# ON THE FUNDAMENTAL THEOREM OF ALGEBRA. PROOF BASED ON TOPOLOGY

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**Abstract.** In this paper we focus on the fundamental theorem of algebra, one of the most important results in mathematics. The work consists of two parts: the history of the fundamental theorem of algebra and the proof of this theorem, based on topology.

## I. The history of the fundamental theorem of algebra.

The fundamental theorem of algebra, stating that any polynomial with a positive effective degree has at least a root in the complex number field, was first given by Jirarom in 1629 and Descartes in 1637; Of course their formulation was different from the one that is used nowadays.

The fundamental theorem of algebra was the focus of research for many algebraists during the last century. The impossibility of finding explicit formulae for the roots of polynomial equations of degree higher than 4 with real numbers as coefficients which can be now explained by the theory of Abel and Galois, gave rise to the natural question whether these equations do always have complex or real roots.

The first proof of the fundamental theorem of algebra was given by D'Alembert in 1746. Although the scientists of the $18^{th}$ century did not find any drawbacks in this proof, it still seemed to be too analytical. The wider known rigorous proofs are nowadays based on the following methods and techniques

-the theorem of calculating the limit under the integral
-the symmetrical functions property and the Kronecker-Steinitz theorem
-the Liouville theorem
-the variation of the variable of a function along a closed curve

All these proofs are using more or less the tools of mathematical analysis. They can be classified with respect to the tools that are used as :

1) Proofs based on an elementary mathematical tools.
2) Proofs based on topology and the thoery of functions.

The aim of this paper is to present the proof based on the properties of the topological degree.

Let $R^n$ be the *n* dimensional real euclidean space, $G$ an open bounded subset of $R^n$,
$\overline{G}$ the closure of G and F(G) the boundary of G
$T^r : G \to R^n$ a transform of the closure $\overline{G}$ in $R^n$
$z^1,...,z^n$ the coordinates of the vector $z \in R^n$
$T^1,...,T^n$ the projections of the transform T on the coordinates from $R^n$

$|z| = \left[\sum_{i=1}^{n}\left(z^i\right)^2\right]^{\frac{1}{2}}$ the euclidean norm of z.

J(T) the Jacobian of T , which exists when the partial $\partial T^j / \partial z^i \quad (i = 1,2,...,n)$ are well defined.

The transform $T : \bar{G} \to R^n$ is said to be differentiable if it is continuous on $\overline{G}$ and the partial derivatives $\partial T^j / \partial z^i \quad (i = 1,2,...,n)$ exists and they are continuous on G.

Let $T : \overline{G} \to R^n$ be a differentiable transform and $z_0 \in R^n$, a vector such that $T(z) \neq z_0$ for $z \in Fr(G)$.

Let then $\Phi : [0,\infty) \to R$ be a continuous function such that $\int_{R^n} \Phi(|z|) dz^1 dz^2 \cdot \ldots \cdot dz^n = 1$ and $\Phi(r) = 0$ for $r \in [0,\alpha) \cup [\beta,\infty)$ where $0 < \alpha < \beta < \min\{|T(z) - z_0|, z \in Fr(G)\}$.

The degree of the differentiable transform T, with respect to the subset G and the vector $z_0$, is defined as the number $d(T,G,z_0)$ given by the multiple integral: $d(T,Gz_0) = \int_G \Phi(T(z) - z_0) J(T(z)) dz^1 ... dz^n$ . In order for the degree $d(T,G,z_0)$ to be corectly defined it has to be independent on the function $\Phi$. If we denote by $d\Phi_1(T,G,z_0)$ and $d\Phi_2(T,G,z_0)$ the degrees constructed with two different functions $\Phi_1$ and $\Phi_2$ then $d\Phi_1(T,G,z_0) = d\Phi_2(T,G,z_0)$.

Let $T_1, T_2 : G \to R^n$ be two differentiable transforms and $z_0 \in R^n$ a vector such that $T_z(z) \neq z_0$ for $z \in Fr(G)$ and $i = 1,...,n$, If $|T_1(z) - T_2(z)| < \gamma$ for $z \in \bar{G}$, then $d(T_1,G,z_0) = d(T_2,G,z_0)$.

Let $T : G \to R^n$ be a continuous transformation and $z \in R^n$ a vector such that $T(z) \neq z_0$ for $z \in Fr(G)$.

Let $T_m$ be a sequence of differentiable transforms with $T_m : G \to R^n$ such that $T_m(z) = z_0$ for $z \in Fr(G)$, $m > m_0$ and $T_m(z) \to T(z)$ uniformly on $\overline{G}$ when $m \to \infty$ . The degree of the continuous transform *T* with respect to the subset *G* is defined as $d(T,G,z_0) = \lim_{m\to\infty} d(T_m,G,z_0)$.

**Theorem.** Let $T : G \to R^n$ be a continuous transform and $z_0 \in R^n$ a vector such that $T(z) \neq z_0$ for $z \in Fr(G)$. If $d(T, G, z_0) \neq 0$ then there is a $z_1 \in G$ such that $T(z_1) = z_0$.

**Corollary.** Let $T_1, T_2 : G \to R^n$ be two continuous transforms and $z_0 \in R^n$. If for every $z \in Fr(G)$ the following inequality holds

$$|T_1(z) - T_2(z)| < |T_1(z) - z_0|,$$

then

$$d(T_1, G, z_0) = d(T_2, G, z_0).$$

**II. Proof based on the properties of the topological degree.**

**Theorem**. Let *n* be a positive integer, $f(z)$ a polynomial of degree *n* with complex coefficients, of the form $f(z) = z^n + a_{n-1}z^{n-1} + ... + a_0$ with $n \geq 1$,

We assume $r_0 > 0$, to be large enough such that for $z = r_0$ we have

$|z^n - f(z)| \leq |a_{n-1}| r_0^{n-1} + |a_{n-2}| r_0^{n-2} + ... + |a_0|$, $r_0^n = |z|^n$.

We consider the transforms $T_1, T_2$, $T_1 : \bar{G} \to R^2$, $T_2 : \bar{G} \to R^2$, where $G = \{z = (z^1, z^2) \in R^2, (z_1)^2 + (z_2)^2 < r_0^2\}$, given by the equations,

$T_1(z) = P^1(z^1z^2) \cdot P^2(z^1z^2)$

$T_1(z) = Q^1(z^1z^2) \cdot Q^2(z^1z^2)$ where $P^1(z^1z^2)$ şi $P^2(z^1z^2)$ are the real and the imaginary part of the expression $z^n$, with $z = z^1 + iz^2$, and $Q^1(z^1z^2)$ and $Q^2(z^1z^2)$ the real and the imaginary part of the polynomial $f(z)$.

We have $d(T_2, G, 0) = d(T_1, G, 0) > 0$. Then we obtain that there is $z_1 = (z_1^2 \cdot z_2^2) \in G$ such that $T_2(z_1) = 0$, which means that the polynomial $f(z)$ has at least a complex root $z_1 = z_1^2 + z_2^2$ which completes the proof.

**REFERENCES**

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