NONCOMMUTATIVE GEOMETRY AND THE DIFRACTION ONE DIMENSIONAL NETWORK

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Abstract. Noncommutative geometry extend the notion from classical differential geometry from differential manifold to discrete spaces and even noncommutative spaces which are given by noncommutative algebra (over **R** or **C**). Such an algebra A replace the commutative algebra of function of class C^∞ over a smooth manifold.

In this work I present some aspects about the calculus of the distance in the noncommutative geometry case. An important role in the distance calculus is playing by the Dirac operator.

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1. Introduction

Definition 1.1_ Let V be a finite dimensional vector space over the scalar field $\mathbf{K}$, where **K**=**R** or **C**.
A *Quadratic form* on V is a mapping $Q:V\to \mathrm{K}$ such that:

1) $Q(\lambda v)=\lambda^2 Q(v)$

2) The associated form $B(v,w)=\frac{1}{2}\{Q(v)+Q(w)-Q(v-w)\}$ $v,w\in V$ is biliniar

In this case (V,Q) is a *Quadratic space.*

Definition 1.2 The pair (A,ν) is said to be a *Clifford algebra* for the quadratic space (V,Q) when :

1) A is generated as an algebra by $\{\nu(v)\mid v\in V\}$ and $\{\lambda 1\mid \lambda\in K\}$

2) $((\nu(v))^2=-Q(v)1$, $v\in V$

Example:1) The **R**-algebra of complex numbers **C** is generated by 1 and $\boldsymbol{i}$, verifying the relation:
$\boldsymbol{i}^2=-1$, it is a Clifford algebra for the quadratic space (**R**,Q) and the Clifford map $\boldsymbol{c}$, where Q :$\mathbf{R}\to\mathbf{R}$ and c: $\mathbf{R}\to\mathbf{C}$ are given by:

$Q(x) = -x^2$, $c(x) = ix$

2)When we take $Cl_{\mathbf{R}}$ ($\mathbf{R}^{p+q}$, Q) where Q is the quadratic form

$Q(x) = x_1^2 + ... + x_p^2 - x_{p+1}^2 .... - x_{p+q}^2$

we use the notation $Cl(p,q)$, we put $Cl(n) \equiv Cl(0,n)$ and $Cl^*(n) \equiv Cl(n,0)$

Hence, for the universal real Clifford algebra Cl_C (V,-Q) over the vector space V=$\mathbf{R}^n$, where Q comes from the biliniar form of the usual euclidian product in $\mathbf{R}^n$, we use the notation $Cl(n)$, that means that if we take an orthonormal basis $e_1, e_2, ..., e_n$ in $\mathbf{R}^n$, we have: $e_i e_j + e_j e_i = -2\delta_{ij} 1$

We have for instance:

$Cl(1) = \mathrm{C},\ Cl(2) = \mathrm{H},\ Cl(3) = \mathrm{H} \oplus \mathrm{H}$

3)Let be the Pauli matrices in $\mathbf{C}^{2x2}$:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the associated matrices would be:

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We have:

$\sigma_0^2 = \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \boldsymbol{I}$, $e_0^2 = \boldsymbol{I}$, $e_1^2 = e_2^2 = e_3^2 = -\boldsymbol{I}$

and $\sigma_j \sigma_k = -i\sigma_l$, $e_j e_k = e_l$, where $\{j,k,l\}$ are cyclic permutation of the set $\{1,2,3\}$.

Let $A_{0,0} = \{\lambda\sigma_0, \lambda \in \mathrm{R}\}$, $A_{1,0} = \left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} \| x, y \in \mathrm{R} \right\}$,

$A_{0,1} = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \| x, y \in \mathrm{R} \right\}$,

$A_{0,2} = \left\{ \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \| x_j \in \mathrm{R},\ j = \overline{0,3} \right\} = \left\{ \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix} \| z_j \in \mathrm{C} \right\}$

Then we have: $A_{0,0} \cong R,\ A_{1,0} \cong R \oplus R,\ A_{0,1} \cong C,\ A_{0,2} \cong H$.

2. Dirac operator relative to a vector bundle

Given a riemannian manifold M, an operator $\Delta \in Diff^{(2)}(E,E)$ of order 2 is said to be a *generalized laplacian* if $\sigma_2(\Delta)(\xi_p) = -\|\xi_p\|_2^p I_{E_p}$ where $\|...\|_g$ is the metric norm of M , and an operator of order 1 , $D \in Diff^{(1)}(E,E)$ is said to be a *Dirac operator* relative to the vector bundle $(E,M,K^{\succ})$ whenever D^2 is a generalized laplacian. If D is a Dirac operator we can define for any $\xi_p \in T_p^*M$ an K -endomorphism $c(\xi_p)$ of E_p given by $c(\xi_p) = \sigma_1(D)(\xi_p)$ or, alternatively we define a map:

$\gamma_p : T_p^*M \times E_p \to E_p$

$(\xi_p, u_p) \to c(\xi_p)(u_p) \equiv c(\xi_p)u_p$

for the tensorization of T_p^*M by **C** we use the above relation $T_C^*M \equiv (T_p^*M) \otimes_R C$, so if we don't want to precise if $K = R$ or **C** we will write $T_K^*M \equiv (T_p^*M) \otimes_R K$, and obviously $T_R^*M = T_p^*M$, note that the map γ_p could be defined using T_K^*M instead of T_p^*M .

The endomorphisms $c(\xi_p)$ verify the condition $c(\xi_p)^2 = -\|\xi_p\|_g^2 I_{E_p}$, equivalent to $c(\xi_p)c(\eta_p) + c(\eta_p)c(\xi_p) = -2g(\xi_p, \eta_p) I_{E_p}$, this means that we got for any $p \in M$ a K -bilinear map $\gamma_p : (T_K^*M) \times E_p \to E_p$ that is a K -linear map $(T_K^*M) \otimes_K E_p \to E_p$, with the preceding property.

The set of K -linear operators $c(\xi_p) : E_p \to E_p$ generates an associative and unital sub algebra A of the $K-$algebra $End_K(E_p)$ (with $1 \equiv I_{E_p}$ as element one) and there exist a K -linear map $c : T_K^*M \to A$ given by $\xi_p \to c(\xi_p)$ and verifying

$c(\xi_p)c(\eta_p) + c(\eta_p)c(\xi_p) = -2g(\xi_p, \eta_p)1$

This means that A is a Clifford algebra for the quadratic space $(T_K^*M, -g)$ and c is the Clifford mapping. The vector bundle E is a *Clifford vector bundle* and $\gamma : T_K^*M \times E \to E$, given by $\gamma(\xi,u)\big|_p = \gamma_p(\xi_p, u_p) = c(\xi_p)u_p$ is the coresponding *Clifford action.*

The following property exhibit the compatibility of D with the Clifford action:
$D(sf) = c(df)s + D(s)f \qquad (f \in \mathrm{F_K}(M), \quad s \in \Gamma(M,E))$
or alternatively, making $F_K(M)$ act multiplicatively in $\Gamma(M,E)$ through $\overline{f}(s) = sf$, we will have :
$D(sf) - D(s)f = (D \circ \overline{f})(s) - (\overline{f} \circ D)(s) = ((D \circ \overline{f}) - (\overline{f} \circ D))(s) = [D, \overline{f}] = c(df)s$,
that is:
$[D, \overline{f}] = c(df)$
or we can write using the previus conventions:
$[D, \overline{f}] = c(df)$.

We can say that any Dirac operator, relative to the vector bundle (E, M, K^n) , with a riemannian manifold in the basis, determines in this vector bundle a structure of Clifford bundle compatible with D in the preceding sense.
Any Clifford vector bundle on a riemannian manifold endowed with a covariant derivative

$$\overline{\nabla} : \Gamma(M,E) \to \Gamma(M, E \otimes T_K^* M)$$

compatible with a Clifford action $\gamma \; : T_K^* M \otimes E \to E$ in a sense to be precise later, is associated to a Dirac operator acting on the sections of this vector bundle and compatible with the Clifford action, we will call such a vector bundle a *Clifford-Weyl bundle.*

The compatibility of $\overline{\nabla}$ with the Clifford action $\xi_p s = c(\xi_p)s$ means the following:

$$\overline{\nabla}_X(\omega \cdot s) = \nabla_X \omega \cdot s + \omega \cdot \overline{\nabla}_X s \quad (\omega \in \Omega^1(M,K))$$

or alternatively,

$$\overline{\nabla}_X(c(\omega)s) = c(\nabla_X \omega)s + c(\omega)\overline{\nabla}_X s \quad (\omega \in \Omega^1(M,K))$$

where $\nabla : \aleph_K(M) \to \aleph_K(M) \otimes \Omega^1(M,K)$ or else $\nabla_X : \aleph_K(M) \to \aleph_K(M)$ for any $X \in \aleph_X(M)$, is the Levi Cevita covariant derivative in M, determining a covariant derivative $\nabla : \Omega^1(M,K) \to \Omega^1(M,K) \otimes \Omega^1(M,K)$ or actually, $\nabla_X : \Omega^1(M,K) \to \Omega^1(M,K)$ for any $X \in \aleph_K(M)$.
The set $\Gamma(M,E)$ of the sections of a Clifford-Weyl vector bundle has a structure of $(Cl_K(T^* M), F_K(M))$ -bimodule, with the following properties of compatibility:
$\overline{\nabla}(sf) = \overline{\nabla}(s)f + s \otimes df$
$\overline{\nabla}(\omega \cdot s) = (\nabla \omega) \cdot s + \omega \cdot \overline{\nabla} s$

Next, I will prove the folowing result, which is the distance between two diferent points on the straight line, in noncommutative geometry case:

$$d_C(x,y) = \sup_{f\in A}\{|f(x)-f(y)| : \|[D,f]\| < 1\} = \sup_{f\in A}\{|f(x)-f(y)| : \|f'\| < 1\} = |x-y|$$

We know that $[D,\overline{f}] : A \to A$

$$g \to [D,\overline{f}](g) = (D\circ\overline{f} - \overline{f}\circ D)(g) =$$
$$= D(fg) - f(Dg) = (fg)' - fg' = f'g + fg' - fg' = \overline{f'}$$

where I use the definition $\overline{f}(s) = sf$.

In general for an operator $T : E \to E$, we have : $\|T\| = \sup_{\|\xi\|\le 1}\|T(\xi)\|$

In this case

$$\|[D,f]\| = \sup_{\|\xi\|\le 1}\|f'\xi\| = \sup_{\|\xi\|\le 1}\|f'\|\xi = \|f'\|$$

and using the inequality:

$$|f(x)-f(y)| \le \|f'\|\cdot|x-y|$$

I prove the distance relation.

Now, I will use the Dirac operator to recover the distance between atoms in a periodical one dimensional diffraction network.
Let assume that we have a network in which, between two atom we have the same distance.
So, we can represent the network in this way:

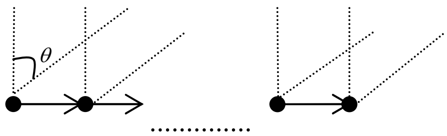
Using this network we can construct the incidence matrix puting the element 1 when we have a link between two atom and puting 0 if we don’t have link.
So, the incidence matrix would be:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & ... & 0 \\ 0 & 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & 0 & ... & 0 \end{pmatrix}.$$

We define:

$$D_N = \begin{pmatrix} 0 & \frac{\sin\theta}{\lambda} & 0 & 0 & ... & 0 \\ 0 & 0 & \frac{\sin\theta}{\lambda} & 0 & ... & 0 \\ 0 & 0 & 0 & \frac{\sin\theta}{\lambda} & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & \frac{\sin\theta}{\lambda} \\ 0 & 0 & 0 & 0 & ... & 0 \end{pmatrix},$$

where θ is the angle between two wave and λ is the wave lenght.



Let be $N = \{1,2,...,n\}$ the set of atoms.
We will denote with A, the algebra of all maps $f : N \to C$.
The function f is represented as

$$f \to \widetilde{f} = \begin{pmatrix} f_1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & f_2 & 0 & 0 & \ldots & 0 \\ 0 & 0 & f_3 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & f_n \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \in C^n,$$

where $f(n) = f_n$, and with this representation we can construct the function

$$\bar{f} = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}.$$

We can associate to the complex function f, a real function F with the properties: $F_1 = 0$ and $F_{k+1} = F_k + |f_{k+1} - f_k|$.

We have the operator

$$\hat{D} = \begin{pmatrix} 0 & D_N^{tr} \\ D_N & 0 \end{pmatrix}$$

We know that in general, the distance in noncommutative geometry is given by:

$$d_C(x,y) = \sup_{f \in A}\{|f(x) - f(y)| : \|[D, f]\| < 1\}$$

In our case we can compute, and is easy to prove that

$\left\|\left[\hat{D}, \bar{f}\right]\psi\right\| = \left\|\left[\hat{D}, F\right]\psi\right\|$ for $\psi \in C^n$.

So, after easy computation we will get:

$$\left\|\left[\hat{D}, \bar{f}\right]\right\| = \max\left\{\frac{\sin\theta}{\lambda}|f_2 - f_1|, \ldots, \frac{\sin\theta}{\lambda}|f_N - f_{N-1}|\right\},$$

and using the condition $\|[D, f]\| < 1$, we get:

$$d(i, i+n) = \frac{\lambda}{\sin\theta} + \ldots + \frac{\lambda}{\sin\theta} = \frac{n\lambda}{\sin\theta}.$$

We can find the same result if we start from physics.

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