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ON PARAMETERS ESTIMATION OF STATIONARY AR(1) WITH NONZERO MEAN ALPHA-STABLE INNOVATIONS IN THE CASE $\alpha \in]1,2]$

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Abstract. Most of articles on stationary first order autoregressive processes in model given by :

$$X_n = \lambda X_{n-1} + Z_n, \quad n \in \mathbb{Z}$$

with i.i.d. alpha-stable innovations in the case $\alpha > 1$, consider a common mean centered on zero. Whereas one doesn't know the data so indeed are centered or not. In this synthesis, we are going to omit this assumption to take the innovations that haven't zero of mean and we will use obtained results in the i.i.d. case for estimating the parameters of a stable AR(1) via the residuals estimators.

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1. Introdution

Consider a first order autoregressive model defined by :

$$X_n = \lambda X_{n-1} + Z_n, \quad n \in \mathbb{Z} \tag{1}$$

where λ is AR(1) parameter such as $|\lambda| < 1$ .The sequence (Z_n) of innovations is supposed independent and identically distributed (i.i.d.) and has common distribution G is a Levy-stable law with stability index $\alpha_z \in]1,2]$ indicated by $S_{\alpha_z}(\mu_z, \beta_z, \gamma_z)$; consequently , it satisfies a standard tail regularity and balance condition, i.e. :

$$1 - G(z) \sim p_z C_{\alpha_z} z^{-\alpha_z} \quad , \quad G(-z) \sim q_z C_{\alpha_z} z^{-\alpha_z} \tag{2}$$

and

$$\frac{1 - G(z)}{1 - G(z) + G(-z)} \sim p_z \quad , \quad \frac{G(-z)}{1 - G(z) + G(-z)} \sim q_z \tag{3}$$

as $z \to \infty$, p_z and q_z are non-negative constants with $p_z + q_z = 1$ and $C_{\alpha_z} > 0$ is some constant.(the notation $a(t) \sim b(t)$ denote the fact that $a(t)/b(t) \to 1$ as $t \to \infty$)

It is well known that in this setup, the distribution of the sequence (X_n) has the same type that the one of the innovations i.e. :

$$X_n \sim S_{\alpha_x}(\mu_x, \beta_x, \gamma_x)$$

where $\alpha_x, \mu_x, \beta_x, \gamma_x$ are its stability index, mean, skewness and dispersion parameters; and consequently, in the same way, this last satisfies a standard tail regularity and balance condition, i.e. :

$$1 - F(x) \sim p_x C_{\alpha_x} x^{-\alpha} \quad , \quad F(-x) \sim q_x C_{\alpha_x} x^{-\alpha} \tag{4}$$

and

$$\frac{1 - F(x)}{1 - F(x) + F(-x)} \sim p_x \quad , \quad \frac{F(-x)}{1 - F(x) + F(x)} \sim q_x \tag{5}$$

as $x \to \infty$ and where F designed distribution function of X_n, p_x and q_x are non-negative constants with $p_x + q_x = 1$ and $C_{\alpha_x} > 0$. Only that, both distributions have the same characteristic exponent : $\alpha_z = \alpha_x$ that we will note simply α (see [10]).

we are going to try in this synthesis, after having estimated the autoregressive coefficient to apply the known enough results on the random variables i.i.d. to residuals, while starting from a process finite realization $X_0, X_1, X_2, ..., X_n$ in order to estimate the AR(1) parameter and those of its distribution.

2. The AR(1) parameter

Let us consider a finite sequence $X_0, X_1, X_2, ..., X_n$ of real random variables which we suppose verifying the autoregressive stable AR(1) model given by (1).

Generally, for an unspecified α-stable law, the well known consistent estimator of λ (see [12]) is given by :

$$\widehat{\lambda}_n = \frac{\sum_{i=1}^{n} X_i X_{i-1}}{\sum_{i=1}^{n} X_{i-1}^2} \tag{6}$$

However, when the mean exists i.e. $\alpha > 1$ the estimator for λ is replaced by its mean corrected version [12] :

$$\widetilde{\lambda}_n = \frac{\sum_{i=1}^{n} (X_i - \bar{X}_n)(X_{i-1} - \bar{X}_n)}{\sum_{i=1}^{n} (X_{i-1} - \overline{X}_n)^2} \tag{7}$$

where : $\bar{X}_n = (n+1)^{-1} \sum_{i=0}^{n} X_i$.

It is shown that under conditions (4), (5), the stationarity condition $\lambda < 1$ and for some other condition, the estimator given by (7) is also consistent (see corollary of theorem 4.2. in [12]) i.e.

$$\widetilde{\lambda}_n \xrightarrow{p} \lambda.$$

and more precisely [5]

$$\widetilde{\lambda}_n - \lambda = O_p([n/\log n])^{1/\alpha} = o_p(n^{1/\theta}) \quad \text{for} \quad \text{all} \quad \theta > \alpha$$

Furthermore, in the case $\alpha = 2$, both estimators (6) and (7) have limiting normal distributions [5]. But, in the case $1 < \alpha < 2$ the limit distributions of these estimators are complex and they are presented each one, in the form as the ratio of two stable laws with specific parameters multiplied by some constant which depend on α, for more details, see [13].

3. Parameters Estimation of AR(1) Stable Distribution

3.1. Levy-stable distributions

The rich class of Levy-stable distributions was introduced and characterized by Paul Levy, about 1925 in his study of normalized sums of independent random variables. It is a class of distributions that allow skewness and fat tails; it includes those of Gaussian and Cauchy and has many intriguing mathematical properties. They were suggested like models for many types of physical and economic systems.

The drawback for these distributions is the lack of explicit formulas for their densities allowing their use, except three cases, in which, one knows their formulas (Gaussian, Cauchy and Levy distributions). Luckily, now there are reliable computer programs to compute Levy-stable distribution functions, densities and quantiles see for example [33] and [35] . Thus, it is possible to use Levy-stable models in various practical fields.

3.2. Characteristic function of Levy-stable distributions

Such distributions are known via their characteristic function and they are generally described by four parameters : a characteristic exponent (index of stability, tail exponent) $\alpha \in]0, 2]$, a skewness parameter $\beta \in [-1, 1]$, a dispersion parameter $\gamma \in]0, \infty[$, a location parameter $\mu \in]-\infty, +\infty[$ and they are indicate by $S_\alpha(\mu, \beta, \gamma)$.

When $\alpha > 1$, the mean of distribution exits and is equal to μ and in this case one will note it by "m"; When the skewness parameter β is positive, the distribution is skewed to right. When $\beta = 0$, the distribution is symmetric about m and otherwise, it's skewed to left. As α is close to 2, β loses its effect and the distribution approaches the Gaussian distribution without being concerned with value of β.

The characteristic function representation of Levy-stable distribution under our condition $\alpha > 1$ is given by:

$$\varphi_Z(t) = \exp\left\{imt - \gamma|t|^\alpha\left(1 - i\beta\tan(\frac{\alpha\pi}{2})sgn(t)\right)\right\} \quad (8)$$

Note that the parameter of dispersion γ is sometimes replaced by what is called the "scale parameter" $\sigma > 0$ (see[45]) with $\gamma = \sigma^\alpha$ and if $\gamma = 1$ and $m = 0$ the distribution is called a "standard Levy-stable distribution". When $\beta = 0$ and $m = 0$ i.e. $\varphi_Z(t) = \exp\{-\gamma|t|^\alpha\}$ then the distribution is noted $S\alpha S(\gamma)$ and it is called a symmetric α-stable distribution.

3.3. Estimating the parameters for an i.i.d. sample

There are at least five principal approaches used for estimating parameters of a Levy-stable distribution $S_\alpha(\mu, \beta, \gamma)$ on the basis of an observed i.i.d sample $Z_1, Z_2, ...Z_n$:

3.3.1. Extreme value approach

This one is especially used to estimate the tail index which is equivalent to inverse of characteristic exponent in the case of stable distributions. The idea is based on the well known following result :

Theorem 1. *Suppose $Z_1, Z_2, ..., Z_n$ are i.i.d. from distribution G which verifying a regular variation condition*

$$1 - G(z) = z^{-\alpha}L(z), \quad z > 0$$

with $L(z)$ is a slower varying function $L(tz)/L(z) \to 1$ as $t \to \infty$

Let $0 < Z_{1,n} < Z_{2,n} < ... < Z_{n,n}$ be the order statistics.Then, the Hill estimator defined by expression

$$H_{k,n} = k^{-1}\sum_{i=1}^{k}\log Z_{n-k+i,n} - \log Z_{n-k,n}$$

is consistent for tail index parameter i.e.

$$H_{k,n} \xrightarrow{p} \alpha^{-1}$$

when $n \to \infty$, $k \to \infty$ and $k/n \to 0$.

The consistency (weak or strong) and normality asymptotic in both cases, i.i.d. model and linear model, of Hill's estimator and its extensions (Dekkers-Einmahl-de Hann's estimator) have been proved by many authors as : Pickands [42], Mason [30], Hall [23], Davis and Resnick [11], Csorgo and al [8], Goldie and Smith [20], Hsing [27], de Hann and Resnick ([?], [36]), Resnick and Starica [43], Datta and

McCormick [10], de Hann and Peng [14] etc.

Once the characteristic exponent was estimating by extreme values theory, one can then estimate the other parameters of Levy-stable distribution like mean, skewness and dispersion parameter.

Indeed, in the situation where $\alpha > 1$ we have :

1. *Mean's estimator:*

   As regards the location parameter which is equal the mean m of distribution we have the Peng's estimator [39] :

$$m_n^P(k) = \widehat{m}_n^-(k) + \widehat{m}_n(k) + \widehat{m}_n^+(k)$$

   where :
   $\widehat{m}_n^-(k) := (k/n)Z_{k,n}\ \widehat{\alpha}_n^-/(\widehat{\alpha}_n^- - 1)$,
   $\widehat{m}_n^+(k) := (k/n)Z_{n-k+1,n}\ \widehat{\alpha}_n^+/(\widehat{\alpha}_n^+ - 1)$,
   and the trimmed mean $\widehat{m}_n(k) := n^{-1}\sum_{i=k+1}^{n-k} Z_{i,n}$
   with :

$$\widehat{\alpha}_n^- := \left\{\frac{1}{k}\sum_{i=1}^{k}\log^+(-Z_{i,n}) - \log^+(-Z_{k,n})\right\}^{-1} = 1/H_{k,n}^- \qquad (9)$$

   and :

$$\widehat{\alpha}_n^+ := \left\{\frac{1}{k}\sum_{i=1}^{k}\log^+(Z_{n-i+1,n}) - \log^+(Z_{n-k+1,n})\right\}^{-1} = 1/H_{k,n}^+ \qquad (10)$$

   $H_{k,n}^-$ and $H_{k,n}^+$ are respectively the Hill estimators of the tail index corresponding to each of the two extremities left and right-hand side of Levy-distribution $S_\alpha(m, \beta, \gamma)$. (Here $\log^+ z := \log(z \vee 1)$, z is a real).

   The Peng estimator is asymptotically normal under some conditions [39] and the strong limiting behavior of $m_n^P(k)$ has been studied by Necir [36] to construct a sequential test with power 1 for the mean of Levy-stable distribution.

2. *Dispersion's estimator:*

   As regards the dispersion parameter γ we have the Meraghni and Necir's estimator [31]:

$$\widehat{\gamma}_n(k) := \left(k\pi/2n\Gamma(\widehat{\alpha}_n)\sin\frac{\pi\widehat{\alpha}_n}{2}\right)Y_{n-k,n}^{\widehat{\alpha}_n}$$

where $\widehat{\alpha}_n = 1/H_{k,n}$ the Hill's estimator of exponent characteristic and $Y_{n-k,n}$ denote the order statistic $|Z|_{n-k,n}$ of the sequence $|Z_1|, |Z_2|, ..., |Z_n|$.

It is shown in [31] that $\widehat{\gamma}_n(k)$ is a consistent estimator for the dispersion parameter γ and if the distribution function belongs to Hall's class of models [22] and under some condition, this estimator is asymptotically normal.

3. *asymmetry's estimator:*

As regards the skewness parameter $\beta = 2p-1$, we have the following estimator due to de Hann and Preira in [17] for balance parameter p :

$$\widehat{p}_n = k^{-1}\sum_{i=1}^n 1_{\{Z_i>|Z|_{n-k,n}\}}$$

where $k = k(n) \to \infty$ and $k/n \to 0$ $(n \to \infty)$, wich is consistent under (3) and the condition which is always verified by the stable distributions :

$$\lim_{z\to\infty}\frac{1-G(z)+G(-z)}{\int_{-z}^{z}t^2dG(t)} = \frac{2-\alpha}{\alpha}.$$

and in addition, under some others conditions (see [17]) and when $0 < p < 1$ then :

$$\sqrt{k}(\widehat{p}_n - p) \longrightarrow \mathcal{N}(0, \sqrt{p(1-p)})$$

3.3.2. The regression approach on sample characteristic function

The idea of the use of the characteristic function sampled to approach the theoretical characteristic function as well as possible, is justified by the fact that there exists a bijective mapping between the distributions functions and their transforms of Fourier-Stieltjes; And the first which proposed this method was Press in 1972 in his article [40] on the estimate of the parameters of a stable distribution called the method of moments, based on transformations of the characteristic function. Comes then, the proposal made by Paulson, Holcomb and Leitch in [41], called method of minimum of distance between the theoretical characteristic function and that of its sampled function.

Koutrouvilis in [28] propose a method based on the regression applied to the function characteristic.

Starting from the general expression of the characteristic function of the stable law given by 8, one can obtain the following writing:

$$\log(-\log|\varphi_Z(t)|^2) = \log(2\gamma) + \alpha \log|t|. \tag{11}$$

which is a function in α. Consequently, one can adjust a linear regression:

$$x_k = C + \alpha y_k + \epsilon_k \tag{12}$$

by posing $x_k = \log(-\log|\widehat{\varphi}_Z(t_k)|^2)$ where $t_k = k\pi/25$, $k = 1, 2, ..., K$, $9 \leq K \leq 134$ according to the proposal of Koutrouvelis, ϵ_k denotes an error term, $C = \log(2\gamma)$ and $y_k = \log|t_k|$, which makes it possible to obtain $\widehat{\alpha}$ and $\widehat{\gamma}$.

Estimators of β and m can be obtained from :

$$\arctan\left(\frac{Im(\varphi_Z)(t)}{Re(\varphi_Z)(t)}\right) = mt + \beta\gamma \tan\frac{\pi\alpha}{2} sgn(t)|t|^{\alpha} \tag{13}$$

by taking $h_n(t) = \arctan\left(\frac{Im(\widehat{\varphi}_Z)(t)}{Re(\widehat{\varphi}_Z)(t)}\right)$ where $\widehat{\varphi}_Z$ is the sample characteristic function :

$$\widehat{\varphi}_Z(t) = \left(n^{-1}\sum_{j=1}^{n}\cos(tz_j)\right) + i\left(n^{-1}\sum_{j=1}^{n}\cos(tz_j)\right)$$

and $s = h_n(u) + \pi k_n(u)$ (the integer $k_n(u)$ makes it possible to consider the other values of the function arctan). Then, one can adjust a linear regression :

$$s_j = mu_j + \beta\widehat{\gamma} \tan\tfrac{\pi\widehat{\alpha}}{2} sgn(u_j)|u_j|^{\widehat{\alpha}} + \eta_j.$$

where $u_j = \frac{\pi j}{50}$, $j = 1, 2, ..., L$ for a suitable L (see [28]) and η_j denotes an error term.

The asymptotic convergence and the normality of the estimators of least squares in a linear regression are well-known.The principal disadvantage of this method is that the results are unsatisfactory when the sample is not standardized [48].To mitigate this problem, Koutrouvelis in [29] proposed another alternative known as "method of iterative regression" whose results are much better for a greater area of parametric space.

3.3.3. L-moments approach The principal idea in this approach which is based on the notion of the "weighted moment" balanced by the law itself, initiated by Greenwood et al. [21] and valid even if only the moment of first order exists, consists in considering linear combinations of these weighted moments.

Let us recall by this occasion the definition of the weighted moments, for any random variable X of distribution function F by:

$$M_{p,r,s} = E\{Z^p[F(Z)]^r[1-F(Z)]^s\}$$

where p, r and s are integers.

In particular for $p = 1$ and $s = 0$, we have $\delta_r := M_{1,r,0} = \int zF(z)^r dF(z)$.

Also let us recall that the weighted moments admit an interpretation similar to that of the ordinary moments like the measures of location, dispersion, asymmetry, kurtosis and other aspects on the shape of the distributions or the samples.

Now, the L-moments are then defined by:

$$\begin{aligned}
\kappa_1 &= \delta_0 \\
\kappa_2 &= 2\ \delta_1 - \delta_0 \\
\kappa_3 &= 6\delta_2 - 6\delta_1 + \delta_0 \\
\kappa_4 &= 20\delta_3 - 30\delta_2 + 12\delta_1 - \delta_0
\end{aligned}$$

as linear combinations with the shifted coefficients of Legendre polynomials [24].

Furthemore, the empirical weighted moments for a sample $Z_1, Z_2, ..., Z_n$ ordered in ascending order, are defined by:

$$m_0 = \frac{1}{n}\sum_{i=1}^{n} Z_i \quad , \quad m_r = \frac{1}{n}\sum_{i=r+1}^{n} \frac{(i-1)(i-2)\ldots(i-r)}{(n-1)(n-2)\ldots(n-r)} Z_i$$

The first L-moments of a sample are defined as for a random variable by :

$$\begin{aligned}
l_1 &= m_0 \\
l_2 &= 2m_1 - m_0 \\
l_3 &= 6m_2 - 6m_1 + m_0 \\
l_4 &= 20m_3 - 30m_2 + 12m_1 - m_0
\end{aligned}$$

One will equalize then, the empirical L-moments at the theoretical L-moments, which enables us to obtain estimators for the four parameters of the distribution $S(\alpha, \mu, \beta, \gamma)$, by solving the system of equations :

$$\begin{cases}
\kappa_1(\alpha, \mu, \beta, \gamma) = l_1 \\
\kappa_2(\alpha, \mu, \beta, \gamma) = l_2 \\
\kappa_3(\alpha, \mu, \beta, \gamma) = l_3 \\
\kappa_4(\alpha, \mu, \beta, \gamma) = l_4
\end{cases}$$

Finally should it be said that the L-moments have a sense as soon as the moment of order one exists even if the other moments miss, such as the stable laws in our case where $\alpha > 1$, and that the asymptotic approximation of the empirical distributions is better for the L-moments than for the ordinary moments [25] as they are also less sensitive to the aberrant data ([44],[47]).

This approach has the same advantages as the method of the traditional moments such as consistency and asymptotic normality, therefore it is a method of very general estimating; it is even very robust and less demanding and, in certain situations, it gives estimators more effective than the estimators of maximum of likelihood [25]. However, the main problem of this approach is that do not exist precise and firm expressions for the theoretical L-moments what generates difficulties on the resolution of the system of equation and inevitably leads to approximation errors.

3.3.4. Maximum Likelihood approach

It is one of the methods most used in statistics, it allows obtaining a consistent estimator and, if it is unique, it is asymptotically without bias, effective and normal; Only, this method remains often difficult to implement because the difficulty lies mainly in the calculation of the likelihood probability,

$$L(\alpha,\mu,\beta,\gamma) = \sum_{i=1}^{n} \log f(Z_i,\alpha,\mu,\beta,\gamma)$$

which must be made in an approached way, with numerical methods of optimization, this on the one hand.

On the other hand, with regard to the stable laws, there are no simple and firm formulas expressing their densities, except in some known cases, which still poses problem in the estimate of their parameters. However, there were attempts on behalf of several mathematicians, making object of calculation on stable laws such as Holt and Crow (1973, [26]) which provided tables of values of the density for various values of α and β, Worsdale (1975, [50]) and Panton (1992, [38]) which provided tables of the functions of distributions of the symmetrical stable laws; Mc Culloch and Panton (1998,[33]) gave tables of the densities and quantiles for completely asymmetrical stable laws; Zolotarev in [51] then, Noland (1996,[35]), this last which obtained integral representations for the densities and the distribution functions as well as the quantiles in a precise way in all parametric space, and all that implies that the cost calculation is significant, without counting the error of approximation induced by the integral formula. finally, let us note that in a comparative study, Ojeda (2001)[37] noticed that the methods based on the maximum of likelihood are most accurate except that them are slowest compared to others; The same remark was observed by a simulative study by Stoyanov and Racheva-Iotova [46] and confirmed by Weron in his calculations made on the value at Risk [49].

3.3.5. Quantiles approach

The work of McCulloch [32] was a generalization of the Fama and Roll approach to provide consistent estimators of all parameters with β is in its full permissible ranges i.e. $[-1,+1]$ but α is only in the range $[0.6, 2.0]$.

The idea is to start with n independent drawing values $z_1, z_2, ..., z_n$ of a distribution $S_\alpha(m,\beta,\gamma)$ to initially estimate the only parameters α and β using simple index of five pre-determined sample quantiles, by considering :

$$l_\alpha = \frac{z_{0.95} - z_{0.05}}{z_{0.75} - z_{0.25}} \quad , \quad l_\beta = \frac{z_{0.95} + z_{0.05} - 2z_{0.50}}{z_{0.95} - z_{0.05}}$$

for which it is shown that they do not depend on both γ and m, and the first one is a strictly decreasing function of α for different values of β, and the second one is strictly increasing function in β for each α. These are thus invertible functions whose inverse functions are respectively :

$$\alpha = \varphi_1(l_\alpha, l_\beta) \text{ and } \beta = \varphi_2(l_\alpha, l_\beta)$$

(Here z_p designing the p-th quantile of the distribution.)

Considering now their consistent estimators (see[32]) :

$$\widehat{l}_\alpha = \frac{\widehat{z}_{0.95} - \widehat{z}_{0.05}}{\widehat{z}_{0.75} - \widehat{z}_{0.25}} \quad , \quad \widehat{l}_\beta = \frac{\widehat{z}_{0.95} + \widehat{z}_{0.05} - 2\widehat{z}_{0.50}}{\widehat{z}_{0.95} - \widehat{z}_{0.05}}$$

which allow to estimate consistently the parameters α and β like :

$$\widehat{\alpha} = \varphi_1(\widehat{l}_\alpha, \widehat{l}_\beta) \ \text{ and } \ \widehat{\beta} = \varphi_2(\widehat{l}_\alpha, \widehat{l}_\beta)$$

(Here $\widehat{z}_p$ designing the p-th empirical quantile).

Remark 1. *In order to avoid a false asymmetry of the small samples, a correction is necessary while arranging in the ascending order the* $z_k := z_{q(k)}$ *with* $q(k) = (2k-1)/2n$ *then we carry out a linear interpolation to obtain* $\widehat{z}_p$ *from* $\widehat{z}_{q(k)}$ *and* $\widehat{z}_{q(k+1)}$ *where*
$q(k) \leq p \leq q(k+1)$.

In a second phase, one will estimate the remainder of the parameters by using the following index

$$l_\gamma = \frac{z_{0.75} - z_{0.25}}{\gamma^{1/\alpha}} := \varphi_3(\alpha,\beta) \quad , \quad l_m = \frac{m - z_{0.5}}{\gamma^{1/\alpha}} := \varphi_4(\alpha,\beta)$$

which give also the consistent estimators [32]:

$$\widehat{\gamma} = \left(\frac{\widehat{z}_{0.75} - \widehat{z}_{0.25}}{\varphi_3(\widehat{\alpha},\widehat{\beta})}\right)^{\widehat{\alpha}} \quad , \quad \widehat{m} = \widehat{\gamma}^{1/\widehat{\alpha}} \ \varphi_4(\widehat{\alpha},\widehat{\beta}) + \widehat{z}_{0.5}$$

As the estimator $\widehat{z}_p$ is consistent and asymptotically normal for z_p and that the functions φ_i are continuous, then the estimators of our stable law parameters are consistent and asymptotically normal.

Also let us note that the functions defined above $\varphi_1, \varphi_2, \varphi_3$ and φ_4 can be calculated on a network of points, thus forming tables like those of DuMouchel [18] and being used as references for our calculation of the estimates of the whole of the parameters of the stable distribution.

Only that this method based on the empirical quantiles goes with values of α pertaining to interval $[0.6, 2.0]$ (see [18]), what corresponds well for our case since $\alpha > 1$.

3.4. Estimating the parameters for the stable AR(1)

After having found an estimate of the autoregression parameter by using (7), we can calculate now n residuals via the recursion :

$$\widehat{Z}_k = X_k - \widehat{\lambda} X_{k-1}, \qquad k = 1, 2, ..., n$$

From this finite sequence of residuals, we can carry out the estimate of the whole of distribution parameters of innovations i.e. α_z, m_z, β_z and γ_z.

Once they are estimated we use the following properties [45] in order to find estimators for the AR(1) distribution parameters :

Property 1. Let $Z_1 \sim S_\alpha(\mu_1, \beta_1, \gamma_1)$ and $Z_2 \sim S_\alpha(\mu_2, \beta_2, \gamma_2)$ be independent stable random variables. Then,

$$Z_1 + Z_2 \sim S_\alpha(\mu, \beta, \gamma)$$

where,

$$\mu = \mu_1 + \mu_2 \ , \ \beta = \frac{\beta_1\gamma_1 + \beta_2\gamma_2}{\gamma_1 + \gamma_2} \ , \ \ \gamma = \gamma_1 + \gamma_2$$

Property 2. Let $Z \sim S_\alpha(\mu, \beta, \gamma)$ with $\alpha > 1$ and $c \in \mathbb{R}$. Then,

$$cZ \sim S_\alpha(c\mu, sgn(c)\beta, |c|^\alpha \gamma)$$

Indeed, On the basis of the expression : $X_n = \lambda X_{n-1} + Z_n$ with $|\lambda| < 1$ and $\alpha > 1$ and the fact that X_{n-1} is independent of Z_n, we have the following relationships between the different estimators :

- $\widehat{\alpha}_X = \widehat{\alpha}_z = \widehat{\alpha}$
- $\widehat{m}_X = \dfrac{\widehat{m}_z}{1 - \widehat{\lambda}}$

- $\widehat{\beta}_X = \begin{cases} \widehat{\beta}_z & 0 \leq \widehat{\lambda} < 1 \\ \dfrac{1-|\widehat{\lambda}|^{\widehat{\alpha}}}{1+|\widehat{\lambda}|^{\widehat{\alpha}}}\widehat{\beta}_z & -1 < \widehat{\lambda} < 0 \end{cases}$
- $\widehat{\gamma}_X = \dfrac{\widehat{\gamma}_z}{1-|\widehat{\lambda}|^{\widehat{\alpha}}}$

4. Simulation

In this section, one will work on synthetic data by generating several sample with several observations of a process AR(1) α-stable to which one will apply our approach to estimate the parameter of the process AR(1) considered, then one will collect the whole of the results obtained in a table on which one will specify also the errors: absolute, relative and the mean squared ones.

On the examples which follow, we took various values for the parameters : α and β by pushing them even with controversial limits of the point of considering simulation, while fixing the coefficient of autocorrelation λ at the value of 1.2 as in the first eight cases ; and in the second time, more precisely in the last both cases, we increased his value towards limits close to 1.

N.B. We preferred to work here with the scale parameter $\sigma = \gamma^{1/\alpha}$ instead of the dispersion parameter and we notice $S(\alpha, \beta, \sigma, m)$ instead of $S_\alpha(m, \beta, \gamma)$ and this in accordance with the notation used in R for the stable distributions .

In each table, we indicate:

- In top, the equation of AR(1) process in which one specified the theoretical value of the coefficient λ and theoretical values of its theoric distribution $S(\alpha, \beta_x, \sigma_x, m_x)$ which parameters are calculate by the last formulas from those of the innovations.
- In bottom and on the first line, one mentioned in the left part, the estimates (the mean values of estimates) of the five parameters of AR(1) i.e. $\widehat{\lambda}$ and $\widehat{\alpha}$, $\widehat{m}_x$, $\widehat{\beta}_x$, $\widehat{\sigma}_x$ calculated by our approach and in the right part, the estimates $\tilde{\alpha}$, $\tilde{m}_x$, $\tilde{\beta}_x$, $\tilde{\sigma}_x$ of these same parameters obtained, as comparison, directly on the sample $X_1, X_2, ..., X_n$ i.e., without passing by the residues, and this, for various sizes of samples ($n = 500, 1000, 10000$) and for one hundred replications ($r = 100$) of each one.
- In addition and in bottom, we are indicate their corresponding errors: absolute errors (AE), relative errors (RE) and mean square errors (MSE).

4.1. Comments

As a whole, the results are very satisfactory by comparing them with the values which the direct estimation of the set of the parameters could provide on the origine sample $X_1, X_2, ..., X_n$.

We have some comments that here:

- In the first series and for the theoretical AR(1) parameters :

$$\lambda = 0.2, \quad \alpha = 1.2, \quad \beta_x = 0.1, \quad \sigma_x = 1.1695, \quad m_x = 6.25$$

which obtained from innovation theoric parameters $\alpha_z = \alpha = 1,2$, $m_z = 5$, $\beta_z = 0,1$, $\sigma_z = 1$ by applied formulas. we have generally, concerning the absolute errors, the estimates of our approach are about the thousandths near, on the other hand those being on the right are hundredth near. Even notices on the relative errors. For the MSE errors of this approach are better in the majority of the cases.

- In the second series and for the theoretical AR(1) parameters :

$$\lambda = 0.2, \quad \alpha = 1.5, \quad \beta_x = 0.1, \quad \sigma_x = 1.0982, \quad m_x = 6.25$$

where α increased value, one notices the same thing for this case too.

- In the third series and for the theoretical AR(1) parameters :

$$\lambda = 0.2, \quad \alpha = 1.2, \quad \beta_x = 0.5, \quad \sigma_x = 1.1695, \quad m_x = 6.25$$

where this time β increased value, one remarks the same thing for the absolute and relative errors. Concerning the MSE, they are better except perhaps for the average in the sample of 500 observations.

- In the forth series and for the theoretical AR(1) parameters :

$$\lambda = 0.2, \quad \alpha = 1.2, \quad \beta_x = 0.9, \quad \sigma_x = 1.1695, \quad m_x = 6.25$$

where β is almost $+1$ (almost totally right skewed), we have the same remark of the preceding case.

- In the fifth series and for the theoretical AR(1) parameters :

$$\lambda = 0.2, \quad \alpha = 1.2, \quad \beta_x = -0.9, \quad \sigma_x = 1.1695, \quad m_x = 6.25$$

where β is almost -1 (almost totally left skewed), we have the same remark of the preceding case.

- In the sixth series and for the theoretical AR(1) parameters :

$$\lambda = 0.2, \quad \alpha = 1.7, \quad \beta_x = 0.1, \quad \sigma_x = 1.0693, \quad m_x = 6.25$$

where α increased value. There are practically the same performances for both approaches.

- In the seventh series and for the theoretical AR(1) parameters :

$$\lambda = 0.2, \quad \alpha = 1.8, \quad \beta_x = 0.1, \quad \sigma_x = 1.0584, \quad m_x = 6.25$$

where α increased value more, there are not great changes and we have even notices that previously.

- In the eighth series and for the theoretical AR(1) parameters :

$$\lambda = 0.2, \quad \alpha = 1.9, \quad \beta_x = 0.1, \quad \sigma_x = 1.0493, \quad m_x = 6.25$$

where α is close to 1, the absolute and relative errors are better in the majority of the cases; For the MSE errors, the values are very close to each other.

- In the ninth series and for the theoretical AR(1) parameters :

$$\lambda = 0.8, \quad \alpha = 1.2, \quad \beta_x = 0.1, \quad \sigma_x = 4.2568, \quad m_x = 25$$

where this time the autoregressive coefficient λ which increases value towards 1, we remark that the errors for our approach are better except two cases of the mean.

- In the last series and for the theoretical AR(1) parameters :

$$\lambda = 0.95, \quad \alpha = 1.8, \quad \beta_x = 0.1, \quad \sigma_x = 11,3387, \quad m_x = 100$$

where the autoregressive coefficient λ is close to 1, we remark that in spite of the instability sometimes of the results, the errors for our approach are better except for some cases.

AR(1): $X_n = 0.2X_{n-1} + Z_n$			
Theoretical values of $S(\alpha, \beta_x, \sigma_x, m_x)$			
α	β_x	σ_x	m_x
1.2000	0.1000	1.1695	6.2500

	Estimated values via $\{\widehat{Z}_k\}$					Estimated values directly on $\{X_k\}$			
	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\beta}_x$	$\widehat{\sigma}_x$	$\widehat{m}_x$	$\tilde{\alpha}$	$\tilde{\beta}_x$	$\tilde{\sigma}_x$	$\tilde{m}_x$
	$r = 100, n = 500$								
	0.1993	1.2013	0.0763	1.1722	6.2726	1.2011	0.0704	1.1364	6.2850
AE	0.0007	0.0013	0.0237	0.0027	0.0226	0.0011	0.0296	0.0331	0.0350
RE	0.0035	0.0011	0.2370	0.0023	0.0036	0.0009	0.2960	0.0283	0.0056
MSE	0.0015	0.0055	0.0192	0.0087	0.1429	0.0049	0.0255	0.0062	0.0144
	$r = 100, n = 1000$								
	0.1993	1.2175	0.0996	1.1651	6.2460	1.2032	0.0817	1.1346	6.2863
AE	0.0007	0.0175	0.0004	0.0044	0.0040	0.0031	0.0183	0.0349	0.0363
RE	0.0035	0.0146	0.0040	0.0038	0.0060	0.0027	0.1830	0.0299	0.0058
MSE	0.0006	0.0028	0.0075	0.0024	0.0480	0.0024	0.0121	0.0034	0.0056
	$r = 100, n = 10000$								
	0.1991	1.1998	0.0997	1.1681	6.2458	1.2003	0.0992	1.1384	6.2842
AE	0.0009	0.0002	0.0003	0.0014	0.0042	0.0003	0.0008	0.0311	0.0342
RE	0.0045	0.0002	0.0030	0.0012	0.0007	0.0002	0.0080	0.0266	0.0055
MSE	4.7e -05	0.0003	0.0008	0.0003	0.0028	0.0004	0.0012	0.0013	0.0016

Table 1: For $\lambda = 0.2$ as autoregressive parameter and $\alpha_z = 1.2$, $\beta_z = 0.1$, $\sigma_z = 1$, $m_z = 5$ as theoretical parameters of the distribution $S(\alpha_z, \beta_z, \sigma_z, m_z)$ of the innovations.

AR(1): $X_n = 0.2X_{n-1} + Z_n$			
Theoretical values of $S(\alpha, \beta_x, \sigma_x, m_x)$			
α	β_x	σ_x	m_x
1.5000	0.1000	1.0982	6.2500

	Estimated values via $\{\widehat{Z}_k\}$					Estimated values directly on $\{X_k\}$			
	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\beta}_x$	$\widehat{\sigma}_x$	$\widehat{m}_x$	$\tilde{\alpha}$	$\tilde{\beta}_x$	$\tilde{\sigma}_x$	$\tilde{m}_x$
	$r = 100, n = 500$								
	0.1968	1.5135	0.0802	1.0953	6.2339	1.4978	0.0534	1.0648	6.2709
AE	0.0032	0.0135	0.0198	0.0029	0.0161	0.0022	0.0466	0.0334	0.0209
RE	0.0160	0.0090	0.1980	0.0027	0.0026	0.0015	0.4660	0.0304	0.0033
MSE	0.0009	0.0093	0.0271	0.0042	0.0646	0.0102	0.0329	0.0049	0.0096
	$r = 100, n = 1000$								
	0.1986	1.5019	0.1024	1.0975	6.2440	1.4917	0.1029	1.0651	6.2614
AE	0.0014	0.0019	0.0024	0.0007	0.0060	0.0083	0.0029	0.0331	0.0114
RE	0.0070	0.0013	0.0240	0.0007	0.0010	0.0055	0.0290	0.0302	0.0018
MSE	0.0006	0.0038	0.0090	0.0018	0.0389	0.0044	0.0122	0.0030	0.0045
	$r = 100, n = 10000$								
	0.2000	1.5036	0.0998	1.0963	6.2529	1.5026	0.1002	1.0627	6.2699
AE	0.0000	0.0036	0.0002	0.0019	0.0029	0.0026	0.0002	0.0355	0.0199
RE	0.0000	0.0024	0.0020	0.0018	0.0005	0.0017	0.0020	0.0324	0.0032
MSE	6 e -05	0.0004	0.0010	0.0002	0.0047	0.0005	0.0012	0.0014	0.0009

Table 2: For $\lambda = 0.2$ as autoregressive parameter and $\alpha_z = 1.5$, $\beta_z = 0.1$, $\sigma_z = 1$, $m_z = 5$ as theoretical parameters of the distribution $S(\alpha_z, \beta_z, \sigma_z, m_z)$ of the innovations.

AR(1): $X_n = 0.2X_{n-1} + Z_n$			
Theoretical values of $S(\alpha, \beta_x, \sigma_x, m_x)$			
α	β_x	γ_x	m_x
1.2000	0.5000	1.1695	6.2500

	Estimated values via $\{\widehat{Z}_k\}$					Estimated values directly on $\{X_k\}$			
	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\beta}_x$	$\widehat{\sigma}_x$	$\widehat{m}_x$	$\tilde{\alpha}$	$\tilde{\beta}_x$	$\tilde{\sigma}_x$	$\tilde{m}_x$
	$r = 100, n = 500$								
	0.2001	1.2051	0.5054	1.1781	6.2736	1.2023	0.4841	1.1535	6.4330
AE	0.0001	0.0051	0.0054	0.0086	0.0236	0.0023	0.0159	0.0160	0.1830
RE	0.0005	0.0043	0.0108	0.0073	0.0038	0.0019	0.0318	0.0137	0.0293
MSE	0.0013	0.0064	0.0110	0.0071	0.1211	0.0073	0.0156	0.0089	0.0432
	$r = 100, n = 1000$								
	0.2029	1.2119	0.4973	1.1691	6.2864	1.2097	0.4836	1.1430	6.4261
AE	0.0029	0.0119	0.0027	0.0004	0.0364	0.0097	0.0164	0.0265	0.1761
RE	0.0145	0.0099	0.0054	0.0004	0.0058	0.0081	0.0328	0.0227	0.0282
MSE	0.0008	0.0041	0.0041	0.0057	0.0878	0.0039	0.0067	0.0048	0.0353
	$r = 100, n = 10000$								
	0.1999	1.2043	0.5044	1.1702	6.2514	1.2035	0.5031	1.1414	6.4205
AE	0.0001	0.0043	0.0044	0.0007	0.0014	0.0035	0.0031	0.0281	0.1705
RE	0.0005	0.0036	0.0088	0.0006	0.0002	0.0029	0.0062	0.0241	0.0273
MSE	2 e -05	0.0005	0.0006	0.0002	0.0018	0.0004	0.0007	0.0011	0.0295

Table 3: For $\lambda = 0.2$ as autoregressive parameter and $\alpha_z = 1.2$, $\beta_z = 0.5$, $\sigma_z = 1$, $m_z = 5$ as theoretical parameters of the distribution $S(\alpha_z, \beta_z, \sigma_z, m_z)$ of the innovations.

AR(1): $X_n = 0.2X_{n-1} + Z_n$			
Theoretical values of $S(\alpha, \beta_x, \sigma_x, m_x)$			
α	β_x	σ_x	m_x
1.2000	0.9000	1.1695	6.2500

	Estimated values via $\{\widehat{Z}_k\}$					Estimated values directly on $\{X_k\}$			
	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\beta}_x$	$\widehat{\sigma}_x$	$\widehat{m}_x$	$\tilde{\alpha}$	$\tilde{\beta}_x$	$\tilde{\sigma}_x$	$\tilde{m}_x$
	$r = 100, n = 500$								
	0.1954	1.1897	0.8818	1.1731	6.2220	1.1918	0.8666	1.1564	6.5661
AE	0.0046	0.0103	0.0182	0.0036	0.0280	0.0082	0.0334	0.0131	0.3161
RE	0.0230	0.0086	0.0202	0.0031	0.0045	0.0068	0.0371	0.0112	0.0506
MSE	0.0013	0.0064	0.0110	0.0071	0.1211	0.0073	0.0156	0.0089	0.0432
	$r = 100, n = 1000$								
	0.1983	1.2041	0.8918	1.1741	6.2576	1.2047	0.8773	1.1435	6.5789
AE	0.0017	0.0041	0.0082	0.0046	0.0076	0.0047	0.0227	0.0260	0.3289
RE	0.0085	0.0034	0.0091	0.0039	0.0012	0.0039	0.0252	0.0223	0.0526
MSE	0.0008	0.0041	0.0041	0.0057	0.0878	0.0039	0.0067	0.0048	0.0353
	$r = 100, n = 10000$								
	0.1991	1.1980	0.9004	1.1675	6.2441	1.2015	0.9048	1.1397	6.5551
AE	0.0009	0.0020	0.0004	0.0020	0.0059	0.0015	0.0048	0.0298	0.3051
RE	0.0045	0.0017	0.0004	0.0017	0.0009	0.0013	0.0053	0.0255	0.0488
MSE	2 e -05	0.0005	0.0006	0.0002	0.0018	0.0004	0.0007	0.0011	0.0295

Table 4: For $\lambda = 0.2$ as autoregressive parameter and $\alpha_z = 1.2$, $\beta_z = 0.9$, $\sigma_z = 1$, $m_z = 5$ as theoretical parameters of the distribution $S(\alpha_z, \beta_z, \sigma_z, m_z)$ of the innovations.

AR(1): $X_n = 0.2X_{n-1} + Z_n$			
Theoretical values of $S(\alpha, \beta_x, \sigma_x, m_x)$			
α	β_x	σ_x	m_x
1.2000	-0.9000	1.1695	6.2500

	Estimated values via $\{\widehat{Z}_k\}$					Estimated values directly on $\{X_k\}$			
	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\beta}_x$	$\widehat{\sigma}_x$	$\widehat{m}_x$	$\tilde{\alpha}$	$\tilde{\beta}_x$	$\tilde{\sigma}_x$	$\tilde{m}_x$
	$r = 100, n = 500$								
	0.1924	1.1881	-0.8769	1.1634	6.1826	1.1965	-0.8837	1.1408	5.9212
AE	0.0076	0.0119	0.0231	0.0061	0.0674	0.0035	0.0163	0.0287	0.3288
RE	0.0380	0.0099	-0.0257	0.0052	0.0108	0.0029	-0.0181	0.0246	0.0526
MSE	0.0006	0.0060	0.0064	0.0045	0.0640	0.0075	0.0065	0.0064	0.1255
	$r = 100, n = 1000$								
	0.2022	1.1910	-0.8894	1.1708	6.2602	1.1802	-0.8810	1.1319	5.9343
AE	0.0022	0.0090	0.0106	0.0013	0.0102	0.0198	0.0190	0.0376	0.3157
RE	0.0110	0.0075	-0.0118	0.0011	0.0016	0.0165	-0.0211	0.0322	0.0505
MSE	0.0005	0.0047	0.0045	0.0035	0.0407	0.0048	0.0048	0.0041	0.1086
	$r = 100, n = 10000$								
	0.1992	1.1215	-0.9066	1.1685	6.2472	1.2038	-0.9114	1.1398	5.9476
AE	0.0008	0.0015	0.0066	0.0010	0.0028	0.0038	0.0114	0.0297	0.3024
RE	0.0040	0.0013	-0.0073	0.0009	0.0004	0.0032	-0.0127	0.0254	0.0484
MSE	2 e -05	0.0005	0.0012	0.0003	0.0025	0.0006	0.0012	0.0011	0.0924

Table 5: For $\lambda = 0.2$ as autoregressive parameter and $\alpha_z = 1.2$, $\beta_z = -0.9$, $\sigma_z = 1$, $m_z = 5$ as theoretical parameters of the distribution $S(\alpha_z, \beta_z, \sigma_z, m_z)$ of the innovations.

AR(1): $X_n = 0.2X_{n-1} + Z_n$			
Theoretical values of $S(\alpha, \beta_x, \sigma_x, m_x)$			
α	β_x	σ_x	m_x
1.7000	0.1000	1.0693	6.2500

	Estimated values via $\{\widehat{Z}_k\}$					Estimated values directly on $\{X_k\}$			
	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\beta}_x$	$\widehat{\sigma}_x$	$\widehat{m}_x$	$\tilde{\alpha}$	$\tilde{\beta}_x$	$\tilde{\sigma}_x$	$\tilde{m}_x$
	$r = 100, n = 500$								
	0.1973	1.6969	0.0865	1.0709	6.2520	1.7009	0.0549	1.0466	6.2712
AE	0.0027	0.0031	0.0135	0.0016	0.0020	0.0009	0.0451	0.0227	0.0212
RE	0.0135	0.0018	0.1350	0.0015	0.0003	0.0005	0.4510	0.0212	0.0034
MSE	0.0011	0.0103	0.0989	0.0034	0.0833	0.0128	0.1274	0.0040	0.0118
	$r = 100, n = 1000$								
	0.2005	1.784	0.1186	1.0684	6.2715	1.7188	0.0921	1.0358	6.2694
AE	0.0005	0.0284	0.0186	0.0009	0.0215	0.0188	0.0079	0.0335	0.0194
RE	0.0025	0.0167	0.1860	0.0009	0.0034	0.0111	0.0790	0.0313	0.0031
MSE	0.0009	0.0073	0.0672	0.0014	0.0856	0.0080	0.0481	0.0025	0.0050
	$r = 100, n = 10000$								
	0.2000	1.7056	0.1048	1.0710	6.2494	1.7072	0.1033	1.0426	6.2615
AE	0.0000	0.0056	0.0048	0.0017	0.0006	0.0072	0.0033	0.0267	0.0115
RE	0.0000	0.0033	0.0480	0.0016	0.0001	0.0042	0.0330	0.0250	0.0018
MSE	5 e -05	0.0007	0.0029	0.0001	0.0042	0.0007	0.0035	0.0008	0.0007

Table 6: For $\lambda = 0.2$ as autoregressive parameter and $\alpha_z = 1.7$, $\beta_z = 0.1$, $\sigma_z = 1$, $m_z = 5$ as theoretical parameters of the distribution $S(\alpha_z, \beta_z, \sigma_z, m_z)$ of the innovations.

AR(1): $X_n = 0.2X_{n-1} + Z_n$			
Theoretical values of $S(\alpha, \beta_x, \sigma_x, m_x)$			
α	β_x	σ_x	m_x
1.8000	0.1000	1.0584	6.2500

	Estimated values via $\{\widehat{Z}_k\}$					Estimated values directly on $\{X_k\}$			
	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\beta}_x$	$\widehat{\sigma}_x$	$\widehat{m}_x$	$\tilde{\alpha}$	$\tilde{\beta}_x$	$\tilde{\sigma}_x$	$\tilde{m}_x$
	$r = 100, n = 500$								
	0.2026	1.7592	0.1085	1.0523	6.2499	1.7759	0.0314	1.0300	6.2540
AE	0.0026	0.0408	0.0085	0.0061	0.0001	0.0241	0.0686	0.0284	0.0040
RE	0.0130	0.0227	0.0850	0.0058	0.0000	0.0134	0.6860	0.0268	0.0006
MSE	0.0013	0.0125	0.1737	0.0028	0.0842	0.0114	0.2334	0.0038	0.0120
	$r = 100, n = 1000$								
	0.2000	1.7999	0.0914	1.0556	6.2549	1.7831	0.0719	1.0234	6.2540
AE	0.0000	0.0001	0.0086	0.0028	0.0049	0.0169	0.0281	0.0350	0.0400
RE	0.0000	0.0001	0.0860	0.0027	0.0008	0.0094	0.2810	0.0331	0.0006
MSE	0.0005	0.0075	0.1047	0.0016	0.0378	0.0069	0.1134	0.0030	0.0075
	$r = 100, n = 10000$								
	0.2000	1.7986	0.0949	1.0568	6.2510	1.8006	0.0950	1.0314	6.2569
AE	0.0000	0.0014	0.0051	0.0016	0.0010	0.0006	0.0050	0.0270	0.0069
RE	0.0000	0.0008	0.0510	0.0015	0.0002	0.0003	0.0500	0.0255	0.0011
MSE	7 e -05	0.0007	0.0064	0.0001	0.0059	0.0008	0.0070	0.0008	0.0007

Table 7: For $\lambda = 0.2$ as autoregressive parameter and $\alpha_z = 1.8$, $\beta_z = 0.1$, $\sigma_z = 1$, $m_z = 5$ as theoretical parameters of the distribution $S(\alpha_z, \beta_z, \sigma, m_z)$ of the innovations.

AR(1): $X_n = 0.2X_{n-1} + Z_n$			
Theoretical values of $S(\alpha, \beta_x, \sigma_x, m_x)$			
α	β_x	σ_x	m_x
1.9000	0.1000	1.0493	6.2500

	Estimated values via $\{\widehat{Z}_k\}$					Estimated values directly on $\{X_k\}$			
	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\beta}_x$	$\widehat{\sigma}_x$	$\widehat{m}_x$	$\tilde{\alpha}$	$\tilde{\beta}_x$	$\tilde{\sigma}_x$	$\tilde{m}_x$
	$r = 100, n = 500$								
	0.1958	1.8229	-0.0023	1.0294	6.2427	1.8226	-0.0251	0.9976	6.2461
AE	0.0042	0.0771	0.1023	0.0199	0.0073	0.0774	0.1251	0.0517	0.0039
RE	0.0210	0.0406	1.0230	0.0190	0.0012	0.0407	1.2510	0.0493	0.0006
MSE	0.0021	0.0149	0.2123	0.0026	0.1609	0.0160	0.1802	0.0042	0.0105
	$r = 100, n = 1000$								
	0.1982	1.8585	0.0204	1.0385	6.2448	1.8537	-0.0293	1.0122	6.2535
AE	0.0018	0.0415	0.0796	0.0108	0.0052	0.0463	0.1293	0.0371	0.0035
RE	0.0090	0.0218	0.7960	0.0103	0.0008	0.0244	1.2930	0.0354	0.0006
MSE	0.0008	0.0067	0.2197	0.0016	0.0665	0.0070	0.2391	0.0028	0.0073
	$r = 100, n = 10000$								
	0.1994	1.8992	0.1104	1.0465	6.2476	1.9030	0.1252	1.0248	6.2534
AE	0.0006	0.0008	0.0104	0.0028	0.0024	0.0030	0.0252	0.0245	0.0034
RE	0.0030	0.0004	0.1040	0.0027	0.0004	0.0016	0.2520	0.0233	0.0005
MSE	0.0001	0.0010	0.0322	0.0001	0.0083	0.0011	0.0446	0.0007	0.0005

Table 8: For $\lambda = 0.2$ as autoregressive parameter and $\alpha_z = 1.9$, $\beta_z = 0.1$, $\sigma_z = 1$, $m_z = 5$ as theoretical parameters of the distribution $S(\alpha_z, \beta_z, \sigma_z, m_z)$ of the innovations.

AR(1): $X_n = 0.8X_{n-1} + Z_n$			
Theoretical values of $S(\alpha, \beta_x, \sigma_x, m_x)$			
α	β_x	σ_x	m_x
1.2000	0.1000	4.2568	25.0000

	Estimated values via $\{\widehat{Z}_k\}$					Estimated values directly on $\{X_k\}$			
	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\beta}_x$	$\widehat{\sigma}_x$	$\widehat{m}_x$	$\tilde{\alpha}$	$\tilde{\beta}_x$	$\tilde{\sigma}_x$	$\tilde{m}_x$
	$r = 100, n = 500$								
	0.7952	1.2095	0.0948	4.1450	24.5708	1.1702	0.0196	3.2878	25.5873
AE	0.0048	0.0095	0.0052	0.1118	0.4292	0.0298	0.0804	0.9690	0.5873
RE	0.0060	0.0079	0.0520	0.0263	0.0172	0.0248	0.8040	0.2276	0.0235
MSE	0.0002	0.0065	0.0162	0.1356	3.8544	0.0167	0.0953	1.1907	0.8200
	$r = 100, n = 1000$								
	0.7973	1.1987	0.0995	4.2682	24.8799	1.1925	0.0552	3.3553	25.4914
AE	0.0027	0.0013	0.0005	0.0114	0.1201	0.0075	0.0448	0.9015	0.4914
RE	0.0034	0.0011	0.0050	0.0027	0.0048	0.0063	0.4480	0.2118	0.0197
MSE	0.0004	0.0026	0.0088	0.1774	4.7007	0.0170	0.0469	0.9329	0.4337
	$r = 100, n = 10000$								
	0.7994	1.2039	0.0983	4.2405	24.9527	1.1908	0.0967	3.3517	25.5133
AE	0.0006	0.0039	0.0017	0.0163	0.0473	0.0092	0.0033	0.9051	0.5133
RE	0.0008	0.0033	0.0170	0.0038	0.0019	0.0077	0.0330	0.2126	0.0205
MSE	1 e -5	0.0002	0.0008	0.0136	0.2231	0.0019	0.0061	0.8315	0.2843

Table 9: For $\lambda = 0.8$ as autoregressive parameter and $\alpha_z = 1.2$, $\beta_z = 0.1$, $\sigma_z = 1$, $m_z = 5$ as theoretical parameters of the distribution $S(\alpha_z, \beta_z, \sigma_z, m_z)$ of the innovations.

AR(1): $X_n = 0.95X_{n-1} + Z_n$			
Theoretical values of $S(\alpha, \beta_x, \sigma_x, m_x)$			
α	β_x	σ_x	m_x
1.8000	0.1000	11.3387	100.0000

	Estimated values via $\{\widehat{Z}_k\}$					Estimated values directly on $\{X_k\}$			
	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\beta}_x$	$\widehat{\sigma}_x$	$\widehat{m}_x$	$\tilde{\alpha}$	$\tilde{\beta}_x$	$\tilde{\sigma}_x$	$\tilde{m}_x$
	$r = 100, n = 500$								
	0.9492	1.7894	-0.0220	11.1422	98.5418	1.0812	-0.5964	3.4889	100.3163
AE	0.0008	0.0106	0.1220	0.1965	1.4582	0.7188	0.6964	7.8498	0.3163
RE	0.0008	0.0059	1.2200	0.0173	0.0146	0.3993	6.9640	0.6923	0.0032
MSE	2 e -06	0.0091	0.2530	1.2581	86.0316	0.5324	0.5050	62.2254	3.7460
	$r = 100, n = 1000$								
	0.9485	1.7881	0.1187	11.0354	97.9825	1.5039	-0.3852	3.8257	100.3365
AE	0.0015	0.0119	0.0187	0.3033	2.0175	0.2961	0.4852	7.5130	0.3365
RE	0.0016	0.0066	0.1870	0.0267	0.0202	0.1645	4.8520	0.6626	0.0034
MSE	2 e - 06	0.0081	0.0900	1.4021	95.7969	0.1270	0.0355	56.6745	1.6978
	$r = 100, n = 10000$								
	0.9495	0.1986	0.0963	16.7515	99.3161	1.1949	0.0780	10.3695	102.8581
AE	0.0005	1.6014	0.0037	5.4128	0.6839	0.6051	0.0220	0.9692	2.8581
RE	0.0005	0.8897	0.0370	0.4774	0.0068	0.3362	0.2200	0.0855	0.0286
MSE	6 e - 06	0.0004	0.0007	0.7132	20.7651	0.0058	0.0247	41.1379	8.8514

Table 10: For $\lambda = 0.95$ as autoregressive parameter and $\alpha_z = 1.8$, $\beta_z = 0.1$, $\sigma_z = 1$, $m_z = 5$ as theoretical parameters of the distribution $S(\alpha_z, \beta_z, \sigma_z, m_z)$ of the innovations.

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