THE PLANE ROTATION OPERATOR AS A MATRIX FUNCTION

by
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Abstract. Formalism in mathematics can offer many simplifications, but it is an instrument which should be carefully treated as it can easily create confusions. Formalism is an instrument which, together with a programming language that allows abstractions (for instance the C language) can create a very strong programming instrument. One example is realizing the matrix functions in C language using pointers.

The present material refers to an example of matrix functions, the exponential matrix function, used as a plane rotation operator.

A matrix function is a function of the form: $Y = F(X)$, where X and Y are quadratic matrices of order n. The relation can also be written as follows:

$$\begin{Vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{Vmatrix} = F\left(\begin{Vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{Vmatrix}\right).$$

In the case of the plane rotations we shall use matrices of order 2 and for rotations in a three-dimension space we use matrices of order 3.

In order to associate a matrix with a linear transformation of a n-dimensional vectorial space V, we must choose a basis $x_1, x_2, ..., x_n$. From the equations:

$$\begin{cases} A(x_1) = a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ A(x_2) = a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ \cdots \\ A(x_n) = a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n \end{cases}$$

or $A(x_j) = \sum_{i=1}^{n} a_{ij}x_i$, for j = 1, 2, ..., n.

we obtain the matrix which represents the transformation :

$$A \rightarrow \|A\| = \begin{Vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{Vmatrix}.$$

If A is the transformation obtained through the rotation of the plane around the origin O with an angle φ and if x_1 and x_2 are two vectors from the basis orthogonal and of length 1, then the representatives of these vectors applied in the origin apply to the representatives of images $A(x_1)$ and $A(x_2)$.

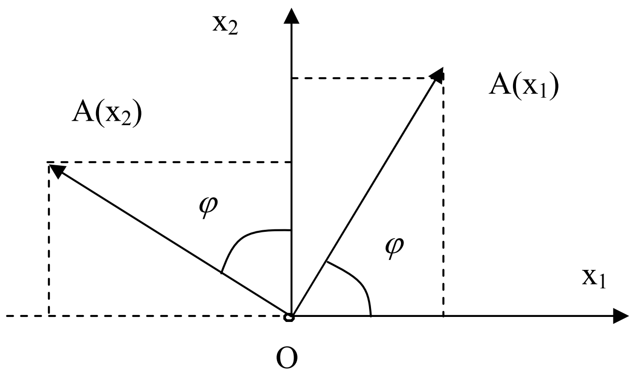

Figure 1.

$A(x_1)$ and $A(x_2)$ verify the following equations:

$$\begin{cases} A(x_1) = \cos(\varphi) \times x_1 + \sin(\varphi) \times x_2 \\ A(x_2) = -\sin(\varphi) \times x_1 + \cos(\varphi) \times x_2 \end{cases}$$

The operator A is represented by the matrix $\|A\|$:

$$\|A\| = \begin{Vmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{Vmatrix}$$

The operator A[-1] is the rotation of angle φ in the opposite direction, that is of angle $-\varphi$.

$$\|A\|^{-1} = \begin{Vmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{Vmatrix} = \begin{Vmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{Vmatrix}$$

we verified the relation $\|A\| \times \|A\|^{-1} = U$, the unity vector.

$$\|A\| \times \|A\|^{-1} = \begin{Vmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{Vmatrix} \times \begin{Vmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{Vmatrix} =$$

$$= \begin{Vmatrix} \cos^2(\varphi) + \sin^2(\varphi) & \cos(\varphi)\sin(\varphi) - \sin(\varphi)\cos(\varphi) \\ \sin(\varphi)\cos(\varphi) - \cos(\varphi)\sin(\varphi) & \sin^2(\varphi) + \cos^2(\varphi) \end{Vmatrix} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} = U$$

Given a vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in plane we can obtain a vector $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ rotated by an angle α towards which we apply the plane rotation operator $Y = F(\alpha) \times X$, where:

$$F(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

The product of two angle rotations α and β is equivalent to an angle rotation $\alpha + \beta$. This is mathematically expressed as follows: $F(\alpha) \times F(\beta) = F(\alpha + \beta)$. The function that verifies this functional equation is the exponential function. This leads us to the conclusion that the rotation operator can be expressed as an exponential. In order to find the exponential function corresponding to this operator we search for a differential equation to verify and then we look for its solutions.

$$F(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \Rightarrow$$

$$F'(\alpha)=\begin{bmatrix}\cos'(\alpha) & -\sin'(\alpha)\\ \sin'(\alpha) & \cos'(\alpha)\end{bmatrix}=\begin{bmatrix}-\sin(\alpha) & -\cos(\alpha)\\ \cos(\alpha) & -\sin(\alpha)\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\times\begin{bmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix}=I\times F(\alpha)$$

where we noted $I=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$.

We obtained the equation: $F'(\alpha)=I\times F(\alpha)$. This equation can also be written $F'(\alpha)=I\times F(\alpha)$. Generally speaking $A\times B\neq B\times A$, but in this case the multiplication is commutative. From the equation:

$$F'(\alpha)=F(\alpha)\times I\Rightarrow$$

$$F^{-1}(\alpha)\times F'(\alpha)=I\Rightarrow$$

$$\int F^{-1}(\alpha)\times F'(\alpha)d\alpha=I\int d\alpha\Rightarrow$$

$$\int F^{-1}(\alpha)\times dF(\alpha)=I\int d\alpha\Rightarrow$$

$$\ln(F(\alpha))=I\alpha+C\Rightarrow$$

$$F(\alpha)=e^{I\alpha+C}$$

for $\alpha=0\Rightarrow F(0)=e^{C}\Rightarrow$

$$F(\alpha)=e^{I\alpha}\times e^{C}=e^{I\alpha}\times F(0)$$

The formula can also be written:

$$\frac{F(\alpha)}{F(0)}=e^{I\alpha}\text{ where }F(0)=e^{C}.$$

We make an assumption $F(0)=e^{0}=U=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Results:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e^{\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}} \Rightarrow$$

$$\ln\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} \Rightarrow$$

We calculate $\ln\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ using the development in a series of exponentials. $\Rightarrow \ln(U + x) = \sum_{n=1}^{\infty} \frac{x}{n}$, where we noted with $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the unit matrix and with x a matrix variable $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$. For $x = O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, where O is zero matrix, results:

$$\ln(U + 0) = \sum_{n=1}^{\infty} \frac{0}{n} = O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore C = O, meaning Ci = 0, for i $\in$ {1, 2, 3, 4}. Thus:

$$F(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} = e^{I\,\alpha}.$$

This formula can also be written in another manner:

$$U \times \cos(\alpha) + I \times \sin(\alpha) = e^{I\alpha},$$

which is another way of writing Euler's formula in the field of matrix functions.

We notice that the following relation is true:

$$I^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -U.$$

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