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THE APPLICATION OF MODIFIED HOMOTOPY ANALYSIS METHOD FOR SOLVING LINEAR AND NON-LINEAR INHOMOGENEOUS KLEIN-GORDON EQUATIONS

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Abstract. Homotopy Analysis method has been applied to solve many functional equations. In this paper, a modified Homotopy Analysis method (mHAM) is presented for solving inhomogeneous linear and nonlinear Klein-Gordon equations. The results reveal that the modified HAM is an effective and convenient method for solving non-linear differential equations. Sometimes, the modified algorithm may give the exact solution for inhomogeneous differential equations by using only two iterations.

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1. Introduction

One of the most important of partial differential equations occurring in applied mathematics is associated with the name of Klein-Gordon. The Klein-Gordon equation plays an important role in mathematical physics such as plasma physics, solid state physics, fluid dynamics and chemical kinetics[1-3]. We consider the Klein-Gordon equation as follows

$$u_{tt} - u_{xx} + N(u(x,t)) = f(x,t), \tag{1}$$

subject to initial conditions

$$u(x,0) = g(x), u_t(x,0) = h(x), \tag{2}$$

where u is a function of x and t, $N(u(x,t))$ is a nonlinear function, and $f(x,t)$ is a known analytic function. There are some methods to obtain approximate solutions of functional equations. One of them is Homotopy Analysis Method. Initially, Homotopy Analysis Method (HAM) proposed by Liao in his Ph.D. thesis [4] which is a powerful method to solve nonlinear problems. In recent years, this method has been successfully employed to solve many types of nonlinear problems in sciences and engineering [5-19]. HAM contains a certain auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. Moreover, by means of the so-called h-curve, a valid region of h can be studied to gain a convergent series solution. More recently, a powerful modification of HAM was proposed in [20-22]. The purpose of the present paper is to apply modified version of HAM to class inhomogeneous Klein-Gordon equations.

2. Basic idea of HAM

To illustrate the basic concept of Homotopy Analysis method, consider the following nonlinear differential equation

$$N[(u(\tau)] = 0, \tag{3}$$

with boundary conditions

$$B(u, \partial u/\partial(n) = 0, \tag{4}$$

where N is a nonlinear operator, τ denotes independent variables, and $u(\tau)$ is an unknown function. By generalizing the traditional Homotopy method, Liao constructs a so- called zero - order deformation equation.

$$(1-p)L[\phi(\tau;p) - u_0(\tau)] = phH(\tau)N[\phi(\tau;p)], \tag{5}$$

where $p[0,1]$ is the embedding parameter, h is a nonzero parameter, $H(\tau)$ is an auxiliary function, L is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of $u(\tau)$, and $\phi(\tau;p)$ is an unknown function. It is important that one has great freedom to choose auxiliary things in HAM. Obviously, when $p = 0$ and $p = 1$ it holds

$$\phi(\tau;0) = u_0(\tau), \phi(\tau;1) = u(\tau).$$

Thus, as p increases from 0 to 1, the solution $\phi(\tau;p)$ varies from initial guesses $u_0(\tau)$ to the solution $u(\tau)$. Expanding in the Taylor series with respect to p, results in

$$\phi(\tau;p) = u_0(\tau) + \sum_{m=0}^{\infty} u_m(\tau)p^m, \tag{6}$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \phi(\tau;p)}{\partial p^m}|_p = 0.$$

If the auxiliary linear operator, the initial guess, and the auxiliary parameter h are so properly chosen, the above series convergent at $p = 1$, then we derive

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau). \tag{7}$$

The vector $\vec{u}$ is defined as follows

$$\vec{u} = \{u_0(\tau), u_1(\tau), u_2(\tau), ...\}.$$

Differentiating Eq. (2), m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, the mth-order deformation is given by

$$L[u_m(\tau) - \chi_m u_{m-1}(\tau)] = hH(\tau)R_m(\vec{u}_{m-1}(\tau)). \tag{8}$$

where

$$R_m(\vec{u}_{m-1}(\tau)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N(\phi(\tau;p))}{\partial p^{m-1}}|_{p=0}, \tag{9}$$

and if $m \leqslant 1$ then $\chi_m = 0$ otherwise, $\chi_m = 1$.

Applying L^{-1} both sides of (8), it can be derived

$$u_m = \chi_m u_{m-1} + hL^{-1}[H(\tau)R_m(u_{m-1})]. \tag{10}$$

This way, it is easy to obtain u_m for $m \geqslant 1$, at Mth-order we have

$$u(\tau) = \sum_{m=0}^{M} u_m(\tau).$$

when $M \to \infty$, an accurate approximation of the original Eq. (3) is obtained.

3. Description of the modified Homotopy Analysis Method

Consider the following nonlinear differential equation

$$N[u(r)] = f(r).$$

The modified form of the HAM can be established based on assumption that the function $f(r)$ can be divided in to several parts, namely,

$$f(r) = \sum_{n=0}^{m} f_n. \tag{11}$$

Then, we can construct the modified mth - order deformation equation,

$$L[u_0(r)] = f_0(r),$$

$$L[u_1(r) - u_0(r)] = h(R_1((u_0) - f_1(r)),$$

$$L[u_m(r) - \chi_m u_{m-1}(r)] = h(R_m(u_{m-1}) - f_m(r)), 2 \leqslant m \leqslant n,$$

$$L[u_m(r) - \chi_m u_{m-1}(r)] = hR_m(u_{m-1}), m > n.$$

By considering $h = -1$, we derive

$$L[u_0(r)] = f_0(r),$$

$$L[u_1(r) - u_0(r)] = -(R_1(u_0) - f_1(r)),$$

$$L[u_m(r) - \chi_m u_{m-1}(r)] = -(R_m(u_{m-1}) - f_m(r)), 2 \leqslant m \leqslant n,$$

$$L[u_m(r) - \chi_m u_{m-1}(r)] = -R_m(u_{m-1}), m > n.$$

Sometimes $f(r)$ may not be finite function. In these cases, Taylor series expansion of $f(r)$ is considered. In this case, we have

$$f(r) = \sum_{n=0}^{\infty} f_n. \tag{12}$$

Then

$$L[u_0(r)] = f_0(r),$$

$$L[u_1(r) - u_0(r)] = -(R_1(u_0) - f_1(r)),$$

$$L[u_m(r) - \chi_m u_{m-1}(r)] = -(R_m(u_{m-1}) - f_m(r)), m \geqslant 2.$$

4. Numerical application

In this section, modified HAM is applied to find appropriate solutions of Klein- Gordon equations. The numerical results are very encouraging.

Example 1
Consider inhomogeneous linear Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 2sin(x), \tag{13}$$

with initial conditions

$$u(x, 0) = sin(x), u_t(x, 0) = 1. \tag{14}$$

The exact solution of (13) is $u = sin(x) + sin(t)$. We choose the linear operator as follows

$$L[\phi(\tau;p)] = \frac{\partial^2\phi(\tau;p)}{\partial t^2},$$

with the property $L[c_0 + c_1 t] = 0$, where c_0 ,c_1 are constants of integration and we define a nonlinear operator as the following form

$$N[\phi(\tau;p)] = \frac{\partial^2\phi(\tau;p)}{\partial t^2} - \frac{\partial^2\phi(\tau;p)}{\partial x^2} + \phi(\tau;p).$$

By taking $f_0(r) = 2sin(x)$, and $f_1(r) = 0$, we derive

$$L[u_1(t) - u_0(t)] = -(N[u_0(t)] - 0),$$

$$L[u_m(t) - u_{m-1}(t)] = -(N[u_{m-1}(t)]), m \geqslant 2.$$

That is

$$u_{0_{tt}} = 2sinx, u_0(x,0) = sin(x), u_{0_t}(x,0) = 1,$$

$$u_{1_{tt}} - u_{0_{tt}} = -(u_{0_{tt}} - u_{0_{xx}} + u_0 - 0), u_1(x,0) = 0, u_{1_t}(x,0) = 0,$$

$$(u_m)_{tt} - (u_{m-1})_{tt} = -((u_{m-1})_{tt} - (u_{m-1})_{xx} + u_{m-1}), u_{m-1}(x,0) = 0, u_{m-1_t}(x,0) = 0.$$

So

$$u_0(x,t) = sin(x) + t + t^2 sin(x),$$

$$u_1(x,t) = \frac{-1}{6} sin(x)t^4 - \frac{1}{6}t^3 - t^2 sin(x),$$

$$u_2(x,t) = \frac{1}{90} sin(x)t^6 + \frac{1}{120}t^5 + \frac{1}{6}t^4 sin(x).$$

Now the 3-term approximate solution can be obtained as follows

$$u_0 + u_1 + u_2 = sin(x) + t - \frac{1}{6}t^3 + \frac{1}{90} sin(x)t^6 + \frac{1}{120}t^5.$$

If we choose $f_0(r) = 0$ and $f_1(r) = 2sin(x)$, then we get

$$L[u_0(t)] = 0,$$

$$L[u_1(t) - u_0(t)] = -(N[u_0(t)] - 2sin(x)),$$

$$L[u_m(t) - u_{m-1}(t)] = -(N[u_{m-1}(t)]), m \geqslant 2.$$

So

$$u_{0_{tt}} = 0, u_0(x,0) = sin(x), u_{0_t}(x,0) = 1,$$

$$u_{1_{tt}} - u_{0_{tt}} = -(u_{0_{tt}} - u_{0_{xx}} + u_0 - 0), u_1(x,0) = 0, u_{1_t}(x,0) = 0,$$

$$(u_m)_{tt} - (u_{(m-1)})_{tt} = -((u_{(m-1)})_{tt} - (u_{(m-1)})_{xx} + u_{m-1}), u_{m-1}(x,0) = 0, (u_{m-1})_t(x,0) = 0, m \geqslant 2.$$

Therefore, the following results will be obtained

$$u_0(x,t) = sin(x) + t,$$

$$u_1(x,t) = \frac{-1}{6}t^3,$$

$$u_2(x,t) = \frac{1}{120}t^5,$$

$$u_3(x,t) = \frac{-1}{5040}t^7.$$

Hence, the series solution of (13) is

$$u(x,t) = sin(x) + (t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 + ...) = sin(x) + sin(t),$$

which is exact solution.

Example 2

Consider inhomogeneous non-linear Klein-Gordon equation as follows

$$u_{tt} - u_{xx} + u^2 = 2x^2 - 2t^2 + x^4t^4, \tag{15}$$

with initial conditions

$$u(x,0) = 0, u_t(x,0) = 0.$$

The exact solution of Eq. (15) is $u(x,t) = x^2t^2$.
The linear operator L and nonlinear operator N are selected similar to Example 1.
By considering $f_0(r) = 2x^2 - 2t^2 + x^4t^4$ and $f_1(r) = 0$, we get

$$L[u_0(t)] = 2x^2 - 2t^2 + x^4t^4,$$

$$L[u_1(t) - u_0(t)] = -(N[u_0(t)] - 0),$$

$$L[u_m(t) - u_{m-1}(t)] = -(N[u_{m-1}(t)]), m \geqslant 2.$$

So

$$u_0(x,t) = x^2t^2 - \frac{t^4}{6} + \frac{x^4t^6}{30},$$

$$u_1(x,t) = -\frac{1}{163800}t^14x^8 + \frac{1}{11880}t^12x^4 - \frac{1}{1350}t^10 + \frac{11}{840}t^8x^2 - \frac{x^4t^6}{30} + \frac{t^4}{6}.$$

Now the 2-term approximate solution,will be derived as follows

$$u_0 + u_1 = -\frac{1}{163800}t^14x^8 + \frac{1}{11880}t^12x^4 - \frac{1}{1350}t^10 + \frac{11}{840}t^8x^2 + x^2t^2.$$

If we choose $f_0(r) = 2x^2$ and $f_1(r) = -2t^2 + x^4t^4$, then

$$u_0(x,t) = x^2t^2,$$

$$u_1(x,t) = 0,$$

$$u_k(x,t) = 0, k \geqslant 1.$$

Hence, the series solution of (15) is

$$u(x,t) = x^2t^2.$$

Considering $f_0(r) = 2x^2$, $f_1(r) = -2t^2$, and $f_2(r) = x^4t^4$, result in

$$u_0(x,t) = x^2t^2,$$

$$u_1(x,t) = -\frac{x^4t^6}{30},$$

$$u_2(x,t) = +\frac{x^4t^6}{30}.$$

So, the series solution of Eq. (15) will be obtained as follows

$$u(x,t) = x^2t^2.$$

Example 3

Consider inhomogeneous non- linear Klein-Gordon equation

$$u_{tt} - u_{xx} + u^2 = -xcost + x^2cos^2(t), \tag{16}$$

with initial conditions

$$u(x,0) = x, u_t(x,0) = 0.$$

The exact solution of Eq. (16) is $u(x,t) = xcost$. Let's consider $f_0(r) = -xcost$ and $f_1(r) = x^2cos^2(t)$.

So we have

$$L[u_0(t)] = -xcost,$$

$$L[u_1(t) - u_0(t)] = -(N[u_0(t)] - x^2 cos^2(t)),$$
$$L[u_m(t) - u_(m-1)(t)] = -(N[u_(m-1)(t)]), m \geqslant 2.$$

Consequently, solving the above equations, the first few components of the HAM are derived as follows

$$u_0(x,t) = xcos(t),$$
$$u_1(x,t) = 0,$$
$$u_2(x,t) = 0,$$

$$u_k(x,t) = 0, k \geqslant 1.$$

Therefore, the exact solution of Eq. (16) can be obtained as follows

$$u(x,t) = xcos(t).$$

5. Conclusion

In this paper, the modified HAM was applied to solve linear and nonlinear inhomogeneous Klein-Gordon Equations. The main advantage of the modified HAM is that we can accelerate the convergence rate, minimize iterative times, accordingly save computation time and promote the efficiency, if we choose the proper decomposition for the inhomogeneous term. The obtained results suggest that this technique introduces a powerful improvement for solving non-homogeneous differential equations.

References

[1] Debtnath, *Nonlinear Partial Differential Equations for Scientist and Engineers*, Birkhauser, Boston, (1997).

[2] P.G. Drazin, R.S. Johnson, *Solitons: An Introduction* , Cambridge University Press, (1983).

[3] M.J. Ablowitz, P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform*, Cambridge University Press, Cambridge, (1990).

[4] S.J. Liao, *the proposed homotopy analysis technique for the solution of nonlinear problems*, Ph.D. Thesis, Shanghai Jiao Tong University, (1992).

[5] S. Abbasbandy, *the application of homotopy analysis method to nonlinear equations arising in heat transfer*, Physics Letters A, 360 (2006) 109-113.

[6] S. Abbasbandy, A. Shirzadi , *Homotopy analysis method for multiple solutions of the fractional Sturm-Liouville problems*, Numerical Algorithms, 54 (2010) 521-532.

[7] S. Abbasbandy, M. Ashtiani,E. Babolian, *Analytic solution of the Sharma-Tasso-Olver equation by homotopy analysis method*, Zeitschrift fr Naturforschung, 65a (2010) 285-290.

[8] AK. Alomari, MSM. Noorani, R. Nazar, *Explicit series solutions of some linear and nonlinear Schrodinger equations via the homotopy analysis method*, Communications in Nonlinear Science and Numerical Simulation, 14 (2009) 1196-1207.

[9] M. Ayub, A. Rasheed, T. Hayat, *Exact flow of a third grade fluid past a porous plate using homotopy analysis method*, International Journal of Engineering Science, 41 (2003)2091-2103.

[10] AS. Bataineh, MSM. Noorani, I. Hashim, *Solving systems of ODEs by homotopy analysis method*, Communications in Nonlinear Science and Numerical Simulation, 13 (2008) 2060-2070.

[11] AS. Bataineh, MSM. Noorani, I. Hashim, *Homotopy analysis method for singular IVPs of EmdenFowler type*, Communications in Nonlinear Science and Numerical Simulation, 14 (2009) 1121-1131.

[12] SJ. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, Chapman , Hall/CRC: Boca Raton, FL, (2004).

[13] SJ. Liao, *An approximate solution technique which does not depend upon small parameters: a special example*, International Journal of Nonlinear Mechanics, 30 (1995) 371-380.

[14] SJ. Liao, *on the homotopy analysis method for nonlinear problems*, Applied Mathematics and Computation, 147 (2004) 499-513.

[15] SJ. Liao, *Comparison between the homotopy analysis method and homotopy perturbation method*, Applied Mathematics and Computation, 169 (2005) 1186-1194.

[16] A. Molabahrami, F. Khani, *The homotopy analysis method to solve the BurgersHuxley equation*, Nonlinear Analysis: Real World Applications, 10 (2009) 589-600.

[17] S. Nadeem, NS. Akbar, *Peristaltic flow of Sisko fluid in a uniform inclined tube*, Acta Mechanica Sinica, 26 (2010) 675-683.

[18] S. Nadeem, NS. Akbar, *Influence of heat transfer on a peristaltic flow of Johnson Segalman fluid in a non uniformtube*, International Communications in Heat and Mass Transfer, 36 (2009) 1050-1059.

[19] Y. Tan, S. Abbasbandy, *Homotopy analysis method for quadratic Riccati differential equation*, Communications in Nonlinear Science and Numerical Simulation , 13 (2008) 539-546.

[20] A. Sami Bataineh, MSM. Noorani, I. Hashim, *Approximate solutions of singular two- point BVPs by modified homotopy analysis method*, Physics Letters A, 372 (2008) 4062-4066.

[21] A. Sami Bataineh, MSM. Noorani, I. Hashim, *On a new reliable modification of homotopy analysis method*, Communications in Nonlinear Science and Numerical Simulation, 14(2009)409-423.

[22] A. Sami Bataineh, MSM. Noorani, I. Hashim, *Modified homotopy analysis method for solving systems of second-order BVPs*, Communications in Nonlinear Science and Numerical Simulation 14 (2009)430-442.

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