THE UNIVALENT CONDITION FOR AN INTEGRAL OPERATOR ON THE CLASSES $S(\alpha)$ AND T_2

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Abstract.In this paper we present a few conditions of univalency for the operator $F_{\alpha,\beta}$ on the classes of the univalent functions $S(p)$ and T_2. These are actually generalizations(extensions) of certain results published in the paper [1].

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1.Introduction

Let be the class of analytical functions, $A = \{f : f = z + a_2z^2 + ...\}, z \in U$, where U is the unit disk , $U = \{z : |z| < 1\}$ and denote S the class of univalent functions in the unit disk.

Let p the real number with the property $0 < p \leq 2$. We define the class $S(p)$ as the class of the functions $f \in A$, that satisfy the conditions $f(z) \neq 0$ and $\left|(z/f(z))''\right| \leq p, z \in U$ and if $f \in S(p)$ then the following property is true $\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \leq p|z|^2, z \in U$, relation proved in [3].

We denote with T_2 the class of the univalent functions that satisfy the condition $\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1, z \in U$, and also have the property $f''(0) = 0$. These functions have the form $f = z + a_3z^3 + a_4z^4 + ...$ For $0 < \mu < 1$ we have a subclass of functions denoted by $T_{\mu,2}$, containing the functions $f \in T_2$ that satisfy the property $\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < \mu < 1, z \in U$.

Next we present a few well known results related to these classes, results on which shall rely in this paper.

THE SCHWARTZ LEMMA. *Let the analytic function g be regular in the unit disc U and $g(0)=0$. If $|g(z)|\leq 1, \forall z\in U$, then*

$$|g(z)|\leq |z|, \forall z\in U \tag{1}$$

and equality holds only if $g(z)=\varepsilon z$, where $|\varepsilon|=1$.

THEOREM 1.[2]. *Let $\alpha\in\mathbf{C}, \mathrm{Re}\alpha>0$ and $f\in A$. If*

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right|\leq 1, \forall z\in U \tag{2}$$

then $\forall\beta\in, \mathrm{Re}\beta\geq\mathrm{Re}\alpha$, the function

$$F_\beta(z)=\left[\beta\int_0^z t^{\beta-1}f'(t)\,dt\right]^{1/\beta} \tag{3}$$

is univalent.

THEOREM 2.[1]. *Let $f_i\in T_2, f_i(z)=z+a_3^iz^3+a_4^iz^4+..., \forall i=\overline{1,2}$, so that*

$$\alpha,\beta\in\mathbf{C}, \mathrm{Re}\alpha\geq\frac{6}{|\alpha|}, Re\beta\geq Re\alpha. \tag{4}$$

If $|f_i(z)|\leq 1, \forall z\in U, i=\overline{1,2}$ then $\forall\beta\in\mathbf{C}$ the function

$$F_{\alpha,\beta}(z)=\left\{\beta\int_0^z t^{\beta-1}\left(\frac{f_1(t)}{t}\right)^{\frac{1}{\alpha}}\cdot\left(\frac{f_2(t)}{t}\right)^{\frac{1}{\alpha}}dt\right\}^{\frac{1}{\beta}}\in S. \tag{5}$$

THEOREM 3.[1]. *Let $f_i\in T_{2,\mu}$, $f_i(z)=z+a_3^iz^3+a_4^iz^4+..., \forall i=\overline{1,2}$, so that*

$$\alpha,\beta\in\mathbf{C}, \mathrm{Re}\alpha\geq\frac{2(\mu+2)}{|\alpha|}, Re\beta\geq Re\alpha. \tag{6}$$

If $|f_i(z)|\leq 1, \forall z\in U, i=\overline{1,2}$ then $\forall\beta\in\mathbf{C}$ the function

$$F_{\alpha,\beta}(z)=\left\{\beta\int_0^z t^{\beta-1}\left(\frac{f_1(t)}{t}\right)^{\frac{1}{\alpha}}\cdot\left(\frac{f_2(t)}{t}\right)^{\frac{1}{\alpha}}dt\right\}^{\frac{1}{\beta}}\in S. \tag{7}$$

THEOREM 4.[1]. *Let $f_i\in S(p), 0<p<2$, $f_i(z)=z+a_3^iz^3+a_4^iz^4+..., \forall i=\overline{1,2}$, so that*

$$\alpha, \beta \in \mathbf{C}, \mathrm{Re}\alpha \geq \frac{2(p+2)}{|\alpha|}, Re\beta \geq Re\alpha. \tag{8}$$

If $|f_i(z)| \leq 1, \forall z \in U, i = \overline{1,2}$ *then* $\forall \beta \in \mathbf{C}$ *the function*

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \left(\frac{f_1(t)}{t} \right)^{\frac{1}{\alpha}} \cdot \left(\frac{f_2(t)}{t} \right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}} \in S. \tag{9}$$

2.Main results

Theorem 5. *Let* $M \geq 1, f_i \in T_2, f_i(z) = z + a_3^i z^3 + a_4^i z^4 + ..., \forall i = \overline{1,2}$, *so that*

$$\alpha, \beta \in \mathbf{C}, \mathrm{Re}\alpha \geq \frac{(4M+2)}{|\alpha|}, Re\beta \geq Re\alpha. \tag{10}$$

If $|f_i(z)| \leq M, \forall z \in U, i = \overline{1,2}$ *then* $\forall \beta \in \mathbf{C}$ *the function*

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \left(\frac{f_1(t)}{t} \right)^{\frac{1}{\alpha}} \cdot \left(\frac{f_2(t)}{t} \right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}} \in S. \tag{11}$$

Proof.

We consider the function $h(z) = \int_0^z \left(\frac{f_1(t)}{t} \right)^{\frac{1}{\alpha}} \cdot \left(\frac{f_2(t)}{t} \right)^{\frac{1}{\alpha}} dt.$

We observe that $h(0) = h'(0) - 1 = 0$.

By calculating the derivatives of the order I and II for this function we obtain:

$$h'(z) = \left(\frac{f_1(t)}{t} \right)^{\frac{1}{\alpha}} \cdot \left(\frac{f_2(t)}{t} \right)^{\frac{1}{\alpha}} \tag{12}$$

respectively

$$h''(z) = \frac{1}{\alpha} h'(z) \cdot B_1 + \frac{1}{\alpha} h'(z) \cdot B_2 \tag{13}$$

where

$$B_k = \left(\frac{z}{f_k(z)}\right) \cdot \frac{zf_k'(z) - f_k(z)}{z^2}, k = \overline{1,2}. \tag{14}$$

We calculate the fraction $\frac{zh''(z)}{h'(z)}$ and we obtain:

$$\frac{zh''(z)}{h'(z)} = \frac{z \cdot \frac{1}{\alpha} h'(z) \cdot \sum\limits_{k=1}^{2} B_k}{h'(z)} = z \cdot \frac{1}{\alpha} \cdot \sum_{k=1}^{2} B_k, \forall z \in U. \tag{15}$$

Replacing $B_k, k = \overline{1,2}$, in formula (15) we obtain:

$$\frac{zh''(z)}{h'(z)} = \frac{1}{\alpha}\left(\frac{zf_1'(z)}{f_1(z)} - 1\right) + \frac{1}{\alpha}\left(\frac{zf_2'(z)}{f_2(z)} - 1\right). \tag{16}$$

We evaluate the modulus and we multiply in both terms of the relation (16) with $\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}$, obtain:

$$\frac{1 - |z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{1 - |z|^{2\mathrm{Re}\alpha}}{|\alpha|\,\mathrm{Re}\alpha} \sum_{i=1}^{2}\left(\left|\frac{z^2 f_i'(z)}{f_i^2(z)}\right| \frac{|f_i(z)|}{|z|} + 1\right). \tag{17}$$

Because $|f_i(z)| \leq M, \forall z \in U$ $, \forall i = \overline{1,2}$ and applying Schwarz Lemma we obtain that

$$\frac{|f_i(z)|}{|z|} \leq M, \forall z \in U, \forall i = \overline{1,2}. \tag{18}$$

We apply this relation in the above inequality and we obtain:

$$\frac{1 - |z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{1 - |z|^{2\mathrm{Re}\alpha}}{|\alpha|\,\mathrm{Re}\alpha} \sum_{i=1}^{2}\left(\left|\frac{z^2 f_i'(z)}{f_i^2(z)}\right| M + 1\right). \tag{19}$$

But

$$\left|\frac{z^2 f_i'(z)}{f_i^2(z)}\right| = \left|\frac{z^2 f_i'(z)}{f_i^2(z)} - 1 + 1\right| \leq \left|\frac{z^2 f_i'(z)}{f_i^2(z)} - 1\right| + 1, \forall z \in U, \forall i = \overline{1,2}. \tag{20}$$

Because $f \in T_2$, so $\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1$.

Applying this property and (20) in (19), obtain that:

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{\left(1-|z|^{2\mathrm{Re}\alpha}\right)(4M+2)}{|\alpha|\,\mathrm{Re}\alpha} \leq \frac{(4M+2)}{|\alpha|\,\mathrm{Re}\alpha}, \forall z \in U. \quad (21)$$

Because $\mathrm{Re}\alpha > \frac{(4M+2)}{|\alpha|}$, we obtain that

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1, \forall z \in U. \quad (22)$$

Applying Theorem 1 we obtain that $F_{\alpha,\beta} \in S$.

Remark. Theorem 5 is a generalization of the Theorem 2.

Theorem 6. *Let* $M \geq 1$, $f_i \in T_{2,\mu}$, $f_i(z) = z + a_3^i z^3 + a_4^i z^4 + ..., \forall i = \overline{1,2}$, *so that*

$$\alpha, \beta \in \mathbf{C}, \mathrm{Re}\alpha \geq \frac{2(\mu M + M + 1)}{|\alpha|}, Re\beta \geq Re\alpha. \quad (23)$$

If $|f_i(z)| \leq M, \forall z \in U, i = \overline{1,2}$ *then* $\forall \beta \in \mathbf{C}$ *the function*

$$F_{\alpha,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1}\left(\frac{f_1(t)}{t}\right)^{\frac{1}{\alpha}} \cdot \left(\frac{f_2(t)}{t}\right)^{\frac{1}{\alpha}} dt\right\}^{\frac{1}{\beta}} \in S. \quad (24)$$

Proof. Considering the same steps as in the above proof we obtain:

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}\alpha}}{|\alpha|\,\mathrm{Re}\alpha}\sum_{i=1}^{2}\left(\left|\frac{z^2 f_i'(z)}{f_i^2(z)}-1\right| M + M + 1\right). \quad (25)$$

But $f \in T_{2,\mu}$, which implies that $\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < \mu, \forall z \in U$.
In these conditions we obtain:

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{2(\mu M + M + 1)}{|\alpha|\,\mathrm{Re}\alpha}, \forall z \in U. \quad (26)$$

By applying the relation (23) we obtain that $\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1, \forall z \in U$.

So according to the Theorem 1 the function $F_{\alpha,\beta}$ is univalent.

Remark. Theorem 6 is a generalization of the Theorem 3.

Theorem 7. *Let* $M \geq 1, f_i \in S(p), 0 < p < 2,\ f_i(z) = z + a_3^i z^3 + a_4^i z^4 + ..., \forall i = \overline{1,2},$ *so that*

$$\alpha, \beta \in \mathbf{C}, \mathrm{Re}\alpha \geq \frac{2(pM + M + 1)}{|\alpha|}, Re\beta \geq Re\alpha. \tag{27}$$

If $|f_i(z)| \leq M, \forall z \in U, i = \overline{1,2}$ *then* $\forall \beta \in \mathbf{C}$ *the function*

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \left(\frac{f_1(t)}{t} \right)^{\frac{1}{\alpha}} \cdot \left(\frac{f_2(t)}{t} \right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}} \in S. \tag{28}$$

Proof. Considering the same steps as in the above proof we obtain:

$$\frac{1 - |z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{1 - |z|^{2\mathrm{Re}\alpha}}{|\alpha| \mathrm{Re}\alpha} \sum_{i=1}^{2} \left(\left| \frac{z^2 f_i'(z)}{f_i^2(z)} - 1 \right| M + M + 1 \right). \tag{29}$$

But $f \in S(p)$, so

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq p|z|^2, \forall z \in U. \tag{30}$$

In these conditions we obtain:

$$\frac{1 - |z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{1 - |z|^{2\mathrm{Re}\alpha}}{|\alpha| \mathrm{Re}\alpha} \sum_{i=1}^{2} \left(p|z|^2 M + M + 1 \right), \forall z \in U, \tag{31}$$

so,

$$\frac{1 - |z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{2(pM + M + 1)}{|\alpha| \mathrm{Re}\alpha}. \tag{32}$$

By applying in (32) the relation (27) we obtain that $\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma} \left| \frac{zh''(z)}{h'(z)} \right| \leq 1, \forall z \in U$.

So according to the Theorem 1 the function $F_{\alpha,\beta}$ is univalent.

REMARK. Theorem 7 is a generalization of the Theorem 4.

References

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