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POLYNOMIAL IDENTITIES IN SUPERALGEBRAS WITH SUPERINVOLUTIONS [1]

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The talk is a survey on results concerning the nature of the identities fulfilled in some special types of matrix algebras. We consider the superalgebra $M(2)$ of the square matrices of order 4 over a field K of characteristics 0 for which

$$M(2) = A_0 \oplus A_1,\ A_\alpha A_\beta \leq A_{\alpha+\beta}\, (\alpha, \beta \in Z_2).$$

The grading in the general case of the algebra of the square matrices $M_{r+s}(K)$ is determined in the following way:

$$M(r \mid s)_0 = \{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \mid A \in M_r(K), D \in M_s(K)\},$$

$$M(r \mid s)_1 = \{ \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \mid B \in M_{r,s}(K), C \in M_{s,r}(K)\}.$$

For $r = s = n$ we denote this algebra as $M(n)$.

Let A be an associative superalgebra. A superinvolution on A is a Z_2-graded linear map $* : A \to A$ such that, for all $a, b \in A, (a^*)^* = a$ and $(ab)^* = (-1)^{\bar{a}\bar{b}} b^* a^*$, where $\bar{x}$ means the parity of x; $\bar{x} = i$ if $x \in A_i, i = 0, 1$.

Defining two superinvolutions in the considered superalgebra we introduce the notion of symmetric and skew-symmetric due to any of the involutions variables.

We have $x = \frac{x+x^*}{2} + \frac{x-x^*}{2}$ and thus any countable set X could be written as $X = Y \cup Z$, where Y is the set the symmetric elements of X and Z is the set of the skew-symmetric elements. For an algebra R we have $R = R^+ \oplus R^-$, where $R^+ = \{r \mid r^* = r\}$ and $R^- = \{r \mid r^* = -r\}$.

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Definition 1. *The polynomial $f(x_1, ..., x_n)$ from $K\langle Y\rangle$ is an identity for the algebra R in symmetric variables with respect to the considered (super)involution if $f(r_1^+, ..., r_n^+) = 0$ for any $r_i^+ \in R^+, i = 1, ..., n$.*

The polynomial $f(x_1, ..., x_n)$ from $K\langle Z\rangle$ is respectively an identity in skew-symmetric variables (with respect to the considered (super) involution) if $f(r_1^-, ..., r_n^-) = 0$ for any $r_i^- \in R^-, i = 1, ..., n$.

Examples of superinvolutions are **the orthosymplectic superinvolution** *osp*, defined by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{osp} = \begin{bmatrix} H & 0 \\ 0 & K \end{bmatrix}^{-1} \begin{bmatrix} A & -B \\ C & D \end{bmatrix}^{t} \begin{bmatrix} H & 0 \\ 0 & K \end{bmatrix},$$

where H is a symmetric matrix and K is a skew-symmetric one, both invertible, and **the transposition superinvolution** *trp*, defined by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{trp} = \begin{bmatrix} D^t & -B^t \\ C^t & A^t \end{bmatrix}.$$

In the talk, given at Antalya Algebra Days VII [5], it was proved that up to some known superpositions of mappings these are the only two superinvolutions for the superalgeba $M(2)$.

We define a class of homogeneous associative polynomials, called Bergman polynomials [1], and try to give an explicit form for those of them which are polynomial identities either in symmetric or skew-symmetric variables. These are homogeneous and multilinear in $y_1, \ldots, y_n$ polynomials $f(x, y_1, \ldots, y_n)$ from the free associative algebra $K\langle x, y_1, \ldots, y_n\rangle$ which can be written as

$$f(x, y_1, \ldots, y_n) = \sum_{i=(i_1,\ldots,i_n)\in Sym(n)} v(g_i)(x, y_{i_1}, \ldots, y_{i_n}), \tag{1}$$

where $g_i \in K[t_1, \ldots, t_{n+1}]$ are homogeneous polynomials in commuting variables

$$g_i(t_1, \ldots, t_{n+1}) = \sum \alpha_p t_1^{p_1} \ldots t_{n+1}^{p_{n+1}}$$

and

$$v(g_i) = v(g_i)(x, y_{i_1}, \ldots, y_{i_n}) = \sum \alpha_p x^{p_1} y_{i_1} \ldots x^{p_n} y_{i_n} x^{p_{n+1}}. \tag{2}$$

Concerning skew-symmetric variables to any of the above two superinvolutions we could say the following:

Proposition 1.[Lemma 2.9][4]. *If a polynomial $v(g)$ of type* (2) *is an identity for $M(2)$ in skew-symmetric variables with respect to the orthosymplectic superinvolution then the commutative polynomial $(t_1-t_2)(t_1^2-t_3^2)(t_2^2-t_3^2)$ divides the polynomial g.*

Proposition 2.[Lemma 2.8][4].*If a polynomial $v(g)$ of type* (2) *is an identity for $M(2)$ in skew-symmetric variables with respect to the transposition superinvolution then the commutative polynomial $(t_1^2-t_2^2)(t_2^2-t_3^2)(t_1^2-t_3^2)$ divides the polynomial g.*

We could interpret the definition of an identity in symmetric (or skew-symmetric) variables as of a graded identity, i.e.

Definition 2. *A polynomial $f(x_1,...,x_n)$ from the associative algebra $K\langle X\rangle$ is a graded identity for $M(n)$ if $f(r_1,...,r_n)=0$ for any $r_i\in M(n)_0, i=1,...,n$ (respectively for any $r_i\in M(n)_1, i=1,...,n$).*

Now we look for graded identities for $M(2)$.

If we denote by $P71(x,y_1,y_2)$ the associative polynomial from Proposition 1 using the system for computer algebra *Mathematica 5*, we calculate that

$$P71(e_{11}-e_{12}+e_{21}+e_{22},e_{12},e_{21}-e_{11}+e_{12})=-8e_{11}-16e_{21}-8e_{22}.$$

$$P71(e_{13}-e_{23}+e_{14}+e_{32},e_{31}-e_{23},e_{32}-e_{41}+e_{42})=3e_{13}+e_{31}+e_{32}.$$

The first evaluation shows that the considered polynomial is not a graded identity on the even part A_0 while the second result means that $P71(x,y_1,y_2)$ is not a graded identity on the odd part A_1.

The first evaluation shows a result from [4], namely Theorem 2.7 [4] could be improved. We prove the following

Theorem 1. *The least possible degree of a Bergman identity for M_2 in skew-symmetric variables with respect to the orthosymplectic superinvolution is 8.*

Proof: Theorem 2.7 [4] gives 7 as the least possible value of the degree of such an identity and gives its explicit form, namely $P72(x,y_1,y_2)$. Let K be the set of the skew-symmetric elements with respect to the considered superinvolution. Obviously $P72(x,y_1,y_2)$ has to an identity on $K\cap A_0$ as well.

We find the basis of $K\cap A_0$. These are the matrices $e_{11}-e_{33}, e_{22}-e_{44}, e_{12}-e_{43}$ and $e_{21}-e_{34}$. But

$$P72(e_{11}-e_{33}, e_{22}-e_{44}+2\,(e_{21}-e_{34}, e_{12}-e_{43})) = 2e_{11}-2e_{33}.$$

This ends the proof of the theorem.

There is another polynomial of degree 7 interesting for our investigations. The reason for this comes from the symplectic case for $M_4(K)$.

We recall that in the matrix algebra $M_{2n}(K,*)$ the symplectic involution $*$ is defined by

$$\left(\begin{array}{cc} A & B \\ C & D \end{array}\right)^* = \left(\begin{array}{cc} D^t & -B^t \\ -C^t & A^t \end{array}\right),$$

where A, B, C, D are $n\times n$ matrices and t is the usual transpose.

According to [Proposition 3 [3]] the Bergman identity in skew-symmetric variables for the matrix algebra $M(K,*)$ is of minimal degree 7 and its corresponding commutative polynomial is $(t_1^2-t_2^2)(t_2^2-t_3^2)(t_1-t_3)$. We point that this associative polynomial is the linearization in y of the $*$-identity $[[x^2,y]^2,x]=0$ in skew-symmetric variables found by Giambruno and Valenti in [2].

We denote by $P72(x,y_1,y_2)$ this associative polynomial using the system for computer algebra *Mathematica 5*, we calculate that

$$P72(e_{13}-e_{23}+e_{14}+e_{32}, e_{24}-e_{23}, e_{32}-e_{41}+e_{42}) = 2e_{13}-2e_{14}-2e_{23}-2e_{32},$$

meaning that $P72(x,y_1,y_2)$ is not a graded identity for A_1.

The considered polynomials of degree 7 are a step in finding the minimal degree and the explicit form of the Bergman graded identities for $M(2)$.

A very often used after its publication a theorem of Bergman [1] and a well known result that a $*$-identity for $M_{2n}(K,*)$ is an ordinary identity for $M_n(K)$ define the least possible degree as 5 and the corresponding commutative polynomial as $g(t_1,t_2,t_3) = (t_1-t_2)(t_1-t_3)(t_2-t_3)$. This identity is the linearization in y of the well known identity $[[x,y]^2,x]=0$, followed from the Cayley-Hamilton theorem for $M_2(K)$. We denote this polynomial as $P5(x,y_1,y_2)$ and calculate it on the even and on the odd part of $M(2)$. The result is:

$P5(e_{11}, e_{12}, e_{21}) = e_{11}$

$P5(e_{11}-e_{12}+e_{21}+e_{22}, e_{12}, e_{21}-e_{11}+e_{12}) = -10e_{11}-4e_{12}-2e_{22}$

$P5(e_{13}-e_{23}+e_{14}+e_{32}, e_{31}-e_{23}, e_{32}-e_{41}+e_{42}) =$
$2e_{12}+4e_{13}-e_{22}-2e_{23}-2e_{31}-e_{33}-2e_{42}.$

If there is a graded Bergman type identity $P51(x, y_1, y_2)$ of degree 6, the corresponding commutative polynomial has to be

$$(t_1 - t_2)(t_2 - t_3)(t_1 - t_3)(at_1 + bt_2 + ct_3), \quad a, b, c \in K.$$

The explicit form of the polynomial is

$$P51(x, y_1, y_2) = ax.P5(x, y_1, y_2) + cP5(x, y_1, y_2).x + b(P5(x, y_1.x, y_2) + P5(x, y_1, y_2.x).$$

Calculating $P51(e_{11} - e_{12} + e_{21} + e_{22}, e_{12}, e_{21} - e_{11} + e_{12})$ we get the system:

$$\begin{aligned} -10a - 32b - 14c &= 0 \\ -21 + 12b + 6c &= 0 \\ -10a - 4b - 2c &= 0 \\ -6a + 8b - 2c &= 0. \end{aligned}$$

This system has only the trivial solution. This means that there is no Bergman type identity of degree 6 for the even part A_0 of $M(2)$.

Calculating $P51(e_{13} - e_{23} + e_{14} + e_{32}, e_{31} - e_{23}, e_{32} - e_{41} + e_{42})$ we get the system:

$$\begin{aligned} -2a + 3b &= 0 \\ -2a + 2b + 4c &= 0 \\ -a - 2c &= 0 \\ 2a - b &= 0 \\ -b - 2c &= 0 \\ a + c &= 0 \\ -a - c &= 0 \\ -2a + 2b - 2c &= 0 \\ -2c &= 0 \\ 2c &= 0. \end{aligned}$$

The system has the trivial solution only meaning that there is no Bergman identity if degree 6 on the odd part as well.

Thus we come to the main result of the paper, namely

THEOREM 2. *The minimal degree of a Bergman graded identity for $M(2)$ is* 8.

Proof: The above considerations show that we have to start our investigations from 7. A Bergman identity of degree 7 has a corresponding commutative polynomial

$$g(t_1,t_2,t_3)=(t_1-t_2)(t_1-t_3)(t_2-t_3)(at_1^2+bt_2^2+ct_3^2+dt_1t_2+mt_1t_3+nt_2t_3),\quad a,b,c,d,m,n\in K.$$

We could write the Bergman polynomial $P7(x,y_1,y_2)$ in the following way:

Let $G(x,y_1,y_2)=P5(x,y_1.x,y_2)+P5(x,y_1,y_2.x)$, where $P(x,y_1,y_2)$ is the polynomial already defined earlier. Then

$$\begin{aligned} P7(x,y_1,y_2) &= ax^2.P5(x,y_1,y_2)+cP5(x,y_1,y_2)+dx.G(x,y_1,y_2)\\ &+ mx.P5(x,y_1,y_2).x+nG(x,y_1,y_2).x\\ &+ b(G(x,y_1.x,y_2)+G(x,y_1,y_2.x)). \end{aligned}$$

Calculating $P7(e_{13}-e_{23}+e_{14}+e_{32},e_{31}-e_{23},e_{32}-e_{41}+e_{42})$ we get the system:

$$\begin{aligned} c &= 0\\ -a+17b-m &= 0\\ -2a+2b+2d+n &= 0\\ 6b-2m+3n &= 0\\ a-10b+m &= 0\\ 2a-2d+2m &= 0\\ -2b+2m-n &= 0\\ 2a+b-d &= 0\\ 5b-2c-d-2m+2n &= 0\\ a-2b+m &= 0. \end{aligned}$$

This system has a trivial solution only. It means that there is no Bergman identity of degree 7 on the odd part A_1 of $M(2)$.

Now we consider the Bergman polynomial on the even part A_0 of $M(2)$. We calculate:

$$\begin{aligned} &P7(e_{11}-e_{12}+e_{21}+e_{22},e_{12},e_{21}-e_{11}+e_{12}),\\ &P7(e_{11},e_{12},e_{21}),\\ &P7(e_{34}-e_{44}+e_{43}-2e_{33},e_{44}-e_{43},e_{33}+e_{43}-e_{44}). \end{aligned}$$

Thus we get the following system for the coefficients in the presentation of

$P7(x, y_1, y_2)$:

$$\begin{aligned}
-6b - 2c - 7d - 3m - 5n &= 0 \\
a + 24b + 5c + d + 2m + 11n &= 0 \\
-5a + 8b - c - 9d - 4m + n &= 0 \\
-2a + 14b + 5d + m + 3n &= 0 \\
a + b + c + d + m + n &= 0 \\
93a - 80b + 36c + 36d + 103m - 94n &= 0 \\
-110a + 19b - 23c + 5d - 71m + 63n &= 0 \\
-61a - 680b - 205c - 67d - 90m - 255n &= 0 \\
71a + 486b + 128c + 27d + 61m + 157n &= 0.
\end{aligned}$$

This system has a trivial solution only.

This ends the proof of the theorem.

We point again that all the calculations are made using the system for computer algebra *Mathematica 5.0*.

Defining the explicit form of the Bergman graded identities for $M(2)$ is a problem of further investigations.

Conjecture 1. *A Bergman identity for M_2 of minimal degree is a polynomial of type (1) for which $n = 2$ and the corresponding commutative polynomial is $(t_1^2 - t_2^2)(t_2^2 - t_3^2)(t_1^2 - t_3^2)$.*

References

[1] G.M. Bergman, Wild automorphisms of free P.I. algebras and some new identities (1981), preprint.

[2] A. Giambruno, A. Valenti, *On minimal $*$-identities of matrices*, Linear Multilin. Algebra **39** (1995), 309–323.

[3] Ts.G. Rashkova, *Bergman type identities in matrix algebras with involution*, Proceedings of the Union of Scientists - Rousse, ser.5, vol.1, (2001), 26–31.

[4] Ts.G. Rashkova, **-Identities in Matrix Superalgebras with Superinvolution*, Proceedings of the Int. Conf. on Algebras, Modules and Rings, July 2003, Lisbon, World Scientific, 225–236.

[5] Ts.G. Rashkova, *Descripition of the superinvolutions for $M(2)$*, Antalya Algebra Days VII, May 2005,http://www.math.metu.edu.tr/˜antalya/2005/Articles.

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