CEBÎŞEV INEQUALITY FOR RANDOM GRAPHS

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Abstract. Using our new Boolean representing for random graphs, dated since 1995 and introduced in [5], [6], [7], we apply it, at *Cebîşev inequality* and a *weak convergence* in probability.

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1. Preliminaries

We shall consider, as in [1], unoriented simple graphs, with n vertices, M edges, without loops, given by the set:

$$\mathcal{G}(n, M) = \{G = (V(G), E(G)) \Big| \, |V(G)| = n, |E(G)| = M\} \,,$$

whose number of elements is:

$$|\mathcal{G}(n, M)| = \binom{N}{M} \,,$$

where

$$N = \binom{n}{2} \,.$$

If all the simple graphs, without loops and having n vertices, are denoted by:

$$\mathcal{G}(n, M) = \{G = (V(G), E(G)) \Big| \, |V(G)| = n\} \,,$$

their number is obviously:

$$|\mathcal{G}(n)| = \sum_{M=0}^{N} \binom{N}{M} = 2^N .$$

As in [5], [6], [7], we rewrite the set $\mathcal{G}(n)$, using for a graph $G \in \mathcal{G}(n)$, its *state vectorial variable* $x_G \in \{0,1\}^N$, with:

$$x_G = (x_{12}, \ldots, x_{1n}; x_{23}, \ldots, x_{2n}; \ldots; x_{n-2,n-1}, x_{n-2,n}; x_{n-1,n}), \qquad (1)$$

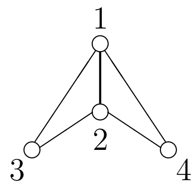
where $x_{ij} \in \{0,1\}, (\forall) i \in \{1, \ldots, n-1\}, (\forall) j \in \{i+1, \ldots, n\}$, and:

$$x_{i,j} = \begin{cases} 1, \text{ iff } [i,j] \text{ exists,} \\ 0, \text{ else.} \end{cases}$$

So:

$$\mathcal{G}(n) = \{x_G \mid x_G \in \{0,1\}^N \text{ respecting (1)}\}.$$

By example, the next graph $G \in \mathcal{G}(n)$, with $|E(G)| = 5$,



will have the following *state vector* $x_G \in \{0,1\}^6$:

$$x_G = (1,1,1;1,1;0) \Longleftrightarrow x_G = (1,1,1,1,1,0) .$$

Passing now to *random graphs*, we remind that there are two cases(see [2], [3], [9]). The simplest is that in which every graph $G \in \mathcal{G}(n, M)$ has the same probability $p(G, n, M)$. The second case is related to the situation in which the edges, of $G \in \mathcal{G}(n)$, are independently chosen, with the same probability $p \in (0,1)$. So, $q = 1 - p \in (0,1)$ will be the probability as an edge does not belong to G.

Looking now at our anterior example, if this graph will be considered as a random one, the edges [1,2], [1,3], [1,4], [2,3], [2,4] will be marked with the probability p and the inexistent edge [3,4] will be considered having the probability $q = 1 - p$.

As in [5], [6], [7], a random graph $G \in \mathcal{G}(n)$ is the following *random vectorial variable*:

$$X_G = (X_{12}, \ldots, X_{1n}; X_{23}, \ldots, X_{2n}; \ldots; X_{n-2,n-1}, X_{n-2,n}; X_{n-1,n}), \qquad (2)$$

where

$$X_{i,j} = \begin{cases} 1, \text{ iff } x_{ij} = 1, \ (\forall)\, i \in \{1, ..., n-1\} \\ 0, \text{ iff } x_{ij} = 0, \ (\forall)\, j \in \{i+1, ..., n\}. \end{cases}$$

These state random variables X_{ij} are surely independent because the state of any edge is not depending on the another else edge. More, because the random variable X_G, which denotes the random graph G, has its components $X_{ij} = 1$, only if the edge $[i, j]$ exists, moment when it receives the probability p, it occurs that for every $i \in \{1, \ldots, n-1\}$, anf for every $j \in \{i+1, \ldots, n\}$,

$$X_{ij} : \begin{pmatrix} 1 & 0 \\ p & q \end{pmatrix}, p, q \in (0, 1). \qquad (3)$$

Let $l(X_G)$ be the *edge numbers* of a random graph $G \in \mathcal{G}(n)$.
We have immediately (see [7]):

$$l(X_G) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}. \qquad (4)$$

Because X_{ij} are independent, obviously $l(X_G) \in B(N, p)$, namely, it is a *binomial random variable*, with the mean value:

$$E(l(X_G)) = Np$$

and with the variance:

$$\sigma(l(X_G)) = \sqrt{Npq}.$$

From here, it occurs that the *average edge number* for the random graphs $G \in \mathcal{G}(n)$ situated in the second case is Np. The set of all these random graphs (see [7]) will be denoted by:

$$\mathcal{R}(n) = \{X_G | G \in \mathcal{G}(n), X_G \text{ as in } (2), X_{ij} \text{ as in } (3)\} .$$

2. Cebîşev inequality

In the following, we shall study Cebîşev inequality (see [8]) applied to random graphs. For $l(X_G) \in B(N,p)$ when its mean-value and its variance are finite, namely for a fixed n and for every $\varepsilon > 0$, $p, q \in (0,1)$, we can write the inequality of Cebîşev:

$$P(\{X_G \Big| \, |l(X_G) - Np| \geq \varepsilon\}) < \frac{Npq}{\varepsilon^2} , \tag{5}$$

or its equivalent form:

$$P(\{X_G \Big| \, |l(X_G) - Np| < \varepsilon\}) \geq \frac{Npq}{\varepsilon^2} . \tag{6}$$

From these, the *rule of 3σ* will be quickly obtained when $\varepsilon = 3\sigma = 3\sqrt{Npq}$. In our case, we shall have:

$$P(\{X_G \Big| \, |l(X_G) - Np| \geq 3\sqrt{Npq}\}) < \frac{1}{9Npq} , \tag{7}$$

$$P(\{X_G \Big| \, |l(X_G) - Np| < 3\sqrt{Npq}\}) \geq \frac{1}{9Npq} . \tag{8}$$

Using the general facts of 3σ-*rule*, we can say that, at least 88,8% of the values, for the *random variable* $l(X_G)$, are in the interval $(Np+3\sqrt{Npq}, Np-3\sqrt{Npq})$. Otherwise, it occurs that the percent of the random graphs which have not Np, as average for their edges, is not bigger than 11,2 %.

From (4), we obviously state that $l(X_G)$ is a random variable, which depends on n. So it is a variable which be thought as $l_n(X_G)$, although for writing facilities, we only note $l(X_G)$. In these conditions (see [4]), passing on limit in (7), we have:

$$0 \leq \lim_{n\to\infty} P(\{X_G \Big| \, |l(X_G) - Np| \geq 3\sqrt{Npq}\}) \leq \lim_{n\to\infty} \frac{1}{9Npq} = 0,$$

or equivalently:

$$\lim_{n\to\infty} P(\{X_G \Big| \, |l(X_G) - Np| \geq 3\sqrt{Npq}\}) = 0\,,$$

from where, it occurs that $(l_n(X_G))_n \equiv (l(X_G))_n$ is a sequence which *converges in probability* to Np, this being a *weak convergence* (see [8]).

References

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