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# COEFFICIENT BOUNDS FOR NEW SUBCLASSES OF BI-UNIVALENT FUNCTIONS USING HADAMARD PRODUCT

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Abstract. The aim of the present paper is to introduce a new subclass of bi-univalent functions defined in the open unit disc using Hadamard product. We obtain estimates on the coefficients $|a_2|$ and $|a_3|$ for functions of this class. Some results related to this work will also be pointed out.

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## 1. Introduction

Let $A$ denote the class of the functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disc $U = \{z \in C : |z| < 1\}$ and satisfy the normalization condition $f(0) = f^{'}(0) = 0$. Let $S$ be the subclass of $A$ consisting of functions of the form (1) which are also univalent in $U$. For $n \in N_0$, we introduce the subclass $Q(n, \delta, \beta, \lambda)$ of $S$ of functions $f$ of the form (1), satisfying the condition

$$Re\left\{ \frac{(1-\lambda)D^k_{n,\delta} f(z) + \lambda D^{k+1}_{n,\delta} f(z)}{z} \right\} > \beta, z \in U, \tag{2}$$

where $D^k_{n,\delta}$ is the differential operator given by Hadamard product between Salagean and Ruscheweyh operators, such as

$$D^k_{n,\delta} f(z) = z + \sum_{n=2}^{\infty} C(\delta, n) n^k a_n z^n.$$

For $k = \delta = 0$, it reduces to the class $Q_\lambda(\beta)$ studied by Ding et al. [3], (see also [4-7]).

Now by having

$$f^{-1}f(z) = z,\ (z \in U),$$

and

$$f^{-1}f(w) = w,\ (|w| < r_0, f(z) \geq \tfrac{1}{4}\ )$$

where $f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^2 - 5a_2a_3 + a_4)w^4 + \ldots$ , we say that a function $f(z) \in A$ is bi-univalent in $U$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $U$.

Let $\Sigma$ denote the class of bi-univalent functions in $U$ given by (1). For a brief history and interesting examples in the class $\Sigma$, see [8]. In fact, Brannan and Taha [9] (see also [11]) introduced certain subclasses of the bi-univalent functions similar to the familiar subclasses $S^*(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order $\alpha(0 \leq \alpha < 1)$, respectively (see [10]). Following the same manner of Brannan and Taha [9] (see also [11]), a function $f \in A$ is in the class of strongly bi-Starlike functions of order $\alpha(0 < \alpha \leq 1)$ if each of the following conditions is satisfied: For $f \in \Sigma$,

$$\left| arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi\alpha}{2},\ \alpha(0 < \alpha \leq 1, z \in U),$$

and

$$\left| arg \left\{ \frac{wg'(w)}{g(w)} \right\} \right| < \frac{\pi\alpha}{2},\ \alpha(0 < \alpha \leq 1, w \in U),$$

where $g$ is the extension of $f^{-1}(z)$ to $U$. Similarly, a function $f \in A$ is in the class $K_\Sigma(\alpha)$ of strongly bi-convex functions of order $\alpha$ if each of the following conditions are satisfied: For $f \in \Sigma$,

$$\left| arg \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \right| < \frac{\pi\alpha}{2},\ \alpha(0 < \alpha \leq 1, z \in U),$$

and

$$\left| arg \left\{ 1 + \frac{wg''(w)}{g'(w)} \right\} \right| < \frac{\pi\alpha}{2},\ \alpha(0 < \alpha \leq 1, w \in U),$$

where $g$ is the extension of to $U$. The classes $S^*_\Sigma(\alpha)$ and $K_\Sigma(\alpha)$ of bi-starlike functions of order $\alpha$ and bi-convex functions of order $\alpha$, corresponding (respectively) to the classes of $S^*(\alpha)$ and $K(\alpha)$ were also introduced analogously. For each of the classes $S^*_\Sigma(\alpha)$ and $K_\Sigma(\alpha)$, it was noted that the estimates obtained for the first two coefficients $|a_2|$ and $|a_3|$ are not sharp (for details, see [9,11]).

The object of the paper is to introduce two new subclasses of the function class $\Sigma$ and to find estimates on the coefficients $|a_2|$ and $|a_3|$ using the same techniques given earlier by Srivastava et al. [8], Frasin and Aouf [12], and Porwal and Darus [2]. In order to prove our main results, we need the following lemma due to [15].

**Lemma 1.** *If $h \in p$ then $|c_k| < 1$, for each $k$, where $p$ is the family of all functions $h$ analytic in $U$ for which $Re\{h(z)\} > 0$, then*

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... \, , z \in U.$$

## 2. Coefficient bounds for the function class $Q_\Sigma(n, \delta, \alpha, \lambda)$

**Definition 1.** *A function $f(z)$ given by (1) is said to be in the class $Q_\Sigma(n, \delta, \alpha, \lambda)$ if the following conditions are satisfied: For $f \in \Sigma$,*

$$\left| arg \frac{(1-\lambda)D^k_{n,\delta} f(z) + \lambda D^{k+1}_{n,\delta} f(z)}{z} \right| < \frac{\pi\alpha}{2}, \alpha(0 < \alpha \leq 1, \lambda \geq 1, z \in U), \tag{3}$$

*and*

$$\left| arg \frac{(1-\lambda)D^k_{n,\delta} g(w) + \lambda D^{k+1}_{n,\delta} g(w)}{w} \right| < \frac{\pi\alpha}{2}, \alpha(0 < \alpha \leq 1, \lambda \geq 1, w \in U), \tag{4}$$

*where the function $g$ is given by*

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^2 - 5a_2 a_3 + a_4)w^4 + ..... \quad . \tag{5}$$

We note that for $k = \delta = 0, \lambda = 1$, the class $Q_\Sigma(n, \delta, \alpha, \lambda)$ reduces to the class $H^\alpha_\Sigma$ introduced and studied by Srivastava et al [8], for $k = \delta = 0$, the class reduces to $Q_\Sigma(\alpha, \lambda)$ introduced and studied by Frasin and Aouf [12]. Also for $\delta = 0$, the class $Q_\Sigma(n, \delta, \alpha, \lambda)$ reduces to $Q_\Sigma(n, \alpha, \lambda)$ studied by Porwal and Darus [2]. We begin by finding the estimates of the coefficients for functions in the class $Q_\Sigma(n, \delta, \alpha, \lambda)$.

**Theorem 2.** *Let the function $f(z)$ given by (1) be in the class $Q_\Sigma(n,\delta,\alpha,\lambda)$, $n\in N_0, 0\leq\beta<1, \lambda\geq 1$. Then*

$$|a_2|\leq 4\alpha\left|\frac{\Gamma(\delta+1)}{\Gamma(\delta+2)}\right|\left[\frac{1}{\sqrt{4^k(1+\lambda)^2+\alpha[2.3^k(1+\lambda)-4^k(1+\lambda)^2]}}\right] \tag{6}$$

*and*

$$|a_3|\leq 12\alpha\frac{\Gamma(\delta+1)}{\Gamma(\delta+3)}\left[\frac{1}{(1-\lambda)3^k+\lambda 3^k(1+\lambda)}+\frac{2\alpha}{[(1-\lambda)2^k+\lambda 2^{k+1}]^2}\right] \tag{7}$$

*Proof.* From (3) and (4), we can write

$$\frac{(1-\lambda)D^k_{n,\delta}f(z)+\lambda D^{k+1}_{n,\delta}f(z)}{z}=[p(z)]^\alpha, \tag{8}$$

and

$$\frac{(1-\lambda)D^k_{n,\delta}g(w)+\lambda D^{k+1}_{n,\delta}g(w)}{w}=[q(w)]^\alpha, \tag{9}$$

respectively, where $p(z)$ and $q(w)$ are in $p$ and have the form

$$p(z)=1+p_1z+p_2z^2+p_3z^3+... \ , \tag{10}$$

and

$$q(w)=1+p_1w+q_2w^2+q_3w^3+... \ . \tag{11}$$

Now, equating the coefficients in (8) and (9), we obtain

$$[(1-\lambda)2^k+\lambda 2^{k+1}]C(\delta,2)a_2=\alpha p_1, \tag{12}$$

$$[(1-\lambda)3^k+\lambda 3^{k+1}]C(\delta,3)a_3=\frac{1}{2}[2\alpha p_2+\alpha(\alpha-1)p_1^2], \tag{13}$$

$$-[(1-\lambda)2^k+\lambda 2^{k+1}]C(\delta,2)a_2=\alpha q_1, \tag{14}$$

$$[(1-\lambda)3^k+\lambda 3^{k+1}](2[C(\delta,2)]^2a_2^2-C(\delta,3)a_3)=\frac{1}{2}[2\alpha q_2+\alpha(\alpha-1)q_1^2]. \tag{15}$$

From (12) and (14), we obtain

$$p_1=-q_1 \tag{16}$$

and

$$2[(1-\lambda)2^k+\lambda 2^{k+1}]^2[C(\delta,2)]^2a_2^2=\alpha^2(p_1^2+q_1^2). \tag{17}$$

Now from (13), (15) and (17), we obtain

$$2[(1-\lambda)3^k+\lambda 3^{k+1}][C(\delta,2)]^2a_2^2 = \alpha(p_2+q_2)+\tfrac{1}{2}[\alpha(\alpha-1)(p_1^2+q_1^2)]$$

$$=\alpha(p_2+q_2)+\tfrac{\alpha(\alpha-1)}{2}.\tfrac{2[(1-\lambda)2^k+\lambda 2^{k+1}]^2[C(\delta,2)]^2a_2^2}{\alpha^2}\ .$$

Therefore we have

$$a_2^2=\frac{\alpha^2(p_2+q_2)}{[4^k(1+\lambda)^2+\alpha[2.3^k(1+\lambda)]-4^k(1+\lambda)^2]]C[(\delta,2)]^2}.$$

Applying Lemma 1 for the coefficients $p_2$ and $q_2$, we immediately have

$$|a_2|\le 4\alpha\left|\frac{\Gamma(\delta+1)}{\Gamma(\delta+2)}\right|\left[\frac{1}{\sqrt{4^k(1+\lambda)^2+\alpha[2.3^k(1+\lambda)-4^k(1+\lambda)^2]}}\right].$$

This gives the bound as asserted in (6).

Next, in order to find the bound on $|a_3|$, we subtract (13) from (15) and obtain

$$2[(1-\lambda)3^k+\lambda 3^{k+1}](C(\delta,3)a_3-C[(\delta,2)]^2a_2^2)$$
$$=\tfrac{1}{2}(2\alpha(p_2-q_2)+\alpha(\alpha-1)(p_1^2-q_1^2)),$$

$$a_3=\frac{\alpha(p_2-q_2)}{2[(1-\lambda)3^k+\lambda 3^{k+1}](C\delta,3)}+\frac{\alpha^2(p_1^2+q_1^2)}{2[(1-\lambda)2^k+\lambda 2^{k+1}]^2(C\delta,3)},$$

$$a_3=\frac{6\alpha(p_2-q_2)\Gamma(\delta+1)}{2[(1-\lambda)3^k+\lambda 3^{k+1}]\Gamma(\delta+3)}+\frac{6\Gamma(\delta+1)(\alpha^2)(p_1^2+q_1^2)}{2[(1-\lambda)2^k+\lambda 2^{k+1}]^2\Gamma(\delta+3)}\ .$$

Applying Lemma 1 for the coefficients $p_2$ and $q_2$, we immediately have

$$|a_3|\le\frac{12\alpha\Gamma(\delta+1)}{[(1-\lambda)3^k+\lambda 3^{k+1}]\Gamma(\delta+3)}+\frac{24\Gamma(\delta+1)\alpha^2}{[(1-\lambda)2^k+\lambda 2^{k+1}]^2\Gamma(\delta+3)},$$

*i.e.*

$$|a_3|\le 12\alpha\frac{\Gamma(\delta+1)}{\Gamma(\delta+3)}\left[\frac{1}{(1-\lambda)3^k+\lambda 3^k(1+\lambda)}+\frac{2\alpha}{[(1-\lambda)2^k+\lambda 2^{k+1}]^2}\right].$$

This completes the proof of Theorem 2.

Putting $\lambda=1, k=\delta=0$, in Theorem 2, we have

**Corollary 3.** *Let $f(z)$ given by (1) be in the class $H_\Sigma^\alpha(0<\alpha\le 1)$. Then*

$$|a_2|\le\alpha\sqrt{\frac{2}{2+\alpha}}$$

*and*

$$|a_3|\le\frac{\alpha(2+3\alpha)}{3}.$$

## 3. Coefficient bounds for the function class $H_\Sigma(n,\delta,\beta,\lambda)$

**Definition 2.** *A function $f(z)$ given by (1) is said to be in the class $H_\Sigma(n,\delta,\beta,\lambda)$ if the following conditions are satisfied:*

$$Re\left\{\frac{(1-\lambda)D^k_{n,\delta}f(z)+\lambda D^{k+1}_{n,\delta}f(z)}{z}\right\}>\beta, z\in U, n\in N_0, 0\le\beta<1, \lambda\ge 1. \quad (18)$$

*and*

$$Re\left\{\frac{(1-\lambda)D^k_{n,\delta}g(w)+\lambda D^{k+1}_{n,\delta}g(w)}{w}\right\}>\beta, w\in U, n\in N_0, 0\le\beta<1, \lambda\ge 1 \quad (19)$$

*where the function $g$ is defined by (5).*

We note that for $k=\delta=0$, and $\lambda=1$, $H_\Sigma(n,\delta,\beta,\lambda)$ the class reduced to the classes $H_\Sigma(\beta)$ studied by Srivastava et al.[8], and for $k=\delta=0$, the class reduced to the classes $H_\Sigma(\beta,\lambda)$ studied by Frasin and Aouf [12].

**Theorem 4.** *Let the function $f(z)$ given by (1) be in the class $H_\Sigma(n,\delta,\beta,\lambda)$, $n\in N_0, 0\le\beta<1, \lambda\ge 1$. Then*

$$|a_2|\le 2\left|\frac{\Gamma(\delta+1)}{\Gamma(\delta+2)}\right|\sqrt{\frac{2(1-\beta)}{(1-\lambda)3^k+\lambda 3^{k+1}}} \quad (20)$$

*and*

$$|a_3|\le\frac{12(1-\beta)\Gamma(\delta+1)}{\Gamma(\delta+3)}\left[\frac{2(1-\beta)}{[(1-\lambda)2^k+\lambda 2^{k+1}]^2}+\frac{1}{(1-\lambda)3^k+\lambda 3^{k+1}}\right]. \quad (21)$$

*Proof.* It follows from (18) and (19) that there exists $p,q\in P$ such that

$$\frac{(1-\lambda)D^k_{n,\delta}f(z)+\lambda D^{k+1}_{n,\delta}f(z)}{z}=\beta+(1-\beta)p(z), \quad (22)$$

and

$$\frac{(1-\lambda)D^k_{n,\delta}g(w)+\lambda D^{k+1}_{n,\delta}g(w)}{w}=\beta+(1-\beta)q(w), \quad (23)$$

where $p(z)$ and $q(w)$ have the forms (10) and (11), respectively. Equating coefficients in (22) and (23) yields

$$[(1-\lambda)2^k+\lambda 2^{k+1}]C(\delta,2)a_2=(1-\beta)p_1, \quad (24)$$

$$[(1-\lambda)3^k+\lambda 3^{k+1}]C(\delta,3)a_3=(1-\beta)p_2, \quad (25)$$

$$-[(1-\lambda)2^k+\lambda 2^{k+1}]C(\delta,2)a_2=(1-\beta)q_1, \tag{26}$$

and

$$[(1-\lambda)3^k+\lambda 3^{k+1}](2[C(\delta,2)]^2a_2^2-C(\delta,3)a_3)=(1-\beta)q_2. \tag{27}$$

From (24) and (26), we have

$$-p_1=q_1 \tag{28}$$

and

$$2[(1-\lambda)2^k+\lambda 2^{k+1}]^2C[(\delta,2)]^2a_2^2=(1-\beta)^2(p_1^2+q_1^2). \tag{29}$$

Also, from (25) and (27), we find that

$$2[(1-\lambda)3^k+\lambda 3^{k+1}]C[(\delta,2)]^2a_2^2=(1-\beta)(p_2+q_2), \tag{30}$$

$$|a_2^2|\le\frac{(1-\beta)(|p_2|+|q_2|)}{2[(1-\lambda)3^k+\lambda 3^{k+1}]C[(\delta,2)]^2}, \tag{31}$$

*i.e.*

$$|a_2|\le 2\left|\frac{\Gamma(\delta+1)}{\Gamma(\delta+2)}\right|\sqrt{\frac{2(1-\beta)}{(1-\lambda)3^k+\lambda 3^{k+1}}}. \tag{32}$$

which is the bound on $|a_2|$ as given in (20).

Next, in order to find the bound on $|a_3|$ by subtracting (27) from (25), we obtain

$$2C(\delta,3)[(1-\lambda)3^k+\lambda 3^{k+1}]a_3=$$

$$2[(1-\lambda)3^k+\lambda 3^{k+1}][C(\delta,2)]^2a_2^2+(1-\beta)(p_2-q_2)$$

or, equivalently

$$a_3=\frac{2[(1-\lambda)3^k+\lambda 3^{k+1}][C(\delta,2)]^2a_2^2}{2C(\delta,3)[(1-\lambda)3^k+\lambda 3^{k+1}]}+\frac{(1-\beta)(p_2-q_2)}{2C(\delta,3)[(1-\lambda)3^k+\lambda 3^{k+1}]}$$

Upon substituting the value of $a_2^2$ from (29), we obtain

$$a_3=\frac{3(1-\beta)^2(p_1^2+q_1^2)\Gamma(\delta+1)}{[(1-\lambda)2^k+\lambda 2^{k+1}]^2\Gamma(\delta+3)}+\frac{3(1-\beta)(p_2-q_2)\Gamma(\delta+1)}{[(1-\lambda)3^k+\lambda 3^{k+1}]\Gamma(\delta+3)}. \tag{33}$$

Applying Lemma 1 for the coefficients $p_1,p_2,q_1$ and $q_2$ we obtain

$$|a_3|\le\frac{12(1-\beta)\Gamma(\delta+1)}{\Gamma(\delta+3)}\left[\frac{2(1-\beta)}{[(1-\lambda)2^k+\lambda 2^{k+1}]^2}+\frac{1}{(1-\lambda)3^k+\lambda 3^{k+1}}\right] \tag{34}$$

which is the bound on $|a_3|$ as asserted in (21).

Putting $\lambda = 1$, $k = \delta = 0$, in Theorem 4, we have the following corollary.

**Corollary 5.** *Let* $fz)$ *given by (1) be in the class* $H_\Sigma(n, \delta, \beta, \lambda), (0 \leq \beta < 1)$*. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3}} \tag{35}$$

*and*

$$|a_3| \leq \frac{(1-\beta)(5-3\beta)}{3}. \tag{36}$$

**Remark 1.** *If we put* $\delta = k = 0$*, in Theorems 2 and 3, we obtain the corresponding results due to Frasin and Aouf [12].*

**Remark 2.** *If we put* $\delta = 0$*, in Theorems 2 and 3, we obtain the corresponding results due to Porwal and Darus [2].*

**Remark 3.** *If we put* $\delta = k = 0, \lambda = 1$*, in Theorems 2 and 3, we obtain the corresponding results due to Srivastava et al [8].*

**Remark 4.** *Similarly, just as stated in [2], it would be nice to find estimates for* $|a_n|, n \geq 4$ *(not necessarily sharp) for the class of functions defined in this work.*

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