Acta Universitatis Apulensis No. 37/2014
ISSN: 1582-5329 pp. 83-91

NEW CLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH THE GENERALIZED HYPERGEOMETRIC FUNCTIONS

K.R. Alhindi and M. Darus

Abstract. Using the generalized hypergeometric function, a new generalized derivative operator $\mathrm{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z)$ is introduced. This operator generalize many well-known operators studied earlier by different authors. By making use of this new operator we derive another class of function denoted by $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$. Coefficient estimate and distortion theorem are investigated. Moreover, the Fekete-Szegö functional $|a_3 - \delta a_2^2|$ for this class is also obtained.

2000 *Mathematics Subject Classification*: 30C45.

Keywords: Fekete-Szegö functional, Hadamard product, Differential operator, Coefficient estimate, Distortion theorem.

1. Introduction

The first time when we are about to study the hypergeometric functions, we refer to John Wallis. We were taught that he was the first mathematician who used hypergeometric series in his book "Arithmetica Infinitorum" (1655). Leonhard Euler was another famously known to use the series. However, the first full systematic treatment was given by Carl Friedrich Gauss (1813), and thereafter by Ernst Kummer (1836). The importance of the hypergeometric theory is stemmed from its applications in many subjects such as, numerical analysis, dynamical system and mathematical physics.

Let $\mathcal{A}$ be the class of analytic functions of the form

$$f(z) = z + \sum_{n=k}^{n=\infty} a_k z^k \qquad z \in (\mathbb{U} = \{z \in C : |z| < 1\}) \tag{1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of *univalent functions*, and $\mathcal{S}(\alpha)$, $\mathcal{C}(\alpha)$ $(0 < \alpha \leq 1)$ denote the subclasses of $\mathcal{A}$ consisting of functions that are starlike of order α and convex of order α in $\mathbb{U}$, respectively.

For two analytic functions $f(z) = z + \sum_{n=k}^{n=\infty} a_k z^k$ and $g(z) = z + \sum_{n=k}^{n=\infty} b_k z^k$ in the open unit disc $\mathbb{U} = \{z \in C : |z| < 1\}$. The Hadamard product (or convolution) $f * g$ of f and g is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{n=k}^{n=\infty} a_k b_k z^k. \tag{2}$$

For complex parameters $\alpha_1, ...\alpha_r$ and $\beta_1, ...\beta_s$ $(\beta_j \neq 0, -1, -1, ...; j = 1...s)$, Dziok and Srivastava [1] defined the generalized hypergeometric function ${}_rF_s(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s; z)$ by

$${}_rF_s(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k ... (\alpha_r)_k}{(\beta_1)_k, ..., (\beta_s)_k} \frac{z^k}{k!}; \tag{3}$$

$$(r \leq s + 1; r, s \in \mathbb{N}_0 = \mathbb{N} \cup 0; z \in \mathcal{U}), \tag{4}$$

where $(x)_k$ is the Pochhammer symbol defined, in terms of Gamma function Γ, by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & \text{if} \quad k = 0, \\ x(x+1)...(x+k-1) & \text{if} \quad k \in \mathbb{N}, \end{cases}$$

Dziok and Srivastava [1] defined also the linear operator

$$H(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s) f(z) = z + \sum_{k=n}^{\infty} \Gamma_k a_k z^k \tag{5}$$

where

$$\Gamma_k = \frac{(\alpha_1)_k ... (\alpha_r)_k}{(\beta_1)_k, ..., (\beta_s)_k (k)!} \tag{6}$$

Abbadi and Darus [2] defined the analytic function

$$\Phi^m_{\lambda_1, \lambda_2} = z + \sum_{k=2}^{\infty} \frac{(1 + \lambda_1(k-1))^{m-1}}{(1 + \lambda_2(k-1))^m} z^k, \tag{7}$$

where $m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$ and $\lambda_2 \geq \lambda_1 \geq 0$.

Using the Hadamard product (2), we can derive the generalized derivative operator $\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2}$ as follows

$$\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2} f(z) = z + \sum_{k=2}^{\infty} \frac{(1 + \lambda_1(k-1))^{m-1}}{(1 + \lambda_2(k-1))^m} \Gamma_k a_k z^k \tag{8}$$

where Γ_k is as given in (6).

Remark 1. *When* $(\lambda_1 = \lambda_2 = 0)$, $(\lambda_1 = m = 0)$ *or* $(\lambda_2 = 0$ *and* $m = 1)$ *we get Dziok-Srivistava operator [1].*

Also there are three cases to get the Hohlov operator [3], by giving $(\lambda_1 = \lambda_2 = 0, b_i = 0, a_j = 0)$, $(\lambda_1 = m = 0, b_i = 0, a_j = 0)$ *or* $(\lambda_2 = 0, m = 1, b_i = 0, a_j = 0)$ *where* $(i = 1...r$ *and* $j = 1...s)$.

Putting $(\lambda_1 = \lambda_2 = 0, a_2 = 1, a_3 = ... = a_r = 0, b_2 = ... = b_s = 0)$, $(\lambda_1 = m = 0, a_2 = 1, a_3 = ... = a_r = 0, b_2 = ... = b_s = 0)$ *or* $(\lambda_2 = 0, m = 1, a_2 = 1, a_3 = ... = a_r = 0, b_2 = ... = b_s = 0)$, *we obtain the Carlson-Shaffer operator [4].*

There are six cases to get the Ruscheweyh operator [5] as follows: $(\lambda_1 = \lambda_2 = 0, a_2 = a_3 = ... = a_r = 0, b_1 = b_2 = ... = b_s = 0)$, $(\lambda_1 = m = 0, a_2 = a_3 = ... = a_r = 0, b_1 = b_2 = ... = b_s = 0)$, $(\lambda_2 = 0, m = 1, a_2 = a_3 = ... = a_r = 0, b_1 = b_2 = ... = b_s = 0)$,$(\lambda_1 = \lambda_2 = 0, a_2 = a_3 = ... = a_r = 0, b_2 = ... = b_s = 0)$, $(\lambda_1 = m = 0, a_2 = a_3 = ... = a_r = 0, b_2 = ... = b_s = 0)$ *or* $(\lambda_2 = 0, m = 1, a_2 = a_3 = ... = a_r = 0, b_2 = ... = b_s = 0)$.

If$(\lambda_2 = 0, m = 2, a_2 = a_3 = ... = a_r = 0, b_1 = b_2 = ... = b_s = 0)$, *we get the generalized Ruscheweyh derivative operator as well [6] .*

Moreover, if we put $(a_2 = a_3 = ... = a_r = 0, b_1 = b_2 = ... = b_s = 0)$ *or* $(a_2 = a_3 = ... = a_r = 0, b_1 = b_2 = ... = b_s = 0)$, *we can get Al-Abbadi and Darus operator [2].*

Finally, if $(\lambda_2 = 0, m = m + 1, a_2 = a_3 = ... = a_r = 0, b_1 = b_2 = ... = b_s = 0)$, *we get the generalized Al-Shaqsi and Darus derivative operator [7].*

Definition 1. *Let* $f \in \mathcal{A}$. *Then* $f(z) \in \mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$ *if and only if*

$$\Re\left\{\frac{z\left[\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z)\right]'}{\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z)}\right\} > \eta, 0 \le \eta < 1, z \in \mathcal{U}. \tag{9}$$

In this present paper we study the characterization properties and distrortion theorem of the class $f(z) \in \mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$ in section 2. And in section 3, we determine the sharp upper bound for $|a_2|$ for the same class. Moreover, we calculate the Fekete-Szegö functional $|a_3 - \delta a_2^2|$ for it. For this purpose we need the following Lemma:

Lemma 1. *[8] Let* $p \in \mathcal{P}$, *that is,* p *be analytic in* $\mathcal{U}$, *be given by* $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ *and* $\Re p(z) > 0$ *for* $z \in \mathcal{U}$. *Then*

$$|p_2 - \frac{p_1^2}{2}| \le 2 - \frac{|p_1|^2}{2}$$

and $|p_n| \le 2$ *for all* $n \in N$.

2. General Properties of the Operator $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$

Theorem 2. *Suppose that $f(z) \in \mathcal{A}$. If*

$$\sum_{k=2}^{\infty}(k-\eta)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k| \leq 1-\eta, 0 \leq \eta \leq 1 \tag{10}$$

then $f(z) \in \mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$. The result (10) is sharp.

Proof. Suppose that (10) holds. Since

$$\begin{aligned} 1-\eta &\geq \sum_{k=2}^{\infty}(k-\eta)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k| \\ &\geq \sum_{k=2}^{\infty}\eta\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k| - \sum_{k=2}^{\infty}k\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k| \end{aligned}$$

we deduce that

$$\frac{1+\sum_{k=2}^{\infty}k\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k|}{1+\sum_{k=2}^{\infty}\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k|} > \eta,$$

thus

$$\Re\left\{\frac{z\left[\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z)\right]'}{\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z)}\right\} > \eta, 0 \leq \eta < 1, z \in \mathcal{U}.$$

We note that the assertion is (10) sharp, moreover, the extermal function can be given by

$$f(z) = z + \sum_{k=2}^{\infty}\frac{(1-\eta)}{(k-\eta)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}z^k.$$

Corollary 3. *If the hypotheses of Theorem 2.1 is satisfied. Then*

$$|a_k| \leq \frac{(1-\eta)}{(k-\eta)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}, \forall n \geq 2. \tag{11}$$

We have also this following inclusion result:

Theorem 4. *Let $0 \leq \eta_1 \leq \eta_2 < 1$. Then $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta_1) \supseteq \mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta_2)$.*

Proof. By Theorem 2.1.

Let us introduce the following distrotion theorems.

Theorem 5. *If the hypotheses of Theorem 2.1 be satisfied. Then for* $z \in \mathcal{U}$ *and* $0 \leq \eta < 1$

$$|\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2} f(z)| \geq |z| - \frac{1-\eta}{2-\eta}$$

and

$$|\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2} f(z)| \leq |z| + \frac{1-\eta}{2-\eta}$$

Proof. By using Theorem 2.1, one can verify that

$$(2-\eta)\sum_{k=2}^{\infty}\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k| \leq \sum_{k=2}^{\infty}(k-\eta)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k| \leq 1-\eta$$

then

$$\sum_{k=2}^{\infty}\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k| \leq \frac{1-\eta}{2-\eta}.$$

Thus we obtain

$$\begin{aligned}
|\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2} f(z)| &\leq |z| + \sum_{k=2}^{\infty}\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k||z|^k \\
&\leq |z| + \sum_{k=2}^{\infty}\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k||z|^2 \\
&\leq |z| + \left[\frac{1-\eta}{2-\eta}\right]|z|^2
\end{aligned}$$

The other assertion can be proved as follows:

$$\begin{aligned}
|\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2} f(z)| &= |z + \sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\Gamma_k a_k z^k| \\
&\geq |z| - \sum_{k=2}^{\infty}\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k||z|^k \\
&\geq |z| - \sum_{k=2}^{\infty}\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k||z|^2 \\
&\geq |z| - \left[\frac{1-\eta}{2-\eta}\right]|z|^2
\end{aligned}$$

which completes the proof.

Theorem 6. *Let the assumptions of Theorem 2.1 hold. Then*

$$|f(z)| \leq |z| + \frac{2(1-\eta)(1+\lambda_2)^m.\Gamma(\alpha_1)...\Gamma(\alpha_r).\Gamma(\beta_1+2)...\Gamma(\beta_s+2)}{(2-\eta)(1+\lambda_1)^{m-1}.\Gamma(\beta_1)...\Gamma(\beta_s).\Gamma(\alpha_1+2)...\Gamma(\alpha_r+2)}|z|^2$$

and

$$|f(z)| \geq |z| - \frac{2(1-\eta)(1+\lambda_2)^m.\Gamma(\alpha_1)...\Gamma(\alpha_r).\Gamma(\beta_1+2)...\Gamma(\beta_s+2)}{(2-\eta)(1+\lambda_1)^{m-1}.\Gamma(\beta_1)...\Gamma(\beta_s).\Gamma(\alpha_1+2)...\Gamma(\alpha_r+2)}|z|^2$$

Proof. From Theorem 2.1, one can write

$$\frac{(2-\eta)}{2}\left[\frac{(1+\lambda_1)^{m-1}}{(1+\lambda_2)^m}\right]\frac{\Gamma(\beta_1)...\Gamma(\beta_s).\Gamma(\alpha_1+2)...\Gamma(\alpha_r+2)}{\Gamma(\alpha_1)...\Gamma(\alpha_r).\Gamma(\beta_1+2)...\Gamma(\beta_s+2)}\sum_{k=2}^{\infty}|a_k|$$
$$\leq \sum_{k=2}^{\infty}(k-\eta)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k| \leq 1-\mu$$

then

$$\sum_{k=2}^{\infty}|a_k| \leq \frac{2(1-\eta)(1+\lambda_2(k-1))^m\Gamma(\alpha_1)...\Gamma(\alpha_r).\Gamma(\beta_1+2)...\Gamma(\beta_s+2)}{(2-\eta)(1+\lambda_1)^{m-1}\Gamma(\beta_1)...\Gamma(\beta_s).\Gamma(\alpha_1+2)...\Gamma(\alpha_r+2)}.$$

Thus we obtain

$$\begin{aligned}
|f(z)| &= |z+\sum_{n=k}^{n=\infty} a_k z^k| \\
&\leq |z| + \sum_{n=k}^{n=\infty}|a_k||z|^2 \\
&\leq |z| + \frac{2(1-\eta)(1+\lambda_2)^m.\Gamma(\alpha_1)...\Gamma(\alpha_r).\Gamma(\beta_1+2)...\Gamma(\beta_s+2)}{(2-\eta)(1+\lambda_1)^{m-1}.\Gamma(\beta_1)...\Gamma(\beta_s).\Gamma(\alpha_1+2)...\Gamma(\alpha_r+2)}|z|^2
\end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned}
|f(z)| &= |z+\sum_{n=k}^{n=\infty} a_k z^k| \\
&\geq |z| - \sum_{n=k}^{n=\infty}|a_k||z|^2 \\
&\geq |z| - \frac{2(1-\eta)(1+\lambda_2)^m.\Gamma(\alpha_1)...\Gamma(\alpha_r).\Gamma(\beta_1+2)...\Gamma(\beta_s+2)}{(2-\eta)(1+\lambda_1)^{m-1}.\Gamma(\beta_1)...\Gamma(\beta_s).\Gamma(\alpha_1+2)...\Gamma(\alpha_r+2)}|z|^2.
\end{aligned}$$

This completes the proof.

3. Fekete-Szegö for the Class $\mathrm{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z)$

In this section we obtain sharp upper bound of $|a_2|$ and of the Fekete-Szegö functional $|a_3 - \delta a_2^2|$ for the class $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$.

Theorem 7. *Let the hypothesis of Theorem 2.1 be satisfied. Then*

$$|a_2| \leq \frac{4(1-\eta)(1+\lambda_2)^m\Gamma(\beta_1+2)...\Gamma(\beta_s+2)\Gamma(\alpha_1)...\Gamma(\alpha_r)}{(1+\lambda_1)^{m-1}\Gamma(\alpha_1+2)...\Gamma(\alpha_r+2)\Gamma(\beta_1)...\Gamma(\beta_s)}$$

and the following bound is sharp

$$|a_3 - \delta a_2^2| \leq \frac{3}{2}\left[\frac{(1+2\lambda_2)^m}{(1+2\lambda_1)^{m-1}}\frac{\Gamma(\beta_1+3)...\Gamma(\beta_s+3)\Gamma(\alpha_1)...\Gamma(\alpha_r)}{\Gamma(\alpha_1+3)...\Gamma(\alpha_r+3)\Gamma(\beta_1)...\Gamma(\beta_s}(1-\eta)\right]$$
$$\max\left\{1, \left|1+2(1-\eta)\left[1-\frac{4}{3}\delta\frac{(1+\lambda_2)^2m(1+2\lambda_1)^{m-1}}{(1+\lambda_1)^2(m-1)(1+2\lambda_2)^m}\frac{(\Gamma(\beta_1+2))^2...(\Gamma(\beta_s+2)^2\Gamma(\alpha_1)...\Gamma(\alpha_r)}{(\Gamma(\alpha_1+2))^2...(\Gamma(\alpha_r+2)^2\Gamma(\beta_1)...\Gamma(\beta_s)}\right.\right.\right.$$
$$\left.\left.\left.\frac{\Gamma(\alpha_1+3)...\Gamma(\alpha_r+3)}{\Gamma(\beta_1+3)\Gamma(\beta_s+3)}\right]\right|\right\}$$

for all $\delta \in \mathbb{C}$

Proof. Since $f \in \mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$ then the following condition is satisfied

$$\Re\left\{\frac{z\left[\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z)\right]'}{\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z)}\right\} > \eta, 0 \leq \eta < 1, z \in \mathcal{U}.$$

We can write

$$z\left[\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z)\right]' = \mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z)[(1-\eta)p(z) + \eta, 0 \leq \eta < 1, z \in \mathcal{U},$$

for some $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + ...$
By calculating coefficients we can write:

$$a_2 = A(1-\eta)p_1, a_3 = B[(1-\eta)^2p_1^2 + (1-\eta)p_2]$$

where

$$A = \frac{2(1+\lambda_2)^m\Gamma(\beta_1+2)...\Gamma(\beta_s+2)\Gamma(\alpha_1)...\Gamma(\alpha_r)}{(1+\lambda_1)^{m-1}\Gamma(\alpha_1+2)...\Gamma(\alpha_r+2)\Gamma(\beta_1)...\Gamma(\beta_s)}$$

and

$$B = \frac{3(1+2\lambda_2)^m\Gamma(\beta_1+3)...\Gamma(\beta_s+3)\Gamma(\alpha_1)...\Gamma(\alpha_r)}{(1+2\lambda_1)^{m-1}\Gamma(\alpha_1+3)...\Gamma(\alpha_r+3)\Gamma(\beta_1)...\Gamma(\beta_s)}.$$

Moreover, for

$$C := D\frac{1}{2} + (1-\eta) - \frac{\delta A^2(1-\eta}{B}; D := B(1-\eta),$$

by applying Lemma 1 we have

$$L(x) = 2D + (C - \frac{D}{2})x^2, wherex := |p_1| \le 2.$$

Consequently, we obtain

$$|a_3 - \delta a_2^2| \le \begin{cases} L(0) = 2D & \text{if} \quad C \le \frac{D}{2}, \\ L(2) = 4C & \text{if} \quad C \ge \frac{D}{2}. \end{cases}$$

Equality holds for functions given by

$$\frac{z\left[\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2} f(z)\right]'}{\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2} f(z)} = \frac{1+z^2(1-2\eta)}{1-z^2}$$

and

$$\frac{z\left[\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2} f(z)\right]'}{\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2} f(z)} = \frac{1+z(1-2\eta)}{1-z}.$$

respectively.

Putting $\eta = 0$ we can get the following corollary.

Corollary 8. *Let the hypothesis of Theorem 3.1 be satisfied. Then for* $\eta = 0$

$$|a_2| \le \frac{4(1+\lambda_2)^m \Gamma(\beta_1+2)...\Gamma(\beta_s+2)\Gamma(\alpha_1)...\Gamma(\alpha_r)}{(1+\lambda_1)^{m-1}\Gamma(\alpha_1+2)...\Gamma(\alpha_r+2)\Gamma(\beta_1)...\Gamma(\beta_s)}$$

and

$$|a_3 - \delta a_2^2| \le \frac{3}{2}\left[\frac{(1+2\lambda_2)^m}{(1+2\lambda_1)^{m-1}} \frac{\Gamma(\beta_1+3)...\Gamma(\beta_s+3)\Gamma(\alpha_1)...\Gamma(\alpha_r)}{\Gamma(\alpha_1+3)...\Gamma(\alpha_r+3)\Gamma(\beta_1)...\Gamma(\beta_s}\right]$$

$$\max\left\{1, \left|1+2\left[1-\frac{4}{3}\delta\frac{(1+\lambda_2)^2 m(1+2\lambda_1)^{m-1}}{(1+\lambda_1)^2(m-1)(1+2\lambda_2)^m}\frac{(\Gamma(\beta_1+2))^2...(\Gamma(\beta_s+2)^2\Gamma(\alpha_1)...\Gamma(\alpha_r)}{(\Gamma(\alpha_1+2))^2...(\Gamma(\alpha_r+2)^2\Gamma(\beta_1)...\Gamma(\beta_s)}\right.\right.\right.$$

$$\left.\left.\left.\frac{\Gamma(\alpha_1+3)...\Gamma(\alpha_r+3)}{\Gamma(\beta_1+3)\Gamma(\beta_s+3)}\right]\right|\right\}$$

Acknowledgements. The authors would like to acknowledge and appreciate the financial support received from Universiti Kebangsaan Malaysia under the 'ZA-MALAH' grant and GUP-2012-023.

References

[1] J.Dziok and H.M.Srivastava, *Classes of analytic functions associated with the generalized hypergeometric function*, Applied Mathematics And Computation. 103(1999), 1-13.

[2] M.H.Al-Abbadi and M.Darus, *Differential subordination for new generalized derivative operator*, Acta Universitatis Apulensis, 20 (2009), 265-280.

[3] Yu.E. Hohlov, *Operators and operations in the class of univalent functions*, Matematika , 10(1978) 83-89.

[4] B.C. Carlson and D.B. Shaffer, *Starlike and prestarlike hypergeometric function*, SIAM J. Math. Anal. 15(1984)737-745.

[5] S.Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math soc. 49(1975) 109-115.

[6] K. Al-Shaqsi and M.Darus, *On univalent functions with respect to the k-symmetric points defined by a generalization ruscheweyh derivative operators*,Jour. Anal. Appl., 7(1),(2009),53-61.

[7] K. Al-Shaqsi and M.Darus,*Differential Subordination with generalized derivative operator*,Int. J. Comp. Math. Sci, 2(2), 75-78.

[8] P. L. Duren, *Univalent functions*. Springer-Verlag, New York, 1983.

Khadeejah Rasheed Alhindi
School of mathematical Sciences,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia,
Bangi 43600 UKM, Selangor Darul Ehsan, Malaysia.
email: *m.kh83@yahoo.co.uk*

Maslina Darus
School of mathematical Sciences,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia,
Bangi 43600 UKM, Selangor Darul Ehsan, Malaysia.
email: *maslina@ukm.my*