A RESULT ON THE EXISTENCE OF CRITICAL POINTS

by
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Abstract : The main purpose of this paper is to present a short review on two variants of the so called three critical points theorem. The first variant was given in the context of Finsler manifolds in the paper [2] and the second one is presented in under some locally linking hypotheses.

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1. Preliminaries on the existence of critical points

Let M be a C^1 Banach manifold without boundary (∂M=Ø) and let T(M) be the total space of tangent bundle of M . A continous function

$\| \ \|$:T(M)→ R_+ is a Finsler structure on T(M) if the following conditions are satisfied:

(i) For each x∈M ,the restriction $\| \ \|_x = \| \ \| / Tx(M)$ is an equivalent norm on $Tx(M)$;

(ii) For each $x_0 \in$ M , and k>1 ,there is a trivializing neighbourhood U of x_0 such that $\frac{1}{k} \| \ \|_x \leq \| \ \|_{x0} \leq k \| \ \|_x$ for all x∈U .

M is said to be a Finsler manifold if it is regular (as a topological space) and if it has a Finsler structure on T(M) .

It is known that every paracompact C^1-Banach manifold admits Finsler structures on its tangent bundle and that every C^1-Riemannian manifold is a Finsler manifold .

Suppose that M is connected .For x,y∈M define $\Omega(x,y) = \{\sigma : [0,1] \to M, C^1 such.that \sigma(0) = x, \sigma(1) = y\}$.The length of curve $\sigma \in \Omega(x,y)$is given by

$$l(\sigma) = \int_0^1 \left\| \dot{\sigma}(t) \right\|_{\sigma(t)} dt . \qquad (1)$$

Consider the Finsler metric on M defined as follows

$$d_F(x,y) = \inf\{l(\sigma) : \sigma \in \Omega(x,y)\} . \qquad (2)$$

The pair (M, d_F) is a metric space and the induced topology is equivalent to the topology of the manifold of M (see K.Deimling [4]).

To a given Finsler structure on T(M) there correspond a dual structure on the cotangent bundle $T^*(M)$ given by

$$\|\mu\| = \sup\{|\mu(x)| : \|x\|_p = 1\}, \mu \in T^*(M) . \quad (3)$$

Let $f : M \to \Re$ be a C^1-differentiable mapping .A locally Lipschitz continous vector field v : M$\to$T(M) such that for each x$\in$M the following relations are satisfied: (i) $\|v_x\| \le 2\|(df)_x\|$

(ii) $(df)_x(v_x) \ge \|(df)_x\|^2$

where $\|(df)_x\|$ is given by Finsler structure on $T_x^*(M)$, is called a pseudogradient vector field of f (in short p.g.f. of f) .

If M is a C^2-Finsler manifold and $f : M \to R$ is a C^1-differentiable mapping, then $\nabla(f) \neq \emptyset$, where

$$\nabla(f) = \{v \in X(M) : v \text{ is p.g.f. of } f\}. \qquad (4)$$

Let us note that if M is a Hilbert manifold with the Riemannian structure $\| \ \|$, the norms $\| \ \|_x$ come from inner product by $\| \ \|_x = \langle , \rangle_x^{1/2}$, and we can define a p.g.f. of f by p$\mapsto$ (grad f)(p) , where (grad f)(p) is given via Riesz representation theorem by

$$(df)p(X) = \langle X, (gradf)(p)\rangle_p, \forall X \in T_p(M).$$

Let M be a C^2-Finsler manifold , connected and without boundary . For a C^1-differentiable real-valued function $f : M \to R$, let us define by

$$C(f) = \{p \in M : (df)_p = 0\} \qquad (5)$$

the critical set of f and by B(f)=f(C(f)) the bifurcation set of f . The elements of C(f) are called the critical points of f and the elements of B(f) represent its critical values . If p$\notin$C(f) , s$\notin$B(f) , then p is a regular point and s is regular value of the mapping f .

For s∈R denote by $C_s(f) = C(f) \cap f^{-1}(s)$, the critical point set of f at the level s. It is obvious that s a regular value of f if and only if $C_s(f) =$Ø. We also consider the set $f^s := M_s(f) = f^{-1}((-\infty, s])$.

It is well-known that if s∉B(f) then $f^{-1}(s)$ is Ø or a differentiable submanifold of M , of codimension 1 , and $M_s(f)$ is a differentiable submanifold with boundary of M , of codimension 0 , and $\partial M_s(f) = f^{-1}(s)$.

Suppose that the manifold M and the mapping f satisfy the following hypoteses:
(a) (Completeness) (M, d_F) is a complete metric space, where d_F represents the Finsler metric on M defined by (2) .
(b) (Boundedness from below) If B=inf {f (x) : x∈M} then B>-∞ .
(c) (The Palais-Smale condition) Any sequence $(x_n)_{n\geq 0}$ in M with the properties that $(f(x_n))_{n\geq 0}$ is bounded and $\|(df)_{x_n}\| \to 0$ has a convergent subsequence $(x_{n_k})_{k\geq 0}$, with $x_{n_k} \to p$.

The above conditions (a)-(c) are sometimes called compactness conditions because if M is a compact manifold they are automatically verified. It is clear that the point p, which appears in condition (c) of Palais-Smale, is a critical point of f , p∈C(f) .

Let v∈$\nabla(f)$ be a p.g.f. of f and let $x \in M$ be a fixed point .Because v is locally Lipschitz the following Cauchy problem

$$\begin{cases} \dot{\varphi}(t) = -\upsilon_{\varphi(t)} \\ \varphi(0) = x \end{cases} \qquad (6),$$

has a unique maximal solution $\varphi^v : (\omega_-^v(x), \omega_+^v(x)) \to M$, where $\omega_-^v(x) < 0 < \omega_+^v(x)$. Denote by $\varphi_t^v(X)$ the above solution and by $t \to \varphi_t^v(x)$ the corresponding integral curve of (6). Taking into account the hypotheses (a)-(c) it follows that $\omega_+^v(x) = +\infty$, i.e. $\{\varphi_t^\upsilon\}_{t\geq 0}$ is a semigroup of diffeomorphisms of M (see K.Deimling [4].

For a vector v∈$\mathrm{X}(M)$ let us consider the sets Z(v)={p∈$M : v_p = 0$}, $Fix(\varphi^v) = \{x \in M : \varphi_t^v(x) = x, \forall t \in (\omega_-^v(x), \omega_+^v(x))\}$. It is easy to see that the following relations hold :

$$C(f)=\bigcap_{v\in\nabla(f)} Z(v) \qquad (7)$$

$$C(f)=\bigcap_{v\in\nabla(f)} Fix(\varphi^v) \qquad (8)$$

If $x \notin C(f)$, then $f(\varphi_t^v(x)) < f(x)$ for t>0 and $f(\varphi_t^v(x)) > f(x)$ for t<0 .

The following three critical points type result was proved in the paper [2].

Theorem 1: Let *M be a* C^2*-Finsler manifold, connected and without boundary , and let* $f : M \to R$ *be a* C^1*-differentiable real-valued mapping . Assume that the hypotheses (a)-(c) are satisfied and there exist two local minima points off* f *. Then* f *posses at least three distinct critical points.*

The following two corollaries are obtained from the above important result.

Corollary 1: *Let M be a* C^2*-Finsler manifold, connected and without boundary, and let* $f : M \to R$ *be a* C^1*-differentiable real-valued mapping . Assume that the hypotheses (a)-(c) are satisfied and* f *has a local minimum point which is not a global minimum point. Then* f *posses at least three distinct critical points.*

Corollary 2: *Let* M^m *be a m-dimensional* C^2*-manifold which is closed (i.e. M is compact and without boundary) and connected . If* $f : M \to R$ *is a* C^1*-differentiable real-valued mapping with two local minima points, then* f *posses at least four distinct critical points .*

Remark : The following exemple shows that exist smooth functions $f : R^2 \to R$ having two global minima points and without other critical points,

Hence some supplementary hypotheses are necessary to be imposed. The polynomial function $f(x,y) = (x^2 - x - 1)^2 + (x^2 - 1)^2$ has global minima at the points (1,2) and (-1,0) and it has no other critical points.

2.Existence of critical points under some linking conditions

In what follows we prove the existence of three critical points for a function which is bounded below and has a local linking at 0 .

Definition 1:Let X be a Banach space with a direct sum decomposition $X = X^1 \oplus X^2$.The function $f \in C^1(X,R)$ has a *local linking at 0* , with respect to the pair of subspaces (X^1, X^2) , if , for some r>0 , $\begin{matrix} f(u) \geq 0, u \in X^1, \|u\| \leq r, \\ f(u) \leq 0, u \in X^2, \|u\| \leq r. \end{matrix}$

Remark: If mapping f has a local linking at 0 , then 0 is a critical point of f .
Consider two sequences of subspaces:
$X_0^1 \subset X_1^1 \subset ... \in X^1, X_0^2 \subset X_1^2 \subset ... \subset X^2$, such that $X^j = \overline{\bigcup_{n \in N} X_n^j}, j = 1,2.$
For every multi-index $\alpha = (\alpha_1, \alpha_2) \in N^2$, we denote by X_α the space $X_{\alpha_1}^1 \oplus X_{\alpha_2}^2$.
Let us recall that $\alpha \leq \beta \Leftrightarrow \alpha_1 \leq \beta_1, \alpha_2 \leq \beta_2$.
We say that a sequence $(\alpha_n) \in N^2$ is *admissible* if, for every $\alpha \in N^2$ there is $m \in N$ such that $n \geq m \Rightarrow \alpha_n \geq \alpha$.
For every $f : X \to R$, we denote by f_α the function f restricted to X_α .
We shall use the following compacteness conditions .

Definition 2: Let c$\in R$ and $f \in C^1(X,R)$. The function f satisfies the $(PS)_c^*$ condition if every sequence (u_{α_n}) such that (α_n) is admissible and $u_{\alpha_n} \in X_{\alpha_n}, f(u_{\alpha_n}) \to c, f_{\alpha_n}^{'}(u_{\alpha_n}) \to 0$, contains a subsequence which converges to a critical point of f .

Definition 3: Let $f \in C^1(X,R)$. The function f satisfied the $(PS)^*$ condition if every sequence (u_{α_n}) such that (α_n) is admissible and $u_{\alpha_n} \in X_{\alpha_n}, \sup f(u_{\alpha_n}) \to \infty, f_{\alpha_n}^{'}(u_{\alpha_n}) \to 0$, contains a subsequence which converges to a critical point of f .
Remarks: 1) When $X_n^1 := X, X_n^2 := \{0\}$ for every n$\in N$, the $(PS)_c^*$ conditions is a usual Palais-Smale conditions at the level c.
2) The $(PS)^*$ conditions implies the $(PS)_c^*$ conditions for every c$\in R$.
Let us recall some standard notations in this context:

$$S_\delta = \{u \in X : dist(u,S) \leq \delta\},$$
$$f^c = \{u \in X : f(u) \leq c\},$$
$$K_c = \{u \in X : f(u) = c, f^{'}(u) = 0\}.$$

Lemma 1*: Let f be a function of class* C^1 *defined on a real Banach space* X *. Let* $S \subset X, \varepsilon, \delta > 0$ *and* $c \in R$ *be such that, for every* $u \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}$, *the following inequality holds*

$$\left\| f'(u) \right\| \geq 4\varepsilon / \delta,$$

then there exists $\eta \in C([0,1] \times X, X)$ *such that*

(i) $\eta(0,u) = u, \forall u \in X$,

(ii) $f(\eta(.,u))$ *is non increasing ,* $\forall u \in X$,

(iii) $f(\eta(t,u)) < c, \forall t \in]0,1], \forall u \in f^c \cap S$,

(iv) $\eta(1, f^{c+\varepsilon} \cap S) \subset f^{c-\varepsilon}$,

(v) $\left\| \eta(t,u) - u \right\| \leq \delta, \forall t \in [0,1], \forall u \in X$.

Definition 4: Let A, B be closed subsets of X . By definition, $A \prec^{\infty} B$ if there is $\beta \in N^2$ such that , for every $\alpha \geq \beta$ there exist $\eta_\alpha \in C([0,1] \times X_\alpha, X_\alpha)$ such that

(i) $\eta(0,u) = u, \forall u \in X_\alpha$,

(ii) $\eta(1,u) \in B, \forall u \in A \cap X_\alpha$.

Lemma 2 *Let* $f \in C^1(X,R)$ *and* $c \in R$, $\rho > 0$ *. Let N be an open neighborhood of* K_c. *Assume that f satisfies* $(PS)_c^*$ *. Then, for all* $\varepsilon > 0$ *small enough ,* $f^{c+\varepsilon} \setminus N \prec^{\infty} f^{c-\varepsilon}$. *Moreover the corresponding deformations* $\eta_\alpha : [0,1] \times X_\alpha \to X_\alpha$ *satisfy*

$$\left\| \eta_\alpha(t,u) - u \right\| \leq \rho, \forall t \in [0,1], \forall u \in X_\alpha, \qquad (9)$$

$$f(\eta_\alpha(t,u)) < c, \forall t \in]0,1], \forall u \in f_\alpha^c \setminus N. \qquad (10)$$

Proof The condition $(PS)_c^*$ implies the existence of $\gamma > 0$ and $\beta \in N^2$ such that, for every $\alpha \geq \beta$ and $u \in f_\alpha^{-1}([c-2\gamma, c+2\gamma]) \cap (X_\alpha \setminus N)_{2\gamma}, \left\| f_\alpha'(u) \right\| \geq \gamma$.

It suffices then to choose $\delta := \min\{\gamma/2, \rho, 4\}, 0 < \varepsilon \leq \delta\gamma/4$ and to apply Lemma1 to $S := X_\alpha \setminus N$.

Lemma 3 *Let* $f \in C^1(X,R)$ *be bounded below and let* $d := \inf_X f$. *If* $(PS)_d^*$ *holds then d is a critical value of* f *.*

Proof. If d is not a critical value of f , then , by Lemma2 , there exist $\varepsilon > 0$ such that

$$f^{d+\varepsilon} \prec^{\infty} f^{d-\varepsilon}. \qquad (11)$$

From the definition of d , $f_{\alpha}^{d-\varepsilon}$ is empty for all α and $f_{\alpha}^{d+\varepsilon}$ is non-empty for α large enough . This contradicts (11).

Lemma 4: *Let $f \in C^1(X,R)$ be bounded below . If $(PS)_c^*$ holds for all $c \in R$, then f is coercive .*

Proof. If f bounded below and not coercive then c:=sup{d$\in R$: f^d is bounded} is finite . It is easy to verify that K_c is bounded . Let N be an open bounded neighborhood of K_c . By Lemma2 , there exist $\varepsilon > 0$ such that

$$f^{c+\varepsilon} \setminus N \prec^{\infty} f^{c-\varepsilon}. \qquad (12)$$

Moreover we can assume that the corresponding deformations satisfy (9) with ρ=1. It follows from the definition of c that $f^{c+\varepsilon/2} \setminus N$ is unbounded and that $f^{c-\varepsilon} \subset B(0,R)$ for some R>0 . It follows from (9) and (12) that, for all α large enough , $f_{\alpha}^{c+\varepsilon} \setminus N \subset B(0,R+1)$. But then $f^{c+\varepsilon/2} \setminus N \subset B(0,R+1)$.
This is a contradiction .
The main result in this section is the following:

Theorem 2 *Suppose that $f \in C^1(X,R)$ satisfies the following assumptions*
(A1) f has a local linking at 0 ;
(A2) f satisfies $(PS)^*$;
(A3) f maps bounded sets into bounded sets ;
(A4) f is bounded from below and $d := \inf_X f$.<0 .
Then f has least three critical points .

Proof . 1) We assume that $\dim X^1$ and $\dim X^2$ are positive , since the other cases are similar . By Lemma3 , f achieves its minimum at some point v_0. Supposing K:={0, v_0} to be the critical set of f , we will be led to a contradiction . We may suppose that $r < \|v_0\|/3$ and $B(v_0,r) \subset f^{d/2}$. (13)

By assumption (A2) and Lemma2 , applied to f and to $g := -f$, there exists $\varepsilon \in]0,-d/2[$ such that

$$f^{\varepsilon} \setminus B(0,r/3) \prec^{\infty} f^{-\varepsilon}, \qquad (14)$$

$$g^{\varepsilon} \setminus B(0,r/3) \prec^{\infty} g^{-\varepsilon}, \qquad (15)$$

$$f^{d+\varepsilon} \setminus B(v_0, r) \prec^{\infty} f^{d-\varepsilon} = \emptyset . \quad (16)$$

Moreover , we can assume that the corresponding deformations exists for $\alpha \geq (m_0, m_0)$ and satisfy (9) with ρ= r/2 . Assumption (A2) implies also the existence of $m_1 \geq m_0$ and δ>0 such that , for $\alpha \geq (m_1, m_1)$,

$$u \in f_\alpha'^{-1}([d+\varepsilon, -\varepsilon]) \Rightarrow \| f_\alpha'(u) \| \geq \delta . \quad (17)$$

2) Let us write $\alpha := (n, n)$ where $n \geq m_1$ is fixed . It follows from (13) and (16) that

$$f_\alpha^{d+\varepsilon} \subset X_\alpha \cap B(v_0, r) \subset f_\alpha^{d/2} . \quad (18)$$

Using (14) and (17) , it is easy to construct a deformation $\sigma : [0,1] \times S_n^2 \to X_\alpha$, where $S_n^j := \{ u \in X_n^j : \|u\| = r \}$, j=1,2, such that

$$f(\sigma(t,u)) < 0, \forall t \in]0,1], \forall u \in S_n^2 \text{ and } f(\sigma(1,u)) = d + \varepsilon, \forall u \in S_n^2 . \quad (19)$$

By (18) there exists $\psi \in C(B_n^2, X_\alpha)$, where $B_n^j := \{ u \in X_n^j : \|u\| \leq r \}$, j=1,2 such that $\psi(u) = \sigma(1,u)$,u∈ S_n^2,

$$\psi(B_n^2) \subset X_\alpha \cap B(v_0, r) \subset f_\alpha^{d/2} . \quad (20)$$

Set $Q := [0,1] \times B_n^2$ and define a mapping $\Phi : \partial Q \to f_\alpha^0$ by

$\Phi(t,u) = u, t = 0, u \in B_n^2$,

$\Phi(t,u) = \sigma(t,u), 0 < t < 1, u \in S_n^2$,

$\Phi(t,u) = \psi(u), t = 1, u \in B_n^2$.

Lemma 4 implies the existence of R>0 such that $f^0 \subset B(0,R)$.

Hence there is a continuous extensions of Φ , $\widetilde{\Phi} : Q \to X_\alpha$, such that

$$\sup_Q f(\widetilde{\Phi}) \leq c_0 := \sup_{B(0,R)} f . \quad (21)$$

By assumption (A3) , c_0 is finite .

3) Let η , depending on α ,be the deformation given by (15) . We claim that Φ(∂Q) and $S := \eta(1, S_n^1)$ link nontrivially . We have to prove that , for any extension $\widetilde{\Phi} \in C(Q, X_\alpha)$ of Φ , $\widetilde{\Phi}(Q) \cap S \neq \emptyset$.

Assume , by contradiction , that

$$\eta(1, u_1) \neq \widetilde{\Phi}(t, u_2) \quad (22)$$

for all $u_1 \in S_n^1, u_2 \in B_n^2, t \in [0,1]$. It follows from (15) , (19) and (9) that (22) holds for all $u_1 \in B_n^1, u_2 \in S_n^2, t \in [0,1]$. By (20) we obtain (22) for t =1 and for all

$u_1 \in B_n^1, u_2 \in B_n^2$. Using homotopy invariance and Kronecker property of the degree , we have

$$\deg(F_0, \Omega, 0) = \deg(F_1, \Omega, 0) = 0 \ , \qquad (23)$$

where

$$\Omega := B_n^1 \times B_n^2,$$
$$F_t(u) := \eta(1, u_1) - \tilde{\Phi}(t, u_2).$$

We obtain from (9)

$$\eta(t, u_1) \neq u_2 \qquad (24$$

for all $u_1 \in B_n^1, u_2 \in S_n^2, t \in [0,1]$. It follows from (15) that (24) holds for all $u_1 \in S_n^1, u_2 \in B_n^2, t \in [0,1]$. Let us define on $[0,1] \times \Omega$ the map $G_t(u) := \eta(t, u_1) - u_2$.

Using (23) and homotopy invariance of the degree , we have $0 = \deg(G_1, \Omega, 0) = \deg(G_0, \Omega, 0) = \deg(P_n^1 - P_n^2, \Omega, 0) \neq 0$, a contradiction .

4) Let us define $c := \inf_{\tilde{\Phi} \in \Gamma} \sup_{u \in Q} f(\tilde{\Phi}(u))$, where

$$\Gamma := \{\tilde{\Phi} \in C(Q, X_\alpha) : \tilde{\Phi}(u) = \Phi(u), \forall u \in \partial Q\}$$

It follows from (21) and from the preceding step that $\varepsilon \leq c \leq c_0$.

Assumption (A2) implies the existence of $m_2 \geq m_1$ and γ>0 such that , for $\alpha \geq (m_2, m_2)$,

$$u \in f_\alpha^{-1}([\varepsilon, c_0]) \Rightarrow \|f_\alpha'(u)\| \geq \gamma \ . \qquad (25)$$

By the standard minimax argument , $c \in [\varepsilon, c_0]$ is a critical value of f_α , contrary to (25) .

Corollary3: *Assume that* $f \in C^1(X, R)$ *satisfies (A2) and (A3) . If* f *has a global minimum and a local maximum then* f *has a third critical point*

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