EXPONENTIAL INSTABILITY IN MEAN SQUARE AND ADMISSIBILITY FOR STOCHASTIC VARIATIONAL EQUATION

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Abstract. We associate with a stochastic cocycle $\Theta = (\varphi, \Phi)$, on $Y = \Omega \times H$, a stochastic variational integral equation and we characterize the exponential instability in mean square of stochastic equations in therms of solvability of the associated equation. Thus we obtain a generalization of stochastic case for results obtained by O. Perron [8], in deterministic case.

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1. Introduction

Let $\left(\Omega, F, \{F_t\}_{t\geq 0}, \mathbf{P}\right)$ be a standard filtered probability space and let $\varphi : \mathbf{R}_+ \times \Omega \to \Omega$ a stochastic semiflow on Ω. We consider a stochastic variational equations

$$\frac{du(t)}{dt} = A(\varphi(t,\omega))u(t), \qquad t \geq 0 \tag{1}$$

and the nonhomogeneous equation

$$du(t) = A(\varphi(t,\omega))u(t)dt + B(t)dW(t), \qquad t \geq 0 \tag{2}$$

where A is given by linear operators $A(\omega) \in L(H)$ on Hilbert space H, such that $\omega \to A(\omega)$ is strongly measurable. In addition $A(\omega)$ are infinitesimal generators of an analytic C_0-semigroup on H denoted by $e^{-tA(\omega)}, t \geq 0$ and he function $t \to A(\varphi(t,\omega))$ is Holder continuous with values in $L(H)$ (see Carrabalo in [3]). B is a continuous and bounded in mean square stochastic process on H, and $W(t), t \geq 0$ is a real Wiener process.

If there exists a stochastic cocycle $\Theta = (\varphi, \Phi)$, on $Y = \Omega \times H$ associated with the stochastic equation (1) then for every $\omega \in \Omega$ the mild solution of (2) is given by the stochastic integral variational equation:

$$u(t) = \Phi(t - s, \varphi(s, \omega))u(s) + \int_s^t \Phi(t - \tau, \varphi(\tau, \omega))B(\tau)dW(\tau), \quad \forall t \geq s \geq 0. \tag{3}$$

The existence of the stochastic cocycle associated with the stochastic variational equation (1) is conditioned by specific conditions for the family of linear operators $\{A(\omega)\}_{\omega\in\Omega}$ and was studied in [3, 6, 11] and L. Arnold in [1].

In the stochastic variational case we may associate with a stochastic cocycle $\Theta = (\varphi, \Phi)$ at avery $\omega \in \Omega$ the stochastic integral equation

$$f(t) = \Phi(t-s, \varphi(s, \omega))f(s)+\int_s^t \Phi(t-\tau, \varphi(\tau, \omega))B(\tau)dW(\tau), \quad \forall t \geq s \geq 0 \tag{4}$$

where $B \in I(R_+, H)$- the input space and $f \in O(R_+, H)$ - the output space, and thus the uniform exponential instability in mean square can be expressed in therms of the admissibility of the pair $(I(R_+, H), O(R_+, H))$ for stochastic variational equation, i.e for every $(\omega, B) \in \Omega \times I(R_+, H)$ the stochastic integral equation (4) has a unique solution $f \in O(R_+, H)$.

The main results is a new characterizations for uniform exponential instability in mean square of stochastic variational equations, and thus extends the results from deterministic case, obtained in [5, 10].

2. Preliminaries

Let H be a separable Hilbert space, $L(H)$- the set of all bounded linear operators on H and $\left(\Omega, F, \{F_t\}_{t\geq 0}, \mathbf{P}\right)$ be a standard filtered probability space.

Definition 1 *A stochastic semiflow on* Ω *is a measurable random field* $\varphi : (\boldsymbol{R}_+ \times \Omega, B(R_+) \otimes F) \to (\Omega, F)$ *satisfying the following properties:*

- $\varphi(0, \omega) = \omega,$
- $\varphi(t + s, \omega) = \varphi(t, \varphi(s, \omega))$

for all $(t, s, \omega) \in R_+^2 \times \Omega$.

Definition 2 *A pair $\Theta = (\varphi, \Phi)$ is called stochastic cocycle on $Y = \Omega \times H$ if φ is a stochastic semiflow on Ω and the mapping $\Phi : \boldsymbol{R}_+ \times \Omega \to L(H)$ satisfies the following properties*

- $\Phi(0, \omega) = I$ *(the identity operator on H),*
- $\Phi(t + s, \omega) = \Phi(t, \varphi(s, \omega))\Phi(s, \omega)$,

for all $(t, s, \omega) \in R_+^2 \times \Omega$.

Example 1 *Let H be a real separable Hilbert space and let Ω be the space of all continuous paths $\omega : R_+ \to X$, such that $\omega(0) = 0$ with the compact open topology. Let F_t for $t \geq 0$, be the σ-algebra generated by the set $\{\omega \to \omega(u) \in X$ with $u \leq t\}$ and let F be the associated Borrel σ-algebra to Ω. If $\boldsymbol{P}$ is a Wiener measure on Ω then $\left(\Omega, F, \{F_t\}_{t\geq 0}, \boldsymbol{P}\right)$ is a filtered probability space with the Wiener motion $W(t, \omega) = \omega(t)$ for all $(t, \omega) \in R_+ \times \Omega$.*

Then $\varphi : R_+ \times \Omega \to \Omega$ defined by $\varphi(t, \omega)(\tau) = \omega(t+\tau) - \omega(t)$ is a stochastic semiflow on Ω generated by Wiener shift.

For every $\omega \in \Omega$ we consider the stochastic parabolic system

$$\frac{dy(\xi, t)}{dt} = \omega(t)\frac{\partial^2 y}{\partial^2 \xi}(t, \xi), \quad t > 0, \ \xi \in (0, 1) \tag{5}$$

$$y(0, t) = y(1, t) = 0$$

where $H = L^2(0, 1)$, $Ax = \frac{\partial^2}{\partial^2 \xi}x$ with $D(A) = H_0^1 \cap H^2(0, 1)$. If for every $\omega \in \Omega$ we denote $A(\omega) = \omega(0)A$. The operator A is the infinitesimal generator of an analytic semigroup T(t) on H [7], and the eigenvalues of A are $\lambda_n = -n^2\pi^2$ with the corresponding eigenvectors $\alpha_n = \sqrt{2}cos(\pi n\xi), n \in N^*$. Thus the analytic semigroup on H is

$$T(t)x = \sum_{n=1}^{\infty} 2e^{-n^2\pi^2} cos(\pi n\xi) \int_0^1 x(\tau)cos(\pi n\tau)d\tau, \quad x \in H. \tag{6}$$

Then the stochastic parabolic equation (5) is rewritten of stochastic variational equation

$$dx(t) = A(\varphi(t, \omega))x(t), \quad t > 0, \tag{7}$$

which generates the stochastic cocycle $\Theta(\varphi, \Phi)$ on $H \times \Omega$, where $\Phi : \boldsymbol{R}_+ \times \Omega \to L(H)$, is defined by

$$\Phi(t, \omega)x = T\left(\int_0^t \omega(\tau)dW(\tau)\right)x$$

and $\varphi : R_+ \times \Omega \to \Omega$ is the stochastic semiflow generated by Wiener shift.

Definition 3 *A stochastic cocycle $\Theta = (\varphi, \Phi)$ on Y is with uniform exponential growth in mean square if there exists a constant $M \geq 1$ and $\lambda > 0$ such that:*

$$E||\Phi(t,\omega)x||^2 \leq Me^{\lambda t}E||x||^2, \; for \; all \; t \geq 0, \; and \; \omega \in \Omega.$$

Definition 4 *The stochastic equation (1) is said to be uniformly exponentially instable in mean square if for every $(\omega, t) \in \Omega \times R_+$ the operator $\Phi(t, \omega)$ is invertible and there are two constants $N \geq 1$, $\nu \geq 0$, such that:*

$$E||\Phi(s,\omega)x||^2 \leq Ne^{-\nu(t-s)}E||\Phi(t,\omega)x||^2, \tag{8}$$

for all $t \geq s \geq 0$, and $(\omega, x) \in Y$.

3. Exponential instability in mean square and admissibility

In the next we denote by $C_b(R_+, H)$ the Banach space of all bounded stochastic process $u : R_+ \to H$ with the norm

$$||u||_2 = \left(\sup_{t\geq 0} E||u(t)||^2 \right)^{1/2}.$$

In the all of the paper we have the hypothesis that the stochastic cocycle $\Theta = (\varphi, \Phi)$ is with uniform exponential growth in mean square.

Definition 5 *The pair $(C_b(R_+, H), C_b(R_+, H))$ is said to be admissible for stochastic equation (2), and denoted by (C_b, C_b), if for every $\omega \in \Omega$ and $B \in C_b(R_+, H)$ there exists a unique function $f_B \in C_b(R_+, H)$ such that the pair (f_B, B) satisfies the stochastic integral equation*

$$f_B(t) = \Phi(t-s, \varphi(s,\omega))f_B(s) + \int_s^t \Phi(t-\tau, \varphi(\tau,\omega))B(\tau)dW(\tau), \quad \forall t \geq s \geq 0. \tag{9}$$

Lemma 1 *If the pair (C_b, C_b) is admissible for stochastic equation (2) then $\Phi(t, \omega)$ is invertible, for all $(t, \omega) \in R_+ \times \Omega$.*

Proof. We prove that $\Phi(t,\omega)$, for all $(t,\omega) \in R_+ \times \Omega$, is a bijective mapping. Let $x \in Ker\Phi(t_0,\omega), t_0 > 0$ if we consider that the stochastic process $B \equiv 0$ and $f_B(t) = \Phi(t,\omega)x$ then $f_B \in C_b$ and the pair (f_B, B) satisfies the relation (9), and from the hypothesys we obtain that $f_B \equiv 0$ and since $x = f_B(0) = 0$ it follows that the stochastic process $\Phi(t,\omega)$ is injective.

Let $x \in H$ and let β be a continuous stochastic process with $\int_0^1 \beta(\tau)dW(\tau) = 1$. If denote by

$$B(t) = -\beta(t-t_0)\Phi(t-t_0, \varphi(t_0,\omega))x, \ for\ all\ t \geq t_0$$

$$f_B(t) = \Phi(t-t_0, \varphi(t_0,\omega))x \int_t^\infty \beta(\tau - t_0)dW(\tau)$$

we have that $B, f_B \in C_b$ and the pair (f_B, B) satisfies the relation (9) for all $t \geq t_0 \geq 0$. If define the mapping

$$g(t) = f_B(t+t_0) - \Phi(t, \varphi(t_0,\omega)) \int_{t+t_0}^\infty \beta(\tau)dW(\tau)x$$

we obtain that $g \in C_b$ and we deduce that the pair $(g, 0)$ satisfies the equation (9) and thus from hypothesis it follows that $g = 0$ and so from relation (9) we have $x = f_B(t_0) = \Phi(t_0,\omega)f(0) \in Im\Phi(t_0,\omega)$. Thus the mapping $\Phi(r,\omega)$ is surjective and so is an invertible mapping. □

Remark 1 *If the pair (C_b, C_b) is admissible for stochastic equation (2) then for every $\omega \in \Omega$ we can consider the subspace $D(Q)$ of all stochastic process from $C_b(R_+, H)$ which are solutions of stochastic integral equation (9).*

Lemma 2 *If the pair (C_b, C_b) is admissible for stochastic equation (2) then there exist a positive constant K such that*

$$E||f_B||^2 \leq K\ E||B||^2, \tag{10}$$

for every $f_B \in D(Q)$, where K is independent of B.

Proof. From hypothesis we have that the operator $Q : D(Q) \to C_b(R_+, H)$ is a bijective mapping. In the next we consider the norm

$$|||f||| = ||f||_2 + ||Qf||_2. \tag{11}$$

and we prove D(Q) is a complete space. Let $\{f_B^n\}$ be a sequence, and so from (11) this is a fundamental sequence and is in $C_b(R_+, H)$, and so exists a limit $f_B \in C_b(R_+, H)$ such that for all $t \geq 0$ we have

$$E||f_B^n - f_B||^2 \to 0, \quad n \to \infty.$$

From

$$||Q(f_B^n - f_B^m)||_2 \leq ||Q|||||f_B^n - f_B^m|||$$

result that the sequence $Qf_B^n = B_n$ is fundamental in $C_b(R_+, H)$ and so here exist a limit B(t) such that

$$\sup_{t\geq 0} E||B_n(t) - B(t)||^2 \to 0,$$

for $n \to \infty$, and $B \in C_b(R_+, H)$. We prove that $f_B(t)$ satisfies the equation (9). Thus, since the stochastic process $B(t)$ is continuous and bounded it follows that f_B is F_t-measurable and so exist the integral from equation (9). For all $t \geq 0$ we have

$$E\left\|f_B(t) - \Phi(t-s, \varphi(s,\omega))f_B(s) - \int_s^t \Phi(t-\tau, \varphi(\tau,\omega))B(\tau)dW(\tau)\right\|^2 \leq$$

$$\leq 2E||f_B(t) - f_B^n(t)||^2 +$$

$$+2E\left\|f_B^n(t) - \Phi(t-s, \varphi(s,\omega))f_B(s) - \int_s^t \Phi(t-\tau, \varphi(\tau,\omega))B(\tau)dW(\tau)\right\|^2$$

For $n \to \infty$, the first therm of sum tends to 0, so in the next we estime the second therm. Since $f_B^n(t) \in D(Q)$, for all $n \to \infty$, we have that

$$f_B^n(t) = \Phi(t-s, \varphi(s,\omega))f_B^n(s) + \int_s^t \Phi(t-\tau, \varphi(\tau,\omega))B_n(\tau)dW(\tau), \quad \forall t \geq s \geq 0 \tag{12}$$

and so

$$E\left\|f_B^n(t) - \Phi(t-s, \varphi(s,\omega))f_B(s) - \int_s^t \Phi(t-\tau, \varphi(\tau,\omega))B(\tau)dW(\tau)\right\|^2 \leq$$

$$\leq 2E||\Phi(t-s, \varphi(s,\omega))||^2 E||f_B^n(s) - f_B(s)||^2 +$$

$$+2E\left\|\int_s^t \Phi(t-\tau,\varphi(\tau,\omega))(B_n(\tau)-B(\tau))dW(\tau)\right\|^2 \leq$$

$$\leq 2E||\Phi(t-s,\varphi(s,\omega))||^2 E||f_B^n(s)-f_B(s)||^2+$$

$$+2\int_s^t E||\Phi(t-\tau,\varphi(\tau,\omega))||^2 E||B_n(\tau)-B(\tau)||^2 d\tau$$

From hypothesis obtained that the both therms of sum tends to 0 for $n \to \infty$. So we have that $f_B(t)$ satisfies the equation (9) with probability 1, for all $t \geq 0$. Thus the space $D(Q)$ is complet and from closed graph theorem result that Q^{-1} is a continuous stochastic process and from (11) we have

$$||f_B||_2 \leq |||f_B||| \leq ||Q^{-1}||||B||_2 \leq K||B||_2,$$

and so we obtain the relation (10).□

The main result is a theorem of Perron type [8], and represent a characterization of exponential instability in mean square in therms of admissibility for stochastic variational equation.

Theorem 1 *Let $\Theta = (\varphi, \Phi)$ be a stochastic cocycle on Y with exponential growth in mean square. Then the stochastic variational equation (1) is uniform exponentially instable in mean square if and only if the pair (C_b, C_b) is admissible for stochastic integral equation (2).*

Proof. Necessity Let $B \in C_b(R_+, H)$ and we consider the stochastic process

$$f_B : R_+ \to H, \quad f_B(t) = -\int_t^\infty \Phi(t-\tau,\varphi(\tau,\omega))^{-1}B(\tau)dW(\tau).$$

We have that $f_B \in C_b(R_+, H)$ and that the pair (f_B, B) satisfies the equation (9). Let $\tilde{f}_B \in C_b(R_+, H)$ be such that $(\tilde{f}_B, B)$ satisfies the equation (9). If we denote $g = \tilde{f}_B - f_B$ we obtained that $g(t) = \Phi(t-s,\varphi(s,\omega))g(s)$ for all $t \geq s \geq 0$. Let $s \geq 0$ and from hypothesis we have

$$E||g(s)||^2 \leq Ne^{-\nu(t-s)}E||g(t)||^2 \leq Ne^{-\nu(t-s)}||g||_2$$

and so $g \equiv 0$. This shows that the stochastic process f is uniquely determined and we obtain that the pair (C_b, C_b) is admissible for stochastic integral equation (2).

Sufficiency For every $t_0 > 0$, let $\alpha : R_+ \to [0,1]$ be a stochastic process, with compact support in (t_0, ∞), defined by $\alpha(t) = 1,\ for\ t \in [0, t_0]$, and $\alpha(t) = 0,\ for\ t \geq t_0 + 1$.

If we consider the stochastic processes $B, f_B : R_+ \to H$ defined by

$$B(t) = -\alpha(t)\frac{\Phi(t,\omega)x}{E||\Phi(t,\omega)x||^2},$$

$$f_B(t) = \int_t^\infty \frac{\alpha(\tau)\Phi(t,\omega)x}{E||\Phi(\tau,\omega)x||^2}dW(\tau),$$

then $f_B, B \in C_b(R_+, H)$ and the pair (f_B, B) satisfies the relation (9) for every $t_0 > 0$. Thus, from Lemma 10 it follows that

$$E\left\|\Phi(t,\omega)x\int_t^\infty \frac{\alpha(\tau)}{E||\Phi(\tau,\omega)x||^2}dW(\tau)\right\|^2 \leq K,$$

for all $(t,\omega) \in \mathbf{R}_+ \times \Omega$. So we obtain

$$\int_t^\infty \frac{1}{E||\Phi(\tau,\omega)x||^2}d\tau \leq \frac{K}{E||\Phi(t,\omega)x||^2}. \tag{13}$$

If we consider the function

$$\delta(t) = \int_t^\infty \frac{1}{E||\Phi(s,\omega)x||^2}ds,$$

then it follows that

$$\delta^{'}(t) \leq -\frac{1}{K}\delta(t).$$

Integrating this inequality on $[0,t]$ result

$$\delta(t) \leq e^{-\frac{1}{K}t}\delta(0). \tag{14}$$

Since the stochastic cocycle Φ have uniform exponential growth in mean square, there exists the positive constants M, λ such that

$$E||\Phi(s,\omega)x||^2 \leq Me^{\lambda(s-t)}E||\Phi(t,\omega)x||^2, \tag{15}$$

for all $(\omega, x) \in \Omega \times X$ and $s \geq t$. Thus we obtain

$$\delta(t)E||\Phi(t,\omega)x||^2 = E||\Phi(t,\omega)x||^2\int_t^\infty \frac{1}{E||\Phi(\tau,\omega)x||^2}d\tau \geq$$

$$\geq \int_t^{\infty} \frac{1}{M} e^{-\lambda(\tau - t)} d\tau = L,$$

where L is a positive constant. From the relations (13) and (14) results

$$E||\Phi(t,\omega)x||^2 \geq \frac{L}{\delta(t)} \geq \frac{L}{\delta(0)} e^{\frac{1}{K}t} \geq \frac{L}{K} e^{\frac{1}{K}t} E||x||^2.$$

If denote $N = \frac{K}{L}$, $\nu = \frac{1}{K}$, we have

$$E||x||^2 \leq Ne^{-\nu t} E||\Phi(t,\omega)x||^2, \ \forall \ t \geq 0, \ \forall \ (x,\omega) \in H \times \Omega. \tag{16}$$

Thus, from Lemma 1 and relation (16), we obtain that the stochastic variation equation (1) is uniformly exponentially instable in mean square. □

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