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COMMON FIXED POINTS FOR HYBRID PAIRS OF FUZZY AND CRISP MAPPINGS

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Abstract. We obtained sufficient conditions for existence of common fixed points for hybrid pairs of fuzzy and crisp mappings without completeness.

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1. Introduction and preliminaries

The concept of fuzzy sets was introduced by Zadeh [17] in 1965. Afterwards considerable attention has been given to the study of fuzzy models with numerous applications. The major area concerning fuzzy equation involves the existence, uniqueness, characterization, contraction and approximation of solutions. During last five decades this theory have developed into an area which scientifically as well as from the application point of view is a very valuable contribution to the existing literature [9, 12, 18]. Heilpern [10] started to study fixed point in fuzzy setting. Afterward Beg et al. [1, 2, 3, 4, 5, 6, 7, 8] and many other authors have started work in this direction. The result of studying fixed points theorems of hybrid pairs of fuzzy and non fuzzy mappings is a generalization of these results. In this paper we obtain sufficient conditions for existence of common fixed points for such mappings without the completeness of the involved space. Results of Sharma et al. [16] and Beg and Ahmed [8] are special cases of our results.

Let (X,d) be a metric linear space A *fuzzy set* in X is a function $A : X \to [0,1]$. If A is a fuzzy set and $x \in X$, then the function-value $A(x)$ is called the *grade of membership* of x in A. The collection of all fuzzy sets in X is denoted by $\Im(X)$. Let $A \in \Im(X)$ and $\alpha \in [0,1]$. The *α-level set* of A, denoted by A_α, is defined by

$$A_\alpha = \{x : A(x) \geq \alpha\} \quad \text{if} \quad \alpha \in (0,1], \quad A_0 = \overline{\{x : A(x) > 0\}},$$

whenever $\overline{B}$ is the closure of set (non fuzzy) B.

Definition 1.1. *[10] A fuzzy set A in X is an approximate quantity if and only if its α-level set is a nonempty compact convex subset (non fuzzy) of X for each $\alpha \in [0,1]$ and $sup_{x\in X}A(x) = 1$.*

Following Beg and Ahmed [8] consider a sub collection $W^*(X)$ of $\Im(X)$ such that

$$W^*(X) = \{A \in \Im(X) : A_\alpha \in CP(X), \forall\alpha \in [0,1]\}.$$

where $CP(X)$ be the set of all nonempty compact subsets of X.
It is obvious that $W(X) \subset W^*(X)$. We can rewrite definitions from Heilpern [10] as follow:

Definition 1.2. *Let $A, B \in W^*(X)$, $\alpha \in [0,1]$, Then*

$$\begin{aligned} p_\alpha(A,B) &= \inf\{d(x,y) : x \in A_\alpha, y \in B_\alpha\} \\ \delta_\alpha(A,B) &= \sup\{d(x,y) : x \in A_\alpha, y \in B_\alpha\} \\ \text{and} \qquad D_\alpha(A,B) &= H(A_\alpha, B_\alpha), \end{aligned}$$

where H is the Hausdorff metric between two sets in the collection $CP(X)$.

We also define the following functions

$$\begin{aligned} p(A,B) &= \sup_\alpha p_\alpha(A,B), \\ \delta(A,B) &= \sup_\alpha \delta_\alpha(A,B) \\ \text{and} \qquad D(A,B) &= \sup_\alpha D_\alpha(A,B). \end{aligned}$$

It is noted that p_α is nondecreasing function of α.

Definition 1.3. *Let $A, B \in W^*(X)$. Then A is said to be more accurate than B (or B includes A), denoted by $A \subset B$, if and only if $A(x) \leq B(x)$ for each $x \in X$.*

The relation $\subset$ induces a partial order on $W^*(X)$.

Definition 1.4. *Let X be an arbitrary set and Y be a metric space. Mapping T is said to be a fuzzy mapping if and only if T is a mapping from the set X into $W^*(Y)$, i.e., $T(x) \in W^*(Y)$ for each $x \in X$.*

Let Ψ be the family of real valued lower semi-continuous functions $F : [0,\infty)^6 \to \mathbb{R}$, satisfying the following conditions:

(ψ_1) F is non-increasing in 3^{rd},4^{th},5^{th},6^{th} and non-decreasing in 1^{st} variable,

(ψ_2) there exists $h \in (0,1)$ such that for every $u, v \geq 0$ with

(ψ_{2_1}) $F(u, v, v, u, u + v, 0) \leq 0$

or

(ψ_{2_2}) $F(u, v, u, v, 0, u + v) \leq 0$,

we have $u \leq hv$,

and

(ψ_3) $F(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

Lemma 1.5. *[13] If $A, B \in CP(X)$ and $a \in A$, then there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

Lemma 1.6. *[8] If $\{x_0\} \subset A$ for each $A \in W^*(X)$ and $x_0 \in X$, then $p_\alpha(x_0, B) \leq D_\alpha(A, B)$ for each $B \in W^*(X)$.*

Lemma 1.7. *[8] For all $x, y \in X$ and $A \in W^*(X)$,*

$$p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A).$$

Lemma 1.8. *[8] Let $x \in X$, $A \in W^*(X)$ and $\{x\}$ be a fuzzy set with membership function equal to a characteristic function of the set $\{x\}$. Then $\{x\} \subset A$ if and only if $p_\alpha(x, A) = 0$ for each $\alpha \in [0, 1]$.*

Definition 1.9. *[16] Let I, J be two mappings from a metric space X into itself and T_1, T_2 be fuzzy mappings from X into $W^*(X)$. If for some $x_0 \in X$, there exist $\{y_n\}$ in X such that*

$$\{y_{2n+1}\} = \{Jx_{2n+1}\} \subset T_1 x_{2n}, \quad \{y_{2n+2}\} = \{Ix_{2n+2}\} \subset T_2 x_{2n+1}.$$

then $O(T_1, T_2, I, J, x_0)$ is called the orbit for the mappings (T_1, T_2, I, J)

Definition 1.10. *[16] Metric space X is called x_0 joint orbitally complete, if every Cauchy sequence of each orbit at x_0 is convergent in X.*

Remark 1.11. *[16] Complete space $\Rightarrow$ joint orbitally complete. But the converse is not necessarily true.*

Definition 1.12. *[16] Let I be a mapping from a nonempty subset M of a metric space (X, d) into itself and T be a fuzzy mappings from M into $W(M)$. A hybrid pair (I, T) is called D-compatible if and only if $\{It\} \subset Tt$ for some t in M implies $ITt = TIt$.*

2. Main results

First we rewrite Definition 1.12. for $W^*(M)$ as follows

Definition 2.1. *Let I be a mapping from a nonempty subset M of a metric space (X,d) into itself and T be a fuzzy mappings from M into $W^*(M)$. A hybrid pair (I,T) is called D-compatible if and only if $\{It\} \subset Tt$ for some t in M implies $ITt = TIt$.*

Now we state and prove the following main result.

Theorem 2.2. *Let I, J be two self mappings from a metric space (X,d) into itself and T_1, T_2 be fuzzy mappings from X into $W^*(X)$ such that*
(a); $T_1(X) \subset J(X)$,
(b); $T_2(X) \subset I(X)$,
(c); the pairs (T_1, I) and (T_2, J) are hybrid D-compatible mappings,
(d); $I(x)$ is x_0 joint orbitally complete for some $x_0 \in X$.
***If** there is a $F \in \Psi$ such that for all $x, y \in X$,*

$$F(D(T_1x,T_2y), d(Ix,Jy), p(Ix,T_1x), p(Jy,T_2y), p(Ix,T_2y), p(Jy,T_1x)) \leq 0,$$

then there exists $z \in X$ such that $z = Iz = Jz$ and $\{z\} \subset T_1z \cap T_2z$.

Proof. Let $x_0 \in X$. Construct an orbit $O(T_1, T_2, I, J, x_0)$ with two sequences $\{x_n\}$, $\{y_n\}$ of points in X such that

$$\begin{aligned} \{y_{2n+1}\} &= \{Jx_{2n+1}\} \subset T_1x_{2n}, \\ \{y_{2n+2}\} &= \{Ix_{2n+2}\} \subset T_2x_{2n+1}. \end{aligned}$$

Now we show that $\{y_n\}$ is a Cauchy sequence. Since

$$\{y_1\} = \{Jx_1\} \subset T_1x_0 \text{ and } \{T_1x_0\}_1, \{T_2x_1\}_1 \in CP(X).$$

Using Lemma 1.5. there exists $\{y_2\} = \{Ix_2\} \subset T_2x_1$ such that

$$d(y_1, y_2) \leq D(T_1x_0, T_2x_1).$$

We have from Lemma 1.8. and the property (ψ_1) of F that

$$\begin{aligned} & F(d(y_1,y_2), d(y_0,y_1), d(y_0,y_1), d(y_1,y_2), d(y_0,y_1)+d(y_1,y_2), 0) \\ \leq\ & F(D(T_1x_0,T_2x_1), (d(Ix_0,Jx_1), p(Ix_0,T_1x_0), p(Jx_1,T_2x_1), \\ & p(Ix_0,T_2x_1), p(Jx_1,T_1x_0)) \\ \leq\ & 0. \end{aligned}$$

From the property (ψ_{2_1}) of $F \in \Psi$, there exists $h \in (0,1)$ such that $d(y_1, y_2) \leq hd(y_0, y_1)$. Similarly, one can deduce from the property (ψ_{2_2}) of $F \in \Psi$ that there exists $h \in (0,1)$ such that $d(y_2, y_3) \leq hd(y_1, y_2)$. By induction we obtain,

$$d(y_n, y_{n+1}) \leq h^n d(y_0, y_1).$$

Thus

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d(y_{m-1}, y_m) \\ &\leq h^n d(y_0, y_1) + h^{n+1} d(y_0, y_1) + ... + h^{m-1} d(y_0, y_1) \\ &\leq \frac{h^n}{1-h} d(y_0, y_1). \end{aligned}$$

Therefore $\lim_{n,m\to\infty} d(y_n, y_m) = 0$. Hence $\{y_n\}$ is a Cauchy sequence. As $\{y_{2n}\}$ is a Cauchy sequence in $I(X)$, and $I(X)$ is x_0 joint orbitally complete therefore there exists $z \in X$ such that $y_{2n} \to z = Iu$ .Thus $y_n \to z$ as $n \to \infty$.Next we show that $z \subset T_1 u$.
From Lemma 1.6 and Lemma 1.7 we have

$$p(z, T_1 u) \leq d(z, y_{2n+2}) + D(T_1 u, T_2 x_{2n+1}).$$

Thus

$$\begin{aligned} & F(p(y_{2n+2}, T_1 u), d(z, y_{2n+1}), p(z, T_1 u), d(y_{2n+1}, y_{2n+2}), d(z, y_{2n+2}), p(y_{2n+1}, T_1 u) \\ \leq\ & F(D(T_1 u, T_2 x_{2n+1}), d(Iu, Jx_{2n+1}), p(Iu, T_1 u), p(Jx_{2n+1}, T_2 x_{2n+1}), p(Iu, T_2 x_{2n+1}), \\ & p(Jx_{2n+1}, T_1 u)) \\ \leq\ & 0. \text{ as } n \to \infty. \end{aligned}$$

Therefore

$$F(p(z, T_1 u), 0, p(z, T_1 u), 0, 0, p(z, T_1 u)) \leq 0.$$

By (ψ_{2_2})

$$p(z, T_1 u) \leq h.0 = 0.$$

Thus $z \subset T_1 u$. As $z = Iu \subset T_1 u \subset J(X)$ therefore there exists $\upsilon \in X$ such that $z = J\upsilon$. Similarly $z = J\upsilon \subset T_2 \upsilon$. Since the pair (T_1, I) are hybrid D-compatible and $z = Iu \subset T_1 u$ therefore $Iz = IIu \subset IT_1 u \subset T_1 Iu = T_1 z$.Also $Jz = JJ\upsilon \subset JT_2 \upsilon \subset T_2 J\upsilon = T_2 z$.

Next we show that $z = Iz$. Suppose otherwise i.e. $d(z, Iz) > 0$ then

$$\begin{aligned} & F(d(z,Iz), d(z,Iz), d(Iz,Iz), d(z,z), d(z,Iz), d(z,Iz)) \\ \leq\ & F(D(T_1z, T_2\upsilon), d(Iz, J\upsilon), p(Iz, T_1z), p(J\upsilon, T_2\upsilon), p(Iz, T_2\upsilon), p(J\upsilon, T_1z)) \\ \leq\ & 0. \end{aligned}$$

It further implies

$$F(d(z,Iz), d(z,Iz), 0, 0, d(z,Iz), d(z,Iz)) \leq 0.$$

It contradicts (ψ_3). Thus $d(z, Iz) = 0$. Therefore $z = Iz \subset T_1z$. Similarly $z = Jz \subset T_2z$. Hence $\{z\} \subset T_1z \cap T_2z$.

Theorem 2.3. *Let (X, d) be a metric space and $I, J : X \to X$ and $(T_n, n = 0, 1, 2, ...)$ be fuzzy mappings from X into $W^*(X)$ such that*
(a); $T_i(X) \subset J(X)$,
(b); $T_j(X) \subset I(X)$,
(c); the pairs (T_i, I) and (T_j, J) are hybrid D-compatible mappings,
(d); $I(x)$ is x_0 joint orbitally complete for some $x_0 \in X$.
***If** there is a $F \in \Psi$ such that for all $x, y \in X$,*

$$F(D(T_ix, T_jy), d(Ix, Jy), p(Ix, T_ix), p(Jy, T_jy), p(Ix, T_jy), p(Jy, T_ix)) \leq 0,$$

then there exists $z \in X$ such that $z = Iz = Jz$ and $\{z\} \subset \cap_{n=0}^{\infty} T_nz$.

Proof. Putting $T_i = T_{2n+1}$, $T_j = T_{2n+2}$, $n \in N$ in Theorem 2.2 we obtained the conclusion of Theorem 2.3.

Remark 2.4. *(i). Putting $I = J =$ identity in Theorem 2.2.we have [8, Theorem 2.6].*
(ii). Theorem 2.3 is a generalization of [16, Theorem 1].
(iii). If there is an $F \in \Psi$ such that for each $x, y \in X$,

$$F(\delta(T_1x, T_2y), d(Ix, Jy), p(Ix, T_1x), p(Jy, T_2y), p(Ix, T_2y), p(Jy, T_1x))) \leq 0,$$

then the conclusion of Theorem 2.2 remains valid. This result is considered as a special case of Theorem 2.2 because $D(T_1(x), T_2(y)) \leq \delta(T_1(x), T_2(y))$ [11, page 414].
(iv). Park and Jeong [14, Theorems 3.1 and 3.4] and of Rashwan and Ahmed [15, Theorem 2.1] are special cases of Theorem 2.2.

Remark 2.5. *If there is an $F \in \Psi$ such that, for all $x, y \in X$,*

$$F(\delta(T_ix, T_jy), d(Ix, Jy), p(Ix, T_ix), p(Jy, T_jy), p(Ix, T_jy), p(Jy, T_ix)) \leq 0,$$

for all $n \in N$, then the conclusion of Theorem 2.3 remains valid. This result is a special case of Theorem 2.3 for the same reason as in Remark 2.4 (iii).

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