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AN EOQ MODEL FOR AN ITEM WITH MODIFIED WEIBULL DISTRIBUTION DETERIORATION RATE, EXPONENTIAL DEMAND, SHORTAGES AND PARTIAL BACKLOGGING

Alin Roşca, Natalia Roşca

Abstract. In this paper, we consider an EOQ inventory model for an item under the following assumptions. We assume that the continuous time-dependence of the demand rate is an exponential function and the deterioration rate follows a two-parameter modified Weibull distribution. We also assume that shortages are allowed and during the shortage period the backlogging rate function is an exponential function of the waiting time. Because the proposed model cannot be solved analytically due to its complexity, we used the computer software Matlab 7.0 to find an optimal solution. Further, we consider a numerical example in order to illustrate our model and the solution procedure. A sensitivity analysis with respect to changes in the model parameters is performed to see their effects on the solution.

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1. Introduction

The study of inventory models has kept the attention of researchers for many years. In formulating such models, there are some factors which have to be taken into account: the deterioration of items, the variation of demand rate with time and the backlogging during the shortage period in the inventory.

Some examples of items in which appreciable deterioration can take place during the storage period are food, electronic components, chemicals, etc. This loss is considered when analyzing the Economic Order Quantity (EOQ) models for deteriorating items. Dave and Patel [3] considered an inventory model for deteriorating items with time-varying demand. In their model, a linear increasing demand rate over a finite time horizon and a constant deterioration rate are considered. This model was extended by Sachan [17], to allow for shortages. Inventory models with exponential decay of items or variable proportion of the on-hand inventory gets deteriorated per unit time have been introduced by Ghare and Schrader [5], Misra [13], Shah [20], Tadikamalla [22], etc. In time, many other authors, including Goyal et. al. [8], Hariga [9], Chakrabarti and Chaudhuri [1] developed EOQ inventory models that focused on the effect of deterioration of items with time-varying demands and shortages. Covert and Philip [2], used a two-parameter Weibull distribution to represent the distribution of time of deterioration. This model was extended by Philip [15], considering a three-parameter Weibull distribution for deterioration time of the items. These last two models did not allow for shortages and the demand rate was considered constant. More recently, Wee [23], Jalan and Chaudhuri [12], considered an exponential time-varying demand.

In their literature review, Goyal and Giri [7] indicated that the assumption of constant demand rate, which is the simplest one, is not always applicable to many inventory items, such as: electronic goods, fashionable clothes, etc. Due to this fact, many researchers started to develop inventory models with time-varying demand pattern. Donaldson [4], was the one who established the classical no-shortage inventory model with a linear trend in demand over a finite time-horizon and solved it analytically. Because the procedure of Donaldson's is too complex and computationally complicated, some authors, such as Silver [19] and Ritchie [16], derived simple heuristic procedures for his problem. Mitra et. al. [14] presented a procedure for adjusting the EOQ model for the case of increasing/decreasing linear trend in demand. The shortage and deterioration in inventory were not considered in all these models. However, more recently, Ghosh and Chaudhuri [6] have considered an EOQ model with time-quadratic demand variation, allowing shortages which are completely backlogged. Hollier and Mak [11] were the first who proposed the use of exponentially decreasing demand for an inventory model and obtained optimal replenishment policies under both constant and variable replenishment intervals. Hariga and Benkherouf [10] generalized Hollier and Maks model [11]. Wee [23] developed a deterministic lot size model for deteriorating items where demand decreases exponentially over a fixed time horizon. Su et al. [21] proposed a production inventory model for deteriorating products with an exponentially declining demand over a fixed time horizon.

In daily life, some customers wait for backlogging during the shortage period, some others do not. Therefore, the opportunity cost due to lost sales should be taken into consideration in modeling the inventory problems. In the literature, many authors assume that the shortages are completely backlogged or completely lost. Wee [23], extended the work of Hollier and Mak [11] to allow for shortages and he considered a partial backlogging as a fixed fraction of the demand rate.

However, in some inventory models, especially the ones for fashionable commodities, the backlogging rate is variable, being a decreasing function of the waiting time (i.e, the longer the waiting time, the smaller the backlogging rate).

In this paper, we assume that the continuous time-dependence of the demand rate is an exponential function. We also assume, that the deterioration rate follows a two-parameter modified Weibull distribution. This distribution was considered by Zaindin [24] and Sarhan & Zaindin [18] and generalizes both exponential and two-parameter Weibull distributions. According to Sarhan & Zaindin [18], it is interesting to observe that the modified Weibull distribution has a nice physical interpretation. It represents the lifetime of a series system. This system consists of two independent components. The lifetime of one component follows an exponential distribution and the lifetime of the other one follows a Weibull distribution. Often, deterioration of an item such as electronic goods or complex chemical or food products (having independent components), can occur for more than one reason and the deterioration distribution for each reason can be approximated by an exponential and a Weibull distribution. Hence, the overall deterioration distribution can be considered as a modified Weibull distribution. Further, we assume that shortages are allowed and during the shortage period the backlogging rate function is an exponential function of the waiting time. In the present paper we propose an EOQ inventory model for an item under the above described assumptions. Because the proposed model cannot be solved analytically due to its complexity, we used the computer software Matlab 7.0 to find an optimal solution. The model is illustrated with the help of a numerical example. The sensitivity analysis with respect to changes of all the parameter values of the model is performed to see the effects of these parameters on the solution.

2. Notations and assumptions

The inventory model that we introduce and develop in this paper is based on the following notations and assumptions.

Notations

(i) T - The fixed length of each cycle.

(ii) S - The size of the initial inventory ($S > 0$).

(iii) C_L - The ordering cost per order.

(iv) C_S - The inventory holding cost per unit per unit time.

(v) C_P - The shortage cost per unit per unit time.

(vi) C_D - The cost of each deteriorated unit.

(vii) C_B - The opportunity cost due to lost sales per unit.

(viii) t_1 - Time during which there is no shortage ($0 \leq t_1 \leq T$).

(ix) φ - A constant such that $0 < \varphi < 1$.

Assumptions

(a) A single item is considered, with a deterioration rate which is a function of time given by a modified Weibull distribution with three parameters α, β, γ, denoted by $MWD(\alpha, \beta, \gamma)$. According to Zaidin, the pdf of the $MWD(\alpha, \beta, \gamma)$ is:

$$f(x; \alpha, \beta, \gamma) = (\alpha + \beta\gamma x^{\gamma-1}) \exp\{-\alpha x - \beta x^{\gamma}\}, \quad x \geq 0, \tag{1}$$

and the cdf is:

$$F(x; \alpha, \beta, \gamma) = 1 - \exp\{-\alpha x - \beta x^{\gamma}\}, \tag{2}$$

where $\gamma > 0$, $\alpha, \beta \geq 0$ such that $\alpha + \beta > 0$. Hence, the hazard rate is

$$h(x; \alpha, \beta, \gamma) = \frac{f(x; \alpha, \beta, \gamma)}{1 - F(x; \alpha, \beta, \gamma)} = \alpha + \beta\gamma x^{\gamma-1}, \quad x \geq 0. \tag{3}$$

(b) The supply occurs instantaneously and the lead time is zero.

(c) A deteriorated unit is not repaired or replaced during a cycle.

(e) Shortages are allowed and partially backlogged at a backlogging rate which is variable and is dependent on the length of the waiting time for the next replenishment. The proportion of customers who accept backlogging at time t is decreasing with the waiting time $(T-t)$ for the next replenishment. Hence, we consider the backlogging rate during the shortage period to be an exponential function of the waiting time B, defined as follows:

$$B(T-t) = \exp\{-\delta(T-t)\}, \text{ where } \delta \geq 0 \text{ and } t_1 \leq t < T. \tag{4}$$

(f) The demand rate $D(t)$ is an exponential function of time t:

$$D(t) = A \exp\{-\lambda t\}, \tag{5}$$

where $A > 0$ is the initial demand, and $\alpha > \lambda > 0$ is a constant governing the decreasing rate of demand.

(g) All the involved costs remain constant over time.

3. Mathematical model and solution

Let $I(t)$ be the inventory level at any time t. The inventory is made up from purchased or produced items. During the period $(0, t_1)$ the inventory level diminishes and falls to zero at time $t = t_1$ due to the combined effects of deterioration of the items and market demand. Within the interval (t_1, T) shortages are allowed and they are partially backlogged with backlogging exponential rate function. The instantaneous state of inventory level is governed by two differential equations, one for each of the two different parts of the cycle time T. Therefore, the equations are:

$$\begin{aligned} \frac{dI(t)}{dt} &= -(\alpha + \beta\gamma t^{\gamma-1})I(t) - A\exp\{-\lambda t\}, \qquad 0 \le t \le t_1 \\ &\text{with} \quad I(0) = S \quad \text{and} \quad I(t_1) = 0, \end{aligned} \tag{6}$$

and

$$\begin{aligned} \frac{dI(t)}{dt} &= -A\exp\{-\lambda t\}\exp\{-\delta(T-t)\}, \qquad t_1 \le t \le T \\ &\text{with} \quad I(t_1) = 0. \end{aligned} \tag{7}$$

Next, we solve equation (6), which is a linear ordinary differential equation of first order. Multiplying both sides of (6) by $\exp(\alpha t + \beta t^\gamma)$ and then integrating over $[0, t]$, we get:

$$\begin{aligned} \int_0^t \frac{dI(x)}{dx}\exp(\alpha x + \beta x^\gamma)dx &= -\int_0^t \exp(\alpha x + \beta x^\gamma)(\alpha + \beta\gamma x^{\gamma-1})I(x)dx - \\ &\quad -A\int_0^t \exp((\alpha-\lambda)x + \beta x^\gamma)dx, \qquad 0 \le t \le t_1 \end{aligned} \tag{8}$$

By using the conditions $I(0) = S$ and $I(t_1) = 0$ we obtain the following solution of equation (6)

$$I(t) = \frac{A\Big[\int_0^{t_1}\exp((\alpha-\lambda)x + \beta x^\gamma)dx - \int_0^t \exp((\alpha-\lambda)x + \beta x^\gamma)dx\Big]}{\exp(\alpha t + \beta t^\gamma)}, \qquad 0 \le t \le t_1. \tag{9}$$

Integrating the equation (7) over the interval $[t_1, t]$, we get:

$$I(t) - I(t_1) = -A\int_{t_1}^t \exp(-\delta T)\exp(x(\delta - \lambda))dx. \tag{10}$$

Using the condition $I(t_1) = 0$, we obtain from (10)

$$I(t) = \frac{A\exp(-\delta T)}{\delta - \lambda}\Big(\exp(t_1(\delta-\lambda)) - \exp(t(\delta-\lambda))\Big), \qquad t_1 \le t \le T. \tag{11}$$

We can express the exponential terms in the integral from (9) as an infinite series of powers and thus we obtain:

$$\begin{aligned} \exp\left((\alpha-\lambda)x+\beta x^{\gamma}\right) &= \sum_{n=0}^{\infty}\frac{\left[(\alpha-\lambda)x+\beta x^{\gamma}\right]^{n}}{n!} && (12)\\ &= \sum_{n=0}^{\infty}\frac{(\alpha-\lambda)^{n}}{n!}x^{n}(1+\rho x^{\gamma-1})^{n}, && (13)\end{aligned}$$

where $\rho=\frac{\beta}{\alpha-\lambda}$. Using the binomial identity, we get from the above formula:

$$\exp\left((\alpha-\lambda)x+\beta x^{\gamma}\right)=\sum_{n=0}^{\infty}\frac{(\alpha-\lambda)^{n}}{n!}\sum_{k=0}^{n}\binom{n}{k}\rho^{k}x^{n+k\gamma-k}. \tag{14}$$

Based on relations (9) and (14) we can determine the initial value of the stock S:

$$\begin{aligned} S=I(0) &= A\int_{0}^{t_1}\sum_{n=0}^{\infty}\frac{(\alpha-\lambda)^{n}}{n!}\sum_{k=0}^{n}\binom{n}{k}\rho^{k}x^{n+k\gamma-k}dx\\ &= A\sum_{n=0}^{\infty}\frac{(\alpha-\lambda)^{n}}{n!}\sum_{k=0}^{n}\binom{n}{k}\rho^{k}\frac{t_1^{n+1+k(\gamma-1)}}{n+1+k(\gamma-1)}. && (15)\end{aligned}$$

The average total cost per unit time TC is expressed as the sum of the following costs:

(1) Ordering cost - OC.

(2) Holding cost - HC.

(3) Shortage cost - SC.

(4) Deterioration cost - DC.

(5) Opportunity cost - BC.

In the sequel we deduce these costs. The average inventory holding cost HC in the interval $[0,t_1]$ is

$$\begin{aligned} HC &= \frac{1}{T}C_S A\int_{0}^{t_1}I(t)dt\\ &= \frac{1}{T}C_S A\int_{0}^{t_1}\exp\left(-\alpha t-\beta t^{\gamma}\right)\Big[\int_{t}^{t_1}\exp((\alpha-\lambda)x+\beta x^{\gamma})dx\Big]dt && (16)\end{aligned}$$

Based on relation (14) the last integral from (16) can be written as:

$$\int_t^{t_1} \exp((\alpha-\lambda)x+\beta x^\gamma)dx = \sum_{n=0}^{\infty} \frac{(\alpha-\lambda)^n}{n!} \sum_{k=0}^{n} B(n,k)(t_1^{n+1+k(\gamma-1)} - t^{n+1+k(\gamma-1)}), \tag{17}$$

where

$$B(n,k) = \frac{\binom{n}{k}\rho^k}{n+1+k(\gamma-1)}. \tag{18}$$

Hence, from (16) and (17) we get for the inventory holding cost:

$$\begin{aligned} HC &= \frac{1}{T} C_S A \int_0^{t_1} \exp(-\alpha t - \beta t^\gamma) \\ &\cdot \Big[\sum_{n=0}^{\infty} \frac{(\alpha-\lambda)^n}{n!} \sum_{k=0}^{n} B(n,k)(t_1^{n+1+k(\gamma-1)} - t^{n+1+k(\gamma-1)})\Big] dt \\ &= \frac{1}{T} C_S A \int_0^{t_1} \sum_{n=0}^{\infty} \frac{(\alpha-\lambda)^n}{n!} \\ &\cdot \sum_{k=0}^{n} \exp(-\alpha t - \beta t^\gamma) B(n,k)(t_1^{n+1+k(\gamma-1)} - t^{n+1+k(\gamma-1)}) dt. \end{aligned}$$

By using the Taylor series expansion:

$$\exp(-\alpha t - \beta t^\gamma) = 1 - \alpha t - \beta t^\gamma + \frac{(\alpha t + \beta t^\gamma)^2}{2} - \ldots, \tag{19}$$

which is a valid approximation for small values of $\alpha t + \beta t^\gamma$ and ignoring the terms of order $O((\alpha t + \beta t^\gamma)^2)$, we get for the holding cost:

$$HC = \frac{1}{T} C_S A \int_0^{t_1} \sum_{n=0}^{\infty} \frac{(\alpha-\lambda)^n}{n!} \sum_{k=0}^{n} (1-\alpha t-\beta t^\gamma) B(n,k)(t_1^{n+1+k(\gamma-1)} - t^{n+1+k(\gamma-1)}) dt. \tag{20}$$

After integrating over $[0, t_1]$ and doing some calculations we obtain from (20) that

$$\begin{aligned} HC &= \frac{1}{T} C_S A \sum_{n=0}^{\infty} \frac{(\alpha-\lambda)^n}{n!} \sum_{k=0}^{n} B(n,k) t_1^{n+2+k(\gamma-1)} \Bigg[\frac{n+1+k(\gamma-1)}{n+2+k(\gamma-1)} - \\ & -\beta t_1^\gamma \frac{n+1+k(\gamma-1)}{(\gamma+1)(n+2+k(\gamma-1)+\gamma)} - \alpha t_1 \frac{n+1+k(\gamma-1)}{2(n+3+k(\gamma-1))}\Bigg]. \end{aligned} \tag{21}$$

The length of a shortage period is a part of a cycle time. Hence, we can assume that:

$$t_1 = \varphi T, \quad 0 < \varphi < 1 \tag{22}$$

where φ is a constant to be determined in an optimal manner. Finally, the total holding cost is:

$$\begin{aligned} HC &= \frac{1}{T} C_S A \sum_{n=0}^{\infty} \frac{(\alpha-\lambda)^n}{n!} \sum_{k=0}^{n} B(n,k)(\varphi T)^{n+2+k(\gamma-1)}(n+1+k(\gamma-1)) \cdot \\ &\cdot \Bigg[\frac{1}{n+2+k(\gamma-1)} - \alpha(\varphi T) \frac{1}{2(n+3+k(\gamma-1))} - \\ &- \beta(\varphi T)^{\gamma} \frac{1}{(\gamma+1)(n+2+k(\gamma-1)+\gamma)} \Bigg]. \end{aligned} \tag{23}$$

The shortage cost, over the period $[t_1, T]$ is given by:

$$\begin{aligned} SC &= -\frac{C_P}{T} \int_{t_1}^{T} I(t)dt = \\ &= -\frac{C_P}{T} \int_{t_1}^{T} \frac{A \exp(-\delta T)}{\delta - \lambda} \Big[\exp(t_1(\delta-\lambda)) - \exp(t(\delta-\lambda)) \Big]. \end{aligned} \tag{24}$$

After some calculations and based on (22), the relation (24) becomes:

$$\begin{aligned} SC &= -\frac{C_P}{T} A \frac{\exp\big(-\delta T(1-\varphi) - \lambda\varphi T\big)}{\delta - \lambda} \\ &\cdot \Bigg[T(1-\varphi) - \frac{1}{\delta-\lambda} \exp\big(T(1-\varphi)(\delta-\lambda)\big) + \frac{1}{\delta-\lambda} \Bigg]. \end{aligned}$$

The cost of deterioration DC, is calculated as:

$$DC = \frac{C_D}{T} \Big(I(0) - \int_0^{t_1} A \exp(-\lambda t) dt \Big). \tag{25}$$

Using the relation (15) the cost of deteriorated items in the inventory becomes:

$$DC = \frac{C_D}{T} A \Bigg[\sum_{n=0}^{\infty} \frac{(\alpha-\lambda)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \rho^k \frac{t_1^{n+1+k(\gamma-1)}}{n+1+k(\gamma-1)} + \frac{\exp(-\lambda t_1)}{\lambda} - \frac{1}{\lambda} \Bigg], \tag{26}$$

which based on relation (22) is

$$DC = \frac{C_D}{T} A \left[\sum_{n=0}^{\infty} \frac{(\alpha-\lambda)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \rho^k \frac{(\varphi T)^{n+1+k(\gamma-1)}}{n+1+k(\gamma-1)} + \frac{\exp(-\lambda\varphi T)}{\lambda} - \frac{1}{\lambda} \right]. \quad (27)$$

As the demand is partially backlogged, we have the following opportunity cost:

$$\begin{aligned} BC &= \frac{C_B}{T} \int_{t_1}^{T} D(t)\big(1-\exp(-\delta(T-t))\big)dt \\ &= \frac{C_B}{T} \int_{t_1}^{T} A\exp(-\lambda t)\big(1-\exp(-\delta(T-t))\big)dt \\ &= A\frac{C_B}{T}\frac{1}{\lambda(\delta-\lambda)}\Big[(\delta-\lambda)\exp(-\lambda t_1) - \delta\exp(-\lambda T) + \lambda\exp(-\delta(T-t_1)-\lambda t_1)\Big], \end{aligned}$$

which based on relation (22) is

$$BC = A\frac{C_B}{T}\frac{1}{\lambda(\delta-\lambda)}\Big[(\delta-\lambda)\exp(-\lambda\varphi T) - \delta\exp(-\lambda T) + \lambda\exp\big(-\delta T(1-\varphi)-\lambda\varphi T\big)\Big]. \quad (28)$$

From the analysis carried out so far, we obtain the total inventory cost per unit time as the sum of the ordering cost, holding cost, shortage cost, deterioration cost and opportunity cost as follows:

$$\begin{aligned} TC(\varphi,T) = {} & \frac{C_L}{T} + \frac{1}{T}C_S A \sum_{n=0}^{\infty}\frac{(\alpha-\lambda)^n}{n!}\sum_{k=0}^{n} B(n,k)(\varphi T)^{n+2+k(\gamma-1)}(n+1+k(\gamma-1)) \cdot \\ & \cdot\left[\frac{1}{n+2+k(\gamma-1)} - \alpha(\varphi T)\frac{1}{2(n+3+k(\gamma-1))} - \right. \\ & \left. -\beta(\varphi T)^{\gamma}\frac{1}{(\gamma+1)(n+2+k(\gamma-1)+\gamma)}\right] - \frac{C_P}{T}A\frac{\exp\big(-\delta T(1-\varphi)-\lambda\varphi T\big)}{\delta-\lambda} \cdot \\ & \cdot\left[T(1-\varphi) - \frac{1}{\delta-\lambda}\exp\big(T(1-\varphi)(\delta-\lambda)\big) + \frac{1}{\delta-\lambda}\right] + \\ & + \frac{C_D}{T}A\left[\sum_{n=0}^{\infty}\frac{(\alpha-\lambda)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}\rho^k\frac{(\varphi T)^{n+1+k(\gamma-1)}}{n+1+k(\gamma-1)} + \frac{\exp(-\lambda\varphi T)}{\lambda} - \frac{1}{\lambda}\right] + \\ & + A\frac{C_B}{T}\frac{1}{\lambda(\delta-\lambda)}\Big[(\delta-\lambda)\exp(-\lambda\varphi T) - \delta\exp(-\lambda T) + \\ & + \lambda\exp\big(-\delta T(1-\varphi)-\lambda\varphi T\big)\Big]. \end{aligned} \quad (29)$$

Our objective is to minimize the total inventory cost per unit time. If we treat φ and T as decision variables, the necessary conditions for our optimization problem are:

$$\frac{\partial TC(\varphi,T)}{\partial\varphi}=0 \tag{30}$$

$$\frac{\partial TC(\varphi,T)}{\partial T}=0. \tag{31}$$

After some calculations, the first condition (30) yields:

$$\begin{aligned}
&A\Big\{\sum_{n=0}^{\infty}\frac{(\alpha-\lambda)^n}{n!}\sum_{k=0}^{n}(\varphi T)^{n+k(\gamma-1)}\Big[(n+1\\
&+k(\gamma-1))B(n,k)C_S\varphi T\Big(1-\frac{\alpha\varphi T}{2}-\frac{\beta(\varphi T)^{\gamma}}{\gamma+1}\Big)+\\
&+\binom{n}{k}\rho^k C_D\Big]\Big\}+A\exp\left(-\lambda\varphi T\right)\Big[C_B\Big(\exp\left(-\delta T(1-\varphi)\right)-1\Big)-\\
&-C_D-C_P T(1-\varphi)\exp\left(-\delta(1-\varphi)T\right)\Big]=0.
\end{aligned} \tag{32}$$

The second condition (31) leads to the following equation:

$$\begin{aligned}
&-\frac{C_L}{T^2}+C_S A\varphi^2\sum_{n=0}^{\infty}\frac{(\alpha-\lambda)^n}{n!}\sum_{k=0}^{n}B(n,k)(\varphi T)^{n+k(\gamma-1)}(n+1+k(\gamma-1))\cdot\\
&\cdot\Bigg[\frac{n+1+k(\gamma-1)}{n+2+k(\gamma-1)}-\alpha\varphi T\frac{n+2+k(\gamma-1)}{2(n+3+k(\gamma-1))}\\
&-\beta(\varphi T)^{\gamma}\frac{n+3+k(\gamma-1)}{(\gamma+1)(n+2+k(\gamma-1)+\gamma)}\Bigg]-\\
&-C_P\frac{A}{[(\delta-\lambda)T]^2}\exp\big(-\delta T(1-\varphi)-\lambda\varphi T\big)\big[-\delta(1-\varphi)^2(\delta-\lambda)T^2-\lambda\varphi(1-\varphi)\\
&\cdot(\delta-\lambda)T^2+\lambda T\exp\left(T(1-\varphi)(\delta-\lambda)\right)-\delta T+\varphi T(\delta-\lambda)+\exp\left(T(1-\varphi)(\delta-\lambda)\right)\\
&-1\big]+C_D\frac{A}{\lambda T^2}\Bigg[\sum_{n=0}^{\infty}\frac{(\alpha-\lambda)^n}{n!}\sum_{k=0}^{n}\lambda\binom{n}{k}\rho^k\frac{n+k(\gamma-1)}{n+1+k(\gamma-1)}(\varphi T)^{n+1+k(\gamma-1)}-\\
&-\lambda\varphi T\exp\left(-\lambda\varphi T\right)+1\Bigg]+C_B\frac{A}{\lambda(\delta-\lambda)T^2}\Big[(\delta-\lambda)\exp\left(-\lambda\varphi T\right)(-1-\lambda\varphi T)+\\
&+\lambda\exp\big(-\delta T(1-\varphi)-\lambda\varphi T\big)(-1-\delta(1-\varphi)T-\lambda\varphi T)+\delta\exp\left(-\lambda T\right)(1+\lambda T)\Big]\\
&=0.
\end{aligned} \tag{33}$$

The optimal values φ^* of φ and T^* of T are obtained by solving the equations (32) and (33). The two equations determine a system of non-linear equations, for which we need to employ a numerical method for solving it. This can be done for a given set of parameters by truncating the infinite series that appear in the system.

The sufficient condition that these values minimize the function $TC(\varphi, T)$ is:

$$d^2_{(\varphi^*,T^*)}TC(\varphi, T) > 0. \tag{34}$$

After obtaining the optimal solution, we can use (29) to get the optimal average total cost per unit time as $TC^* = TC(\varphi^*, T^*)$.

4. Numerical example

As we already mentioned the equations (32) and (33) can not be solved analytically. They are solved numerically using the computer software Matlab 7.0, using the following values of the parameters:

$$A = 50,\ \alpha = 0.02,\ \beta = 0.02,\ \gamma = 1.5,\ \delta = 0.04,\ \lambda = 0.07 \quad \text{and}$$
$$C_B = 2,\ C_D = 1,\ C_P = 2,\ C_L = 5,\ C_S = 1.5.$$

We consider the unit time as 'day' and the unit cost \$. Based on this choice of parameters we obtain the following optimal results:

1. Optimum cycle time $T^* = 25.319563$ days;
2. Optimum value $\varphi^* = 0.299647$;
3. Optimum stock period $t_1^* = 7.586931$ days;
4. Optimum average total cost $ATC^* = 234.372537$ \$ per day.

In order to see the importance of choosing φ optimally rather than arbitrarily, we show in Table 1 the results for different values of φ. We observe that, as the value of φ increases to its optimal value, T^* increases while ATC^* decreases. After attaining the optimal value of φ, ATC^* starts increasing.

φ	T^*	ATC^*
0.10	20.246696	311.154129
0.20	21.146436	258.146771
0.21	21.358744	254.468728
0.22	21.602703	250.959856
0.23	21.882071	247.741145
0.24	22.201319	244.820866
0.25	22.565812	242.209322
0.26	22.981990	239.919275
0.27	23.457674	237.966511
0.28	24.002346	236.370551
0.29	24.627468	235.155573
0.299647^*	25.319563^*	234.372537^*
0.32	27.129095	234.381849
0.33	28.218094	234.812206
0.34	29.434659	236.087230
0.35	30.731343	237.986744
0.37	33.115974	243.476767
0.40	34.881277	253.951830

Table 1: Optimal solution with shortage, exponential demand and exponential backlogging rate.

5. Sensitivity Analysis

In this paragraph, we perform a sensitivity analysis of the EOQ model that we proposed. We study the effects of changes in the values of the parameters A, α, β, γ, δ, λ, C_B, C_D, C_P, C_L and C_S on the optimal average total cost ATC^*, optimal cycle time T^* and optimal value φ^*. In order to perform the sensitivity analysis we change each of the parameters by -50%, -25%, 25% and 50% taking one parameter at a time and keeping all the other parameters unchanged. The results that we obtain are presented in Table 2. Based on these results, the conclusions are stated as follows:

(1) T^* and φ^* are insensitive towards changes in parameter A. However, ATC^* is highly sensitive, increasing with the increase in the value of parameter A.

(2) T^*, φ^* and ATC^* are insensitive to changes in parameter α.

(3) T^* and φ^* are lowly sensitive to changes in β, while ATC^* is almost insensitive. T^* and ATC^* increase with the increase in β.

(4) φ^* is moderately sensitive to changes in γ and decreases with the increase in γ. T^* and ATC^* have low sensitivity towards changes in γ, increasing with the increase in γ.

(5) T^*, φ^* and ATC^* are moderately sensitive to changes in δ. Each of T^*, φ^* and ATC^* decreases with the increase in δ.

(6) ATC^* is highly sensitive towards changes in parameter λ and decreases with the increase in λ. φ^* is lowly sensitive to changes in λ, while T^* is moderately sensitive. Also, T^* is decreasing with the increase in λ.

(7) T^*, φ^* and ATC^* are almost insensitive to changes in parameters C_B, C_D and C_L.

(8) φ^* and ATC^* are moderately sensitive to changes in C_P, and they increase as C_P increases. T^* is lowly sensitive to changes in C_P, and increases as C_P increases.

(9) φ^* and ATC^* are moderately sensitive to changes in C_S. φ^* decreases as the parameter C_S increases. ATC^* increases as the parameter C_S increases. T^* is lowly sensitive towards changes in C_S, and decreases as C_S increases.

Parameters change	% change in system parameters	% change in φ^*	% change in T^*	% change in ATC^*
A	-50	0.0543	-0.0802	-49.9610
	-25	0.0182	-0.0267	-24.9805
	+25	-0.0105	0.0160	24.9805
	+50	-0.0177	0.0267	49.9610
α	-50	-0.3644	-1.8976	-0.8226
	-25	-0.1912	-0.9621	-0.4156
	+25	0.2115	0.9841	0.4244
	+50	0.4456	1.9844	0.8578
β	-50	-4.5093	-7.5458	-1.3196
	-25	-0.4709	-2.9601	-1.1132
	+25	-0.9150	1.8359	1.2989
	+50	-2.2996	2.9709	2.5845
γ	-50	15.2606	-2.6062	-5.9776
	-25	10.1138	-1.9005	-3.8433
	+25	-14.2080	2.6315	5.3672
	+50	-27.4720	4.5734	10.7121
δ	-50	9.7617	29.1637	17.2947
	-25	4.1585	13.0508	7.8687
	+25	-3.4876	-10.3843	-6.5882
	+50	-6.5645	-18.7583	-12.1670
λ	-50	-15.7088	42.7110	56.8428
	-25	-5.9040	19.3372	22.8763
	+25	2.0478	-16.0398	-15.8486
	+50	0.5846	-28.7775	-26.8326
C_B	-50	-1.3859	-0.9649	-1.5815
	-25	-0.6898	-0.4785	-0.7876
	+25	0.6844	0.4707	0.7814
	+50	1.3627	0.9336	1.5566
C_D	-50	4.4974	2.9378	-0.9460
	-25	2.2457	1.4493	-0.4895
	+25	-2.2464	-1.4049	0.5245
	+50	-4.4997	-2.7613	1.0863
C_P	-50	-37.5696	-7.9735	-35.4776
	-25	-16.5309	-3.8046	-16.1389
	+25	13.7010	3.2672	13.7707
	+50	25.6324	5.9649	25.6836
C_L	-50	-0.0267	0.0401	-0.0389
	-25	-0.0132	0.0200	-0.0195
	+25	0.0138	-0.0201	0.0194
	+50	0.0273	-0.0401	0.0389
C_S	-50	39.1479	7.0168	-24.4734
	-25	15.7357	2.9880	-10.2364
	+25	-11.6721	-2.2521	7.7951
	+50	-20.7937	-4.0044	13.9557

Table 2: Sensitivity analysis of the model.

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Alin Roşca
Department of Statistics, Forecasts and Mathematics, Faculty of Economics and Business Administration,
Babeş-Bolyai University,
Cluj-Napoca, Romania
email: *alin.rosca@econ.ubbcluj.ro*

Natalia Roşca
Department of Mathematics, Faculty of Mathematics and Computer Science,
Babeş-Bolyai University,
Cluj-Napoca, Romania
email: *natalia@math.ubbcluj.ro*