha f(z)]''}{\alpha+1} = p(z) + zp'(z), \quad z \in U.$$

Then (16) becomes

$$p(z) + zp'(z) \prec h(z), \quad z \in U. \tag{17}$$

Since $p \in \mathcal{H}[1,1]$, we deduce in view of Lemma A that $p(z) \prec q(z)$, where $q(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U$, i.e. $[D_\lambda^\alpha f(z)]' \prec q(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U$, and q is the best dominant.

Example 3. Since $\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, for $h(z) = \frac{z^2+2z}{2(1-z)^2}$, $z \in U$, and $\alpha = 0$, $\lambda = 1$, we deduce that $f'(z) + zf''(z) \prec \frac{z^2+2z}{2(1-z)^2}, \quad z \in U$, implies $f'(z) \prec \frac{1}{2} + \frac{3}{2} \cdot \frac{1}{z(1-z)} + \frac{\ln(z^2-2z+1)}{z}, \quad z \in U$.

Theorem 5. *Let $h \in \mathcal{H}(U)$ be such that $h(0) = 1$ and*

$$\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}, \quad z \in U.$$

If $f \in A$ satisfies the differential subordination

$$[D_\lambda^\alpha f(z)]' \prec h(z), \quad z \in U \tag{18}$$

then

$$\frac{D_\lambda^\alpha f(z)}{z} \prec q(z), \quad z \in U$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function q is convex and is the best dominant.

Proof. A simple application of the differential subordination technique [2, Corollary 2.6.g.2, p.66] yields that the function q is convex.

Let

$$p(z) = \frac{D_\lambda^\alpha f(z)}{z} = \frac{z + \sum_{j=2}^{\infty} p_j z^j}{z} = 1 + \sum_{j=2}^{\infty} p_j z^{j-1}, \tag{19}$$

$z \in U$, $p \in \mathcal{H}(1,1)$.

Differentiating both sides in (19), we obtain that

$$[D_\lambda^\alpha f(z)]' = p(z) + zp'(z), \quad z \in U. \tag{20}$$

Hence (1) becomes $p(z) + zp'(z) \prec h(z), \quad z \in U$.

Since $p \in \mathcal{H}[1, 1]$, we deduce in view of Lemma A that

$$p(z) \prec q(z) = \frac{1}{z} \int_0^z h(t)dt,$$

i.e.

$$\frac{D_\lambda^\alpha f(z)}{z} \prec q(z) = \frac{1}{z} \int_0^z h(t)dt$$

and q is the best dominant.

Example 4. Since Re $\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, for $h(z) = e^{\frac{3}{2}z} - 1$, and $\alpha = 1$, $\lambda = 1$, we deduce that $f'(z) + zf''(z) \prec e^{\frac{3}{2}z} - 1, \quad z \in U$. This subordination yields that $f'(z) \prec \frac{2}{3}\frac{e^{\frac{3}{2}z}}{z} - 1, \quad z \in U$.

References

[1] F.M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Ind. J. Math. Math. Sci., 2004, no.25-28, 1429-1436.

[2] S. S. Miller, P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.

[3] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., 49(1975), 109-115.

[4] Grigore Şt. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, 1013(1983), 362-372.

Authors:

Georgia Irina Oros, Gheorghe Oros
Department of Mathematics, University of Oradea
Str. Universităţii, No.1, 410087 Oradea, Romania
email: georgia_oros_ro@yahoo.co.uk