ON THE FRIEDRICHS ANGLE BETWEEN THE PAST AND THE FUTURE OF SOME Γ-CORRELATED PROCESSES

Ilie Valuşescu

Abstract. In the context of a complete correlated action $\{\mathcal{E},\mathcal{H},\Gamma\}$, the problem of the Friedrichs angle and Dixmier angle is analyzed. Using the operatorial model associated to a Γ-correlated processes, and the obtained Γ-orthogonal projection on a right submodule of the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$, some prediction facts are presented. The positivity of the angles, which give the possibility to obtain the prediction filter is analysed. For a periodically Γ-correlated process it is proved that the positivity of the Friedrichs angle between past and future is preserved to its stationary dilation process. Some remarks on the generalized Friedrichs angle for several subspaces associated to a periodically Γ-correlated process are made.

2000 *Mathematics Subject Classification*: 47N30, 47A20, 60G25.

*Keywords:*Complete correlated actions, Γ-orthogonal projection, Γ-correlated processes, stationary dilation, angle between past and future, Friedrichs angle, generalized Friedrichs angle.

1. Preliminaries

Let $\mathcal{E}$ be a separable Hilbert space, $\mathcal{L}(\mathcal{E})$ the C^*-algebra of all linear bounded operators on $\mathcal{E}$, and $\mathcal{H}$ a right $\mathcal{L}(\mathcal{E})$-module.

By an *action* of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$ we mean the map $\mathcal{L}(\mathcal{E})\times\mathcal{H}$ into $\mathcal{H}$ given by $Ah := hA$ in the sense of the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$. We are writting Ah instead hA to respect the classical notations from the scalar case. A *correlation* of the action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$ is a map Γ from $\mathcal{H}\times\mathcal{H}$ into $\mathcal{L}(\mathcal{E})$ having the properties:

(i) $\Gamma[h,h]\geq 0,\ \ \Gamma[h,h]=0$ implies $h=0$;

(ii) $\Gamma[h,g]^*=\Gamma[g,h]$;

(iii) $\Gamma[h,Ag]=\Gamma[h,g]A$.

By (ii) and (iii), for finite summs of actions of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$, we have the following useful formula

$$\Gamma\Big[\sum_i A_i h_i, \sum_j B_j g_j\Big] = \sum_{i,j} A_i^* \Gamma[h_i, g_j] B_j.$$

A triplet $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ defined as above was called [9] a *correlated action* of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}$.

An example of correlated action can be constructed as follows. Take as the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H} = \mathcal{L}(\mathcal{E}, \mathcal{K})$ – the space of the linear bounded operators from $\mathcal{E}$ into $\mathcal{K}$, where $\mathcal{E}$ and $\mathcal{K}$ are Hilbert spaces. An action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{L}(\mathcal{E}, \mathcal{K})$ is given if we consider $AV := VA$ for each $A \in \mathcal{L}(\mathcal{E})$ and $V \in \mathcal{L}(\mathcal{E}, \mathcal{K})$. It is easy to see that $\Gamma[V_1, V_2] = V_1^* V_2$ is a correlation of the action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{L}(\mathcal{E}, \mathcal{K})$, and the triplet $\{\mathcal{E}, \mathcal{L}(\mathcal{E}, \mathcal{K}), \Gamma\}$ is a correlated action (the *operatorial model*). It was proved [9] that any abstract correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ can be embedded into the operatorial model. Namely, there exists an algebraic embedding $h \to V_h$ of $\mathcal{H}$ into $\mathcal{L}(\mathcal{E}, \mathcal{K})$, where $\mathcal{K}$ is obtained as the Aronsjain reproducing kernel Hilbert space given by a positive definite kernel obtained from the correlation Γ. The generators of $\mathcal{K}$ are elements of the form $\gamma_{(a,h)}$: $\mathcal{E} \times \mathcal{H} \to \mathbb{C}$, where $\gamma_{(a,h)}(b, g) = \langle \Gamma[g, h]a, b\rangle_{\mathcal{E}}$ and the embedding $h \to V_h$ is given by $V_h a = \gamma_{(a,b)}$.

Due to such an embedding of any correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ into the operatorial model, prediction problems can be formulated and solved using operator techniques. In the particular case when the embedding $h \to V_h$ is onto, the correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ is caled a *complete correlated action.* In this paper most of properties are analysed in the complete correlated case.

Even $\mathcal{H}$ is only a right $\mathcal{L}(\mathcal{E})$-module, if $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ is a complete correlated action, it was proved [9] the existence of a Γ-orthogonal projection "on" a right $\mathcal{L}(\mathcal{E})$-submodule $\mathcal{H}_1$ of $\mathcal{H}$, namely, if

$$\mathcal{K}_1 = \bigvee_{x \in \mathcal{H}_1} V_x \mathcal{E} \subset \mathcal{K}, \tag{1}$$

for each $h \in \mathcal{H}$ there exists a unique element $h_1 \in \mathcal{H}$ such that for each $a \in \mathcal{E}$ we have

$$V_{h_1} a \in \mathcal{K}_1 \qquad \text{and} \qquad V_{h-h_1} a \in \mathcal{K}_1^{\perp}. \tag{2}$$

Moreover, we have

$$\Gamma[h - h_1, h - h_1] = \inf_{x \in \mathcal{H}_1} \Gamma[h - x, h - x], \tag{3}$$

where the infimum is taken in the set of all positive operators from $\mathcal{L}(\mathcal{E})$.

Therefore if we put

$$\mathcal{P}_{\mathcal{H}_1} h = h_1, \tag{4}$$

then we can interpret the endomorphism $\mathcal{P}_{\mathcal{H}_1}$ of $\mathcal{H}$ as a Γ-orthogonal projection "on" $\mathcal{H}_1$, since we have $\mathcal{P}^2_{\mathcal{H}_1} = \mathcal{P}_{\mathcal{H}_1}$ and $\Gamma[\mathcal{P}_{\mathcal{H}_1} h, g] = \Gamma[h, \mathcal{P}_{\mathcal{H}_1} g]$.

Let us remark that the unique element h_1 obtained by the Γ-orthogonal projection of $h \in \mathcal{H}$ can belongs not necessary to $\mathcal{H}_1$, but, due to (3) it is close enough to be considered as the best estimation.

The previous result can be generalized to an "orthogonal projection" from $\mathcal{H}^T$ - the cartesian product of T copies of $\mathcal{H}$ on a submodule $\mathcal{M}$ of $\mathcal{H}^T$, as follows. Firstly, the embedding of $\mathcal{H}^T$ into $\mathcal{L}(\mathcal{E}, \mathcal{K}^T)$ is defined by

$$W_X a = (V_{x_1} a, \ldots, V_{x_T} a) \tag{5}$$

for $a \in \mathcal{E}$ and $X = (x_1, \ldots, x_T) \in \mathcal{H}^T$, and then the extended "orthogonal projection" $\mathcal{P}_{\mathcal{M}} X$ it follows with respect to an appropriate correlation [11], considering $\mathcal{K}^T_1 = \bigvee_{X \in \mathcal{M}} W_X \mathcal{E}$ in $\mathcal{K}^T$. The action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{H}^T$ is given by acting on the components, which is a particular case of the matrix action of $\mathcal{L}(\mathcal{E})^{T \times T}$ on $\mathcal{H}^T$ in the sense of the right multiplication.

A Γ-correlated process is a sequence $(f_t)_{t \in G} \subset \mathcal{H}$, where G is $\mathbb{Z}$, $\mathbb{R}$, or more generally a locally compact group. The process (f_t) is *stationary* if $\Gamma[f_s, f_t]$ depends only on $t-s$ and not by s and t separately. For a Γ-correlated process (not necessary stationary) the *past-present* at the moment $t = n$ is the right $\mathcal{L}(\mathcal{E})$-submodule

$$\mathcal{H}^f_n = \Big\{ \sum_k A_k f_k;\ A_k \in \mathcal{L}(\mathcal{E}),\ k \leq n \Big\}, \tag{6}$$

and the *future* is

$$\widetilde{\mathcal{H}}^f_n = \Big\{ \sum_k A_k f_k;\ A_k \in \mathcal{L}(\mathcal{E}),\ k > n \Big\}. \tag{7}$$

By the embedding $h \to V_h$ of $\mathcal{H}$ into $\mathcal{L}(\mathcal{E}, \mathcal{K})$, the corresponding *past* and *future* will be the closed subspaces of $\mathcal{K}$ given by

$$\mathcal{K}^f_n = \bigvee_{j \leq n} V_{f_j} \mathcal{E} \tag{8}$$

and

$$\widetilde{\mathcal{K}}_n^f = \bigvee_{j>n} V_{f_j}\mathcal{E}. \tag{9}$$

Similarly, various processes can be considered in the right $\mathcal{L}(\mathcal{E})$-module, or $\mathcal{L}(\mathcal{E})^{T\times T}$-module $\mathcal{H}^T$, and appropriate past and future can be constructed. Also, various correlations can be done. Between these, especially for the study of periodically correlated processes, the following correlations are of interest. For $X = (x_1, \ldots, x_T)$ and $Y = (y_1, \ldots, y_T)$ from $\mathcal{H}^T$, taking into account the right action of $\mathcal{L}(\mathcal{E})$, respectively of $\mathcal{L}(\mathcal{E})^{T\times T}$ on $\mathcal{H}^T$, it is simply to see that $\Gamma_1 : \mathcal{H}^T \times \mathcal{H}^T \to \mathcal{L}(\mathcal{E})$ and $\Gamma_T : \mathcal{H}^T \times \mathcal{H}^T \to \mathcal{L}(\mathcal{E})^{T\times T}$ defined, respectively, by

$$\Gamma_1[X, Y] = \sum_{k=1}^{T} \Gamma[x_k, y_k] \tag{10}$$

and the matriceal one

$$\Gamma_T[X, Y] = \Big(\Gamma[x_i, y_j]\Big)_{i,j\in\{1,2,\ldots,T\}} \tag{11}$$

are correlations on $\mathcal{H}^T$.

Remember that a process (f_t) is periodically Γ-correlated if there exists a positive T such that $\Gamma[f_{s+T}, f_{t+T}] = \Gamma[f_s, f_t]$.

For a Γ-correlated process (f_t), if we take sequences of consecutive T terms

$$X_n = \big(f_n, f_{n+1}, \ldots, f_{n+T-1}\big), \tag{12}$$

then (X_n) is a stationary Γ_1-correlated process in $\mathcal{H}^T$. Also, taking consecutive blocks of length T

$$X_n^T = \big(f_{nT}, f_{nT+1}, \ldots, f_{nT+T-1}\big), \tag{13}$$

then (X_n^T) is a stationary Γ_T-correlated process in $\mathcal{H}^T$.

From prediction point of view and for the study of periodically Γ-correlated processes, the following result [11] was proved.

Proposition 0.1. *Let $(f_n)_{n\in\mathbb{Z}}$ be a Γ-correlated process in $\mathcal{H}$, $T \geq 2$, (X_n) and (X_n^T) defined by* (12) *and* (13). *The following are equivalent:*

(i) $\{f_n\}$ is periodically Γ-correlated in $\mathcal{H}$, with the period T;

(ii) $\{X_n\}$ is stationary Γ_1-correlated in $\mathcal{H}^T$;

(iii) $\{X_n^T\}$ is stationary Γ_T-correlated in $\mathcal{H}^T$.

A nonstatioary Γ-correlated process (f_t) in $\mathcal{H}$ has a *stationary dilation* if there exists a larger right module H and a stationary process (g_t) in H such that $f_t = \mathcal{P}^H_{\mathcal{H}} g_t$. It is easy to see that each periodically Γ-correlated process $(f_t) \subset \mathcal{H}$ has a stationary Γ_1-correlated dilation $(X_n) \subset \mathcal{H}^T$.

The property of some processes to have a stationary dilation permits us to use some stationary techniques in the study of nonstationary processes. This is the case at least for the processes very close to the stationary processes, such as periodically, harmonizable, or uniformly bounded linearly stationary processes.

2. The Friedrichs angle

The notion of angle between two subspaces of a Hilbert space arises in Friedrichs work [5] and later by Dixmier [4], but implicitly, the origin is much older, starting from the general definition of the scalar product of two vectors $\langle h, g\rangle = \|h\| \, \|g\| \cdot \cos\alpha$. From prediction point of wiew, Helson and Szegő [6] have stated the properties of the angle between the past and the future of a process as the third prediction problem. Starting with the study of Helson and Szegő [6], the results was generalized in various contexts, helping in the characterization of stationary and some nonstationary processes. Here a generalization for the Γ-correlated processes is given, and some results for periodically case are analysed.

Let M_1 and M_2 be two subspaces of a Hilbert space $\mathcal{K}$, and $M = M_1 \cap M_2$. The *Friedrichs angle* between M_1 and M_2 is defined to be the angle in $[0, \pi/2]$ whose cosine is given by

$$c(M_1, M_2) := \sup\{|\langle k_1, k_2\rangle| \,; k_i \in M_i \cap M^{\perp} \cap B_{\mathcal{K}}, \}, \tag{14}$$

where $B_{\mathcal{K}}$ is the unit ball of $\mathcal{K}$.

The *angle* (sometimes called the Dixmier angle) between two subspaces M_1 and M_2 from a Hilbert space $\mathcal{K}$ is given by its cosine

$$\rho(M_1, M_2) := \sup\left\{ |\langle k_1, k_2\rangle| \,; k_i \in M_i \cap B_{\mathcal{K}}, \right\}. \tag{15}$$

By (14) and (15) it follows that $c(M_1, M_2) \leq \rho(M_1, M_2)$. Obviously we have $c(M_1, M_2) = \rho(M_1 \cap M^{\perp}, M_2 \cap M^{\perp})$, and of course $c(M_1, M_2) = c(M_1^{\perp}, M_2^{\perp})$. Various properties of the angles between subspaces in a Hilbert space can be found in [3]. Here some properties of the Friedrichs angle and the generalized

Friedrichs angle [2] are used in the case of processes in a complete correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$. In the context of a complete correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ the Friedrichs angle between the submodules $\mathcal{M}_1$ and $\mathcal{M}_2$ of the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$ is given by its cosine

$$c(\mathcal{M}_1, \mathcal{M}_2) = \sup \big\{ \, |\langle \Gamma[h_1, h_2]a_1, a_2\rangle| \, ; \|\Gamma[h_i, h_i]a_i\| \leq 1, i = 1, 2 \big\},$$

where $h_i \in \mathcal{M}_i \cap \mathcal{M}^{\perp}$, $\mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2$, $\mathcal{M}^{\perp} = \{h \in \mathcal{H}; \Gamma[h, g] = 0, g \in \mathcal{M}\}$, and $a_1, a_2 \in \mathcal{E}$,. The Dixmier angle between the submodules $\mathcal{M}_1$ and $\mathcal{M}_2$ is given by its cosine

$$\rho(\mathcal{M}_1, \mathcal{M}_2) = \sup \big\{ \, |\langle \Gamma[h_1, h_2]a_1, a_2\rangle| \, ; \|\Gamma[h_i, h_i]a_i\| \leq 1, i = 1, 2 \big\},$$

where $h_i \in \mathcal{M}_i$, and $a_1, a_2 \in \mathcal{E}$.

We say that two submodules $\mathcal{M}_1$ and $\mathcal{M}_2$ of the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$ have a *positive angle*, if $\rho(\mathcal{M}_1, \mathcal{M}_2) < 1$, or equivalently, if there exists $\rho < 1$ such that for any $h \in \mathcal{M}_1, g \in \mathcal{M}_2, \ a, b \in \mathcal{E}$

$$|\langle \Gamma[g, h]a, b\rangle_{\mathcal{E}}| \leq \rho \, \|V_h a\| \, \|V_g b\| \, . \tag{16}$$

Analogous, two submodules $\mathcal{M}_1$ and $\mathcal{M}_2$ of the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$ have a *positive Friedrichs angle*, if $c(\mathcal{M}_1, \mathcal{M}_2) < 1$, or equivalently, if there exists $c < 1$ such that for any $h \in \mathcal{M}_1 \cap \mathcal{M}^{\perp}$, $g \in \mathcal{M}_2 \cap \mathcal{M}^{\perp}$, $\mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2$, $a, b \in \mathcal{E}$

$$|\langle \Gamma[g, h]a, b\rangle_{\mathcal{E}}| \leq c \, \|V_h a\| \, \|V_g b\| \, . \tag{17}$$

In the study of prediction problems we are interested in the case when the angle or the Friedrichs angle between past and future is positive, i.e., when $\rho(n) = \rho(\mathcal{H}_n^f, \widetilde{\mathcal{H}}_n^f) < 1$, or $c(n) = c(\mathcal{H}_n^f, \widetilde{\mathcal{H}}_n^f) < 1$.

As a remark, if (f_n) is a stationary Γ-correlated process in $\mathcal{H}$, then, due to the fact that $\Gamma[f_p, f_k] = \Gamma[f_{p+m}, f_{k+m}]$ for each $m \in \mathbb{Z}$, the Dixmier angle and the Friedrichs angle between the past and future is constant, i.e. not depend on the choosing of the present time $t = n$, which is no longer valid for nonstationary processes.

We have seen that a periodically Γ-correlated process $(f_n)_{n\in\mathbb{Z}}$ from $\mathcal{H}$ has a stationary Γ-correlated dilation (X_n) in $\mathcal{H}^T$. In [11] an explicit stationary dilation is constructed which help in obtaining the Wiener filter for prediction and the prediction-error operator function for a periodically Γ-correlated process, in terms of the operator coefficients of its attached maximal function. Here we prove the following result concerning the Friedrichs angle of the stationary dilation of a periodically Γ-correlated process.

PROPOSITION 0.2. *If (f_n) from $\mathcal{H}$ is a periodically Γ-correlated process with a positive Friedrichs angle between its past and future, then the Friedrichs angle between the past and the future of its stationary Γ_1-correlated dilation (X_n) from $\mathcal{H}^T$ it is also positive.*

Proof. Similarly as in (6) and (7), in $\mathcal{H}^T$ the past H_n^X and the future $\tilde{H}_n^X$ for a process $(X_n) \subset \mathcal{H}^T$ is constructed as linear combinations of finite actions of $\mathcal{L}(\mathcal{E})$ on $(X_n) \subset \mathcal{H}^T$. If (f_n) from $\mathcal{H}$ is a periodically Γ-correlated process having a positive Friedrichs angle between its past and future, then at each time $t = n$ there exists $c(n) < 1$ such that

$$|\langle \Gamma[g,h]a,b\rangle_{\mathcal{E}}| \leq c(n)\, \|V_h a\|\, \|V_g b\|$$

for each $h \in \mathcal{H}_n^f \cap \mathcal{M}^{\perp}$, $g \in \tilde{\mathcal{H}}_n^f \cap \mathcal{M}^{\perp}$, where $\mathcal{M} = \mathcal{H}_n^f \cap \tilde{\mathcal{H}}_n^f$ and $a, b \in \mathcal{E}$.

For each element $X = \sum\limits_{k\leq n} A_k X_k$ from $H_n^X \cap M^{\perp}$ and $Y = \sum\limits_{p>n} B_p X_p$ from $\tilde{H}_n^X \cap M^{\perp}$, (where $M = H_n^X \cap \tilde{H}_n^X$) of the process $X_n = (f_n, f_{n+1}, \dots, f_{n+T-1})$, and for any $a, b \in \mathcal{E}$ we have

$$|\langle \Gamma_1[X,Y]a,b\rangle_{\mathcal{E}}| = \left| \left\langle \Gamma_1 \Big[\sum_{p>n} B_p X_p, \sum_{k\leq n} A_k X_k \Big] a, b \right\rangle_{\mathcal{E}} \right| =$$

$$= \left| \sum_{p>n} \sum_{k\leq n} \langle \Gamma_1[B_p X_p, A_k X_k]a, b\rangle_{\mathcal{E}} \right| =$$

$$= \left| \sum_{p>n} \sum_{k\leq n} \sum_{i=0}^{T-1} \langle \Gamma[B_p f_{p+i}, A_k f_{k+i}]a, b\rangle_{\mathcal{E}} \right| =$$

$$= \left| \sum_{p>n} \sum_{k\leq n} \sum_{i=0}^{T-1} \left\langle B_p^* \Gamma[f_{p+i}, f_{k+i}] A_k a, b\right\rangle_{\mathcal{E}} \right| =$$

$$= \left| \sum_{i=0}^{T-1} \left\langle \Gamma \Big[\sum_{p>n} B_p f_{p+i}, \sum_{k\leq n} A_k f_{k+i} \Big] a, b \right\rangle_{\mathcal{E}} \right| \leq$$

$$\leq \sum_{i=0}^{T-1} c_i(n) \left\| \sum_{k\leq n} A_k f_{k+i} a \right\| \left\| \sum_{p>n} B_p f_{p+i} b \right\| \leq$$

$$\leq c(n) \sum_{i=0}^{T-1} \left\| \sum_{k \leq n} A_k f_{k+i} a \right\| \left\| \sum_{p>n} B_p f_{p+i} b \right\| \leq$$

$$\leq c(n) \Big(\sum_{i=0}^{T-1} \left\| \sum_{k \leq n} A_k f_{k+i} a \right\|^2 \Big)^{1/2} \Big(\sum_{i=0}^{T-1} \left\| \sum_{p>n} B_p f_{p+i} b \right\|^2 \Big)^{1/2} =$$

$$= c \left\| \sum_{k \leq n} A_k W_{X_k} a \right\| \left\| \sum_{p>n} B_p W_{X_p} b \right\| = \rho \left\| W_X a \right\| \left\| W_Y b \right\|,$$

where $c(n)$ is the maximum of $c_i(n);\ i = 0, 1, \ldots, T-1$, and we used the embedding $X \to W_X$ of $\mathcal{H}^T$ into $\mathcal{L}(\mathcal{E}, \mathcal{K}^T)$ and the fact that $c(n) = c$ for stationary Γ_1-correlated proces (X_n). Therefore $|\langle \Gamma_1[X,Y]a, b\rangle_{\mathcal{E}}| \leq c \left\| W_X a \right\| \left\| W_Y b \right\|$ for each $X \in H_n^X \cap M^\perp$, $Y \in \tilde{H}_n^X \cap M^\perp$, and the Friedrichs angle between the past and the future of the stationary Γ_1-correlated dilation (X_n) of (f_n) is positive. □

If we take $(X_n) \subset \mathcal{H}^T$ the stationary Γ_1-correlated dilation of a periodically Γ-correlated process $(f_n) \subset \mathcal{H}$, then the Friedrichs angle between the past and the future of (X_n) is given by

$$c(K_n^X, \tilde{K}_n^X) = \sup\{|\langle X, Y\rangle| \,;\, X \in K_n^X \cap M^\perp \cap B_1, Y \in \tilde{K}_n^X \cap M^\perp \cap B_1\},$$

where $M = K_n^X \cap \tilde{K}_n^X$, B_1 is the unit ball in $\mathcal{K}^T$, and K_n^X and $\tilde{K}_n^X$ are the images of the past, respectively of the future from $\mathcal{K}^T$ by the embedding $X \to W_X$ of $\mathcal{H}^T$ into $\mathcal{L}(\mathcal{E}, \mathcal{K}^T)$

$$K_n^X = \bigvee_{k \leq n} W_{X_k} \mathcal{E}, \qquad \tilde{K}_n^X = \bigvee_{j>n} W_{X_j} \mathcal{E}. \tag{18}$$

Even the Friedrichs and Dixmier angles between the past and the future of the stationary process $(X_n) \subset \mathcal{H}^T$ are constant, the angles between various pasts of the components of $X_n = (f_n, f_{n+1}, \ldots, f_{n+T-1})$ are variable and can be characterized by the generalized Friedrichs angle between several subspaces [2]. To do this, let us first remember the following characterization of the Friedrichs angle for two subspaces [2].

Proposition 0.3. *If M_1 and M_2 are closed subspaces of $\mathcal{K}$, then the angle between M_1 and M_2 is given by*

$$\rho(M_1, M_2) = \sup \left\{ \frac{2Re\, \langle m_1, m_2\rangle}{\|m_1\|^2 + \|m_2\|^2} \;;\; m_j \in M_j, (m_1, m_2) \neq (0,0) \right\}$$

and the Friedrichs angle is

$$c(M_1, M_2) = \sup\left\{\frac{2Re\,\langle m_1, m_2\rangle}{\|m_1\|^2 + \|m_2\|^2}\ ;\ m_j \in M_j \cap M^{\perp}, (m_1, m_2) \neq (0,0)\right\}.$$

Then the generalized Friedrichs angle to several subspaces $(M_1, M_2, \ldots, M_T)$ is defined [2] by

$$c(M_1, \ldots, M_T) = \sup\left\{\frac{2}{T-1}\frac{\sum_{j<k} Re\,\langle m_j, m_k\rangle}{\sum_{i=1}^{T}\|m_i\|^2}\right\} \tag{19}$$

for $m_j \in M_j \cap M^{\perp}$, $\sum_{i=1}^{T}\|m_i\|^2 \neq 0$.

In the case of a periodically Γ-correlated process (f_n), since the intersection $M = \bigcap_{i=0}^{T-1} \mathcal{K}_{n+i} = \mathcal{K}_n^f$, we have the generalized Friedrichs angle associated to $(\mathcal{K}_n^f, \mathcal{K}_{n+1}^f, \ldots, \mathcal{K}_{n+T-1}^f)$, corresponding to $X_n = (f_n, f_{n+1}, \ldots, f_{n+T-1})$, defined by its cosine (or *Friedrichs number*):

$$c(\mathcal{K}_n^f, \mathcal{K}_{n+1}^f, \ldots, \mathcal{K}_{n+T-1}^f) = \sup\left\{\frac{2}{T-1}\frac{\sum_{j<p} Re\,\langle k_j, k_p\rangle}{\sum_{i=0}^{T-1}\|k_i\|^2}\right\} \tag{20}$$

for $k_i \in \mathcal{K}_{n+i}^f \cap (\mathcal{K}_n^f)^{\perp}$, $\sum_{i=0}^{T-1}\|k_i\|^2 \neq 0$.

Analoguous, generalizing the angle ρ between two subspaces to T subspaces, a so called *Dixmier number* is obtained

$$\rho(\mathcal{K}_n^f, \mathcal{K}_{n+1}^f, \ldots, \mathcal{K}_{n+T-1}^f) = \sup\left\{\frac{2}{T-1}\frac{\sum_{j<p} Re\,\langle k_j, k_p\rangle}{\sum_{i=0}^{T-1}\|k_i\|^2}\right\}, \tag{21}$$

for $k_i \in \mathcal{K}_{n+i}^f$, $\sum_{i=0}^{T-1}\|k_i\|^2 \neq 0$.

Some more properties of the Friedrichs number and Dixmier number, as well as relations between them, can be found in [2]. Also, other applications of the generalized Friedrichs angle to periodically Γ-correlated process can be found in [12].

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Ilie Valuşescu
Romanian Academy,
Institute of Mathematics "Simion Stoilow"
P.O.Box 1-764, 014700 Bucharest, Romania
e-mail: *Ilie.Valusescu@imar.ro*