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WARPED PRODUCT SUBMANIFOLDS IN QUATERNION SPACE FORMS

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Abstract. B.Y. Chen [3] established a sharp inequality for the warping function of a warped product submanifold in a Riemannian space form in terms of the squared mean curvature. For a survey on warped product submanifolds we refer to [4].
In [8], we established a similar relationship between the warping function f (intrinsic structure) and the squared mean curvature and the holomorphic sectional curvature (extrinsic structures) for warped product submanifolds $M_1 \times_f M_2$ in any complex space form.
In the present paper, we investigate warped product submanifolds in quaternion space forms $\widetilde{M}^m(4c)$. We obtain several estimates of the mean curvature in terms of the warping function, whether $c < 0$, $c = 0$ and $c > 0$, respectively. Equality cases are considered and certain examples are given.
As applications, we derive obstructions to minimal warped product submanifolds in quaternion space forms. As an example, the non-existence of minimal proper warped product submanifolds $M_1 \times_f M_2$ in the m-dimensional quaternion Euclidean space $\mathbf{Q}^m$ with M_1 compact is proved.

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Introduction

The notion of *warped product* plays some important role in differential geometry as well as in physics [3]. For instance, the best relativistic model of the Schwarzschild space-time that describes the out space around a massive star or a black hole is given as a warped product.

One of the most important problems in the theory of submanifolds is the immersibility (or non-immersibility) of a Riemannian manifold in a Euclidean

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space (or, more generally, in a space form). According to a well-known theorem of Nash, every Riemannian manifold can be isometrically immersed in some Euclidean spaces with suficiently high codimension.

Nash's theorem implies, in particular, that every warped product $M_1 \times_f M_2$ can be immersed as a Riemannian submanifold in a certain Euclidean space. Moreover, many important submanifolds in real, complex and quaternion space forms are expressed as warped products.

Every Riemannian manifold of constant curvature c can be locally expressed as a warped product whose warping function satisfies $\Delta f = cf$. For example, $S^n(1)$ is locally isometric to $(0, \pi) \times_{\cos t} S^{n-1}(1)$, $\mathbf{E}^n$ is locally isometric to $(0, \infty) \times_x S^{n-1}(1)$ and $H^n(-1)$ is locally isometric to $\mathbf{R} \times_{e^x} \mathbf{E}^{n-1}$ (see [4]).

1. Preliminaries

Let $\overline{M}^m$ be a $4m$-dimensional Riemannian manifold with metric g. $\overline{M}^m$ is called a *quaternion Kaehlerian manifold* if there exists a 3-dimensional vector space E of tensors of type $(1, 1)$ with local basis of almost Hermitian structures ϕ_1, ϕ_2 and ϕ_3, such that
(i) $\phi_1\phi_2 = -\phi_2\phi_1 = \phi_3$, $\phi_2\phi_3 = -\phi_3\phi_2 = \phi_1$, $\phi_3\phi_1 = -\phi_1\phi_3 = \phi_2$,
(ii) for any local cross-section ϕ of E and any vector X tangent to $\overline{M}$, $\overline{\nabla}_X \phi$ is also a cross-section in E (where $\overline{\nabla}$ denotes the Riemannian connection in $\overline{M}$) or, equivalently, there exist local 1-forms p, q, r such that

$$\overline{\nabla}_X \phi_1 = r(X)\phi_2 - q(X)\phi_3,$$

$$\overline{\nabla}_X \phi_2 = -r(X)\phi_1 + p(X)\phi_3,$$

$$\overline{\nabla}_X \phi_3 = q(X)\phi_1 - p(X)\phi_2.$$

If X is a unit vector in $\overline{M}$, then X, $\phi_1 X$, $\phi_2 X$ and $\phi_3 X$ form an orthonormal set on $\overline{M}$ and one denotes by $Q(X)$ the 4-plane spanned by them. For any orthonormal vectors X, Y on $\overline{M}$, if $Q(X)$ and $Q(Y)$ are orthogonal, the 2-plane $\pi(X, Y)$ spanned by X, Y is called a *totally real plane.* Any 2-plane in $Q(X)$ is called a *quaternionic plane.* The sectional curvature of a quaternionic plane π is called a *quaternionic sectional curvature.* A quaternion Kaehler manifold $\overline{M}$ is a *quaternion space form* if its quaternionic sectional curvatures are constant.

It is well known that a quaternion Kaehlerian manifold $\overline{M}$ is a quaternion space form $\overline{M}(c)$ if and only if its curvature tensor $\overline{R}$ has the following form (see [6])

$$\overline{R}(X,Y)Z = \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + \tag{1}$$

$$+\mathrm{g}(\phi_1 Y, Z)\phi_1 X - g(\phi_1 X, Z)\phi_1 Y + 2g(X, \phi_1 Y)\phi_1 Z +$$

$$+\mathrm{g}(\phi_2 Y, Z)\phi_2 X - g(\phi_2 X, Z)\phi_2 Y + 2g(X, \phi_2 Y)\phi_2 Z +$$

$$+\mathrm{g}(\phi_3 Y, Z)\phi_3 X - g(\phi_3 X, Z)\phi_3 Y + 2g(X, \phi_3 Y)\phi_3 Z\},$$

for vectors X, Y, Z tangent to $\overline{M}$.

A submanifold M of a quaternion Kaehler manifold $\overline{M}$ is called *quaternion* (resp. *totally real)* submanifold if each tangent space of M is carried into itself (resp. the normal space) by each section in E.

The curvature tensor R of M is related to the curvature tensor $\overline{R}$ of $\overline{M}$ by the Gauss equation

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(h(X,Z),h(Y,W)) + g(h(X,W),h(Y,Z)), \tag{2}$$

where h is the second fundamental form of M.

Definition *[1]. A submanifold M of a quaternion Kaehler manifold $\overline{M}$ is called a quaternion CR-submanifold if there exist two orthogonal complementry distributions D and $D^{\perp}$ such that D is invariant under quaternion structures, that is, $\phi_i(D_x) \subseteq D_x$, $i = 1,2,3, \forall x \in M$, and $D^{\perp}$ is totally real, that is, $\phi_i(D_x^{\perp}) \subseteq T_x^{\perp}M$, $i = 1,2,3, \forall i = 1,2,3$.*

A submanifold M of a quaternion Kaehler manifold $\overline{M}$ is a quaternion submanifold (resp. totally real submanifold) if dim $D^{\perp} = 0$ (resp. dim $D = 0$).

For any vector field X tangent to M, we put

$$\phi_i X = P_i X + F_i X, \qquad i = 1,2,3. \tag{3}$$

where $P_i X$ (resp. $F_i X$) denotes tangential (resp. normal) component of $\phi_i X$.

Let M be an n-dimensional submanifold in a quaternion space form $\overline{M}(c)$. Let ∇ be the Riemannian connection of M, h the second fundamental form and R the Riemann curvature tensor of M.

Let $p \in M$ and let $\{e_1, ..., e_n, ..., e_{4m}\}$ be an orthonormal basis of the tangent space $T_p\overline{M}$, such that $e_1, ..., e_n$ are tangent to M at p. One denotes by H the mean curvature vector, that is

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i). \tag{4}$$

For a differentiable function f on M, the Laplacian Δf of f is defined by

$$\Delta f = \sum_{j=1}^{n}\{(\nabla_{e_j} e_j)f - e_j e_j f\}. \tag{5}$$

We recall the following result of Chen for later use.

Lemma 1. *[2]. Let $n \geq 2$ and $a_1, ..., a_n, b$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right).$$

Then $2a_1a_2 \geq b$, with equality holding if and only if

$$a_1 + a_2 = a_3 = ... = a_n.$$

Let M be a quaternion CR-submanifold of a quaternion space form $\overline{M}(c)$. Then from Gauss equation one derives

$$R(X, Y, Z, W) = \frac{c}{4}\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)+$$

$$+\sum_{i=1}^{3}[g(P_iY, Z)g(P_iX, W) - g(P_iX, Z)g(P_iY, W) + 2g(X, P_iY)g(P_iZ, W)]\}$$

$$+g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)).$$

for any vector fields X, Y, Z, W tangent to M.

2. Warped product submanifolds

Chen established a sharp relationship between the warping function f of a warped product $M_1 \times_f M_2$ isometrically immersed in a real space form $\widetilde{M}(c)$ and the squared mean curvature $\|H\|^2$ (see [3]). In [8], we gave a corresponding relationship between the warping function f (intrinsic structure) and the squared mean curvature and the holomorphic sectional curvature (extrinsic structures) for warped product submanifolds $M_1 \times_f M_2$ in any complex space form.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f a positive differentiable function on M_1. The *warped product* of M_1 and M_2 is the Riemannian manifold

$$M_1 \times_f M_2 = (M_1 \times M_2, g),$$

where $g = g_1 + f^2 g_2$ (see, for instance, [3]).

Let $x : M_1 \times_f M_2 \to \overline{M}(c)$ be an isometric immersion of a warped product $M_1 \times_f M_2$ into a quaternion space form $\overline{M}(c)$. We denote by h the second fundamental form of x and $H_i = \frac{1}{n_i} \operatorname{trace} h_i$, where trace h_i is the trace of h restricted to M_i and $n_i = \dim M_i$ $(i = 1, 2)$. The vector fields H_i are called *partial mean curvatures.*

For a warped product $M_1 \times_f M_2$, we denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the distributions given by the vectors tangent to leaves and fibres, respectively. Thus, $\mathcal{D}_1$ is obtained from the tangent vectors of M_1 via the horizontal lift and $\mathcal{D}_2$ by tangent vectors of M_2 via the vertical lift.

Let $M_1 \times_f M_2$ be a warped product submanifold into a quaternion space form $\overline{M}(c)$.

Since $M_1 \times_f M_2$ is a warped product, it is known that

$$\nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)Z, \tag{6}$$

for any vector fields X, Z tangent to M_1, M_2, respectively.

If X and Z are unit vector fields, it follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}\{(\nabla_X X)f - X^2 f\}. \tag{7}$$

Using the above Lemma and the Gauss equation (see [9]), one gets the following.

LEMMA 2. *Let $x : M_1 \times_f M_2 \to \overline{M}(c)$ be an isometric immersion of an n-dimensional warped product into a $4m$-dimensional quaternion space form $\overline{M}(c)$. Then*

$$n_2 \frac{\Delta f}{f} \leq \frac{n^2}{4} \|H\|^2 + n_1 n_2 \frac{c}{4} + 3\frac{c}{4} \sum_{\alpha=1}^{3} \sum_{i=1}^{n_1} \sum_{s=n_1+1}^{n} g^2(P_\alpha e_i, e_s), \quad (8)$$

where Δ *is the Laplacian operator of* M_1.

We distinguish the following cases:

Theorem 1. *Let* $x : M_1 \times_f M_2 \to \overline{M}(c)$ *be an isometric immersion of an* n*-dimensional warped product into a* $4m$*-dimensional quaternion space form* $\overline{M}(c)$ *with* $c < 0$. *Then*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2 + n_1 \frac{c}{4}.$$

Moreover, the equality case holds identically if and only if x *is a mixed totally geodesic immersion,* $n_1 H_1 = n_2 H_2$ *and* $\phi_k \mathcal{D}_1 \perp \mathcal{D}_2$*, for any* $k = 1, 2, 3$.

As applications, one derives certain obstructions to the existence of minimal warped product submanifolds in quaternion hyperbolic space.

Corollary 1.1. *If* f *is a harmonic function on* M_1*, then the warped product* $M_1 \times_f M_2$ *does not admit any isometric minimal immersion into a quaternion hyperbolic space.*

Corollary 1.2. *There do not exist minimal warped product submanifolds in a quaternion hyperbolic space with* M_1 *compact.*

Theorem 2. *Let* $x : M_1 \times_f M_2 \to \overline{M}(c)$ *be an isometric immersion of an* n*-dimensional warped product into a* $4m$*-dimensional flat quaternion space form. Then*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2.$$

Moreover, the equality case holds identically if and only if x *is a mixed totally geodesic immersion and* $n_1 H_1 = n_2 H_2$.

Corollary 2.1. *If* f *is an eigenfunction of Laplacian on* M_1 *with corresponding eigenvalue* $\lambda > 0$*, then the warped product* $M_1 \times_f M_2$ *does not admit any isometric minimal immersion into a quaternion hyperbolic space or a quaternion Euclidean space.*

A warped product is said to be proper if the warping function is non-constant.

Corollary 2.2. *There do not exist minimal proper warped product submanifold in the quaternion Euclidean space* $\mathbf{Q}^m$ *with* M_1 *compact.*

Theorem 3. *Let* $x : M_1 \times_f M_2 \to \overline{M}(c)$ *be an isometric immersion of an* n*-dimensional warped product into a* $4m$*-dimensional quaternion space form* $\overline{M}(c)$ *with* $c > 0$*. Then*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1\frac{c}{4} + 3\frac{c}{4}\min\{\frac{n_1}{n_2}, 1\}.$$

Moreover, the equality case holds identically if and only if x *is a mixed totally geodesic immersion,* $n_1 H_1 = n_2 H_2$ *and* $\phi_k \mathcal{D}_1 \perp \mathcal{D}_2$*, for any* $k = 1, 2, 3$.

Also, Lemma 2 implies another inequality for certain submanifolds (in particular quaternion CR-submanifolds) in quaternion space forms with $c > 0$.

Theorem 4. *Let* $x : M_1 \times_f M_2 \to \overline{M}(c)$ *be an isometric immersion of an* n*-dimensional warped product into a* $4m$*-dimensional quaternion space form* $\overline{M}(c)$ *with* $c > 0$*, such that* $\phi_k \mathcal{D}_1 \perp \mathcal{D}_2$*, for any* $k = 1, 2, 3$*. Then*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1\frac{c}{4}.$$

Moreover, the equality case holds identically if and only if x *is a mixed totally geodesic immersion and* $n_1 H_1 = n_2 H_2$.

Next, we will give some examples which satisfy identically the equality case of the inequality given in Theorem 4.

Example 1. *Let* $\psi : S^n \to S^{4n+3}$ *be an immersion defined by*

$$\psi(x^1, ..., x^{n+1}) = (x^1, 0, 0, 0, x^2, 0, 0, 0, ..., x^{n+1}, 0, 0, 0),$$

and $\pi : S^{4n+3} \to P^n(\mathbf{Q})$ *the Hopf submersion.*

Then $\pi \circ \psi : S^n \to P^n(\mathbf{Q})$ *satisfies the equality case.*

Example 2. *On* $S^{n_1+n_2}$ *let consider the spherical coordinates* $u_1, ..., u_{n_1+n_2}$ *and on* S^{n_1} *the function*

$$f(u_1, ...u_n) = \cos u_1 ... \cos u_{n_1},$$

(f is an eigenfunction of Δ).

Then $S^{n_1+n_2} = S^{n_1} \times_f S^{n_2}$.

Let $\psi : S^{n_1+n_2} \to S^{4(n_1+n_2)+3}$ *be the above standard immersion and* π *the Hopf submersion* $\pi : S^{4(n_1+n_2)+3} \to P^{n_1+n_2}(\mathbf{Q})$.

Then $\pi \circ \psi : S^{n_1+n_2} \to P^{n_1+n_2}(\mathbf{Q})$ *satisfies the equality case.*

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