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# DEFORMATION OF AN LSP-SASAKIAN MANIFOLD

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Abstract. In this paper we shall show LSP Sasakian manifold is invariant under some deformation. Also we shall discuss some properties on LSP Sasakian manifold with the deformation and the behaviour of the Nijenhuis tensor on LSP Sasakian manifold with respect to the same deformation. We shall show that a 4-dimensional LSP Sasakian manifold is invariant under the same deformation.

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## 1. Introduction

An $n$-dimensional differentiable manifold $M$ is said to a Lorentzian Para-Sasakian (LP-Sasakian) manifold if it admits a (1,1)-tensor field $\phi$, a vector field $\xi$ and 1-form $\eta$ and a Lorentzian metric $g$ which satisfies the following relation

$$\phi^2(X) = X + \eta(X)\xi, \tag{1}$$

$$\eta(\xi) = -1, \tag{2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{3}$$

$$g(X, \xi) = \eta(X), \tag{4}$$

$$(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{5}$$

for arbitrary vector fields $X$ and $Y$, where $\nabla$ denotes the covariant differentiation with respect to $g$ [2], [3], [4].

An LP-Sasakian manifold $M$ is said to be Lorentzian Special Para-Sasakian(LSP-Sasakian) manifold if it satisfies

$$F(X, Y) = g(X, Y) + \eta(X)\eta(Y), \tag{6}$$

where $F(X,Y) = g(\phi X, Y)$ is a symmetric (0,2) tensor [1].
It is easily seen that

$$F(X,Y) = g(\phi X, Y) = g(X, \phi Y) = g(\phi X, \phi Y) = F(\phi X, \phi Y). \tag{7}$$

In LSP-Sasakian manifold it can be easily shown that

$$\phi\xi = 0, \tag{8}$$

$$\eta(\phi X) = 0, \tag{9}$$

$$rank(\phi) = n - 1 \tag{10}$$

and

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{11}$$

From (5), it can be shown that,

$$\nabla_X \xi = \phi(X). \tag{12}$$

Let $(M, \phi, \xi, \eta, g)$ be an LSP-Sasakian manifold and $\mu$ be an automorphism, where $\mu = a\xi$ for some real $a$ such that $1 + a > 0$. $D$-deformation is defined in [5], as $D$ be the distribution defined by $\eta = 0$ along with $\mu = a\xi$.

## 2. Some results on LSP-Sasakian manifold under $D$-deformation

In this section we prove:

**Theorem 1.** *In an LSP-Sasakian manifold $(M, \phi, \xi, \eta, g)$ the following relations hold:*

$$L_\mu g = 2aF(X,Y), \tag{13}$$

$$[\mu, \xi] = 0, \tag{14}$$

$$1 - \eta(\mu) > 0, \tag{15}$$

$$F(\xi, \mu) = 2a \tag{16}$$

*and*

$$R(\xi, \mu)\xi = 0, \tag{17}$$

*where $L_\mu$ is the Lie differentiation with respect to $\mu$.*

*Proof.* $(M, \phi, \xi, \eta, g)$ is an LSP-Sasakian manifold and $\mu$ be a vector field over $M$, where $\mu = a\xi$.
Then $(L_\mu g)(X, Y) = a\xi g(X, Y) - g([a\xi, X], Y) - g(X, [a\xi, Y])$.
Again $T(\xi, X) = \nabla_\xi X - \nabla_X \xi - [\xi, X] = 0$ implies

$$[\xi, X] = \nabla_\xi X - \nabla_X \xi. \tag{18}$$

So, $L_\mu g(X, Y) = a(\nabla_\xi g)(X, Y) + a[g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)]$.
Using (7) and (12) we get $L_\mu g = 2aF(X, Y)$.

Next, $[\mu, \xi] = [a\xi, \xi] = a[\xi, \xi] = 0$.
Also $1 - \eta(\mu) = 1 - \eta(a\xi) = 1 - a\eta(\xi) = 1 + a > 0$.
$R(\xi, \mu)\xi = \eta(\mu)\xi - \eta(\xi)\mu = -a\xi + a\xi = 0$
and
$F(\xi, \mu) = g(\xi, \mu) + \eta(\xi)\eta(\mu) = a + a = 2a$.

Let us consider the structure after $D$-deformation denoted by $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ are defined by

$$\tilde{\phi}(X) = \phi(X + \tilde{\eta}(X)\tilde{\xi}), \tag{19}$$

$$\tilde{\eta} = (1 - \eta(\mu))^{-1}\eta, \tag{20}$$

$$\tilde{\xi} = \xi + \mu, \tag{21}$$

$$\tilde{g}(X, Y) = (1 - \eta(\mu))^{-1} g(X + \tilde{\eta}(X)\tilde{\xi}, Y + \tilde{\eta}(Y)\tilde{\xi}) - \tilde{\eta}(X)\tilde{\eta}(Y) \tag{22}$$

and

$$\tilde{F}(X, Y) = (1 - \eta(\mu))^{-1} F(X + \tilde{\eta}(X)\tilde{\xi}, Y + \tilde{\eta}(Y)\tilde{\xi}) - \tilde{\eta}(X)\tilde{\eta}(Y), \tag{23}$$

where $X$ and $Y$ are vector fields over $M$.

**Theorem 2.** *Let $D$ be the deformation on an LSP-Sasakian manifold $(M, \phi, \xi, \eta, g)$, then $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is also an LSP-Sasakian manifold.*

*Proof.* Since $D$ is the distribution defined by $\eta = 0$, i.e., for any $X \in D$, $\eta(X) = 0$. Now we shall show the properties of LSP-Sasakian manifold with respect to the deformation.
Using (20) and (21), we get
$\tilde{\eta}(\tilde{\xi}) = (1 - \eta(\mu))^{-1}\eta(\xi + \mu)$
$\quad = -(1 - \eta(\mu))^{-1}(1 - \eta(\mu))$, since $\eta(\xi) = -1$.
or,

$$\tilde{\eta}(\tilde{\xi}) = -1. \tag{24}$$

Next using (19) and (24), we obtain

$$\tilde{\phi}(\tilde{\xi}) = \phi(\tilde{\xi} + \tilde{\eta}(\tilde{\xi})\tilde{\xi}) = \phi(\tilde{\xi} - \tilde{\xi}) = 0$$

i.e,

$$\tilde{\phi}(\tilde{\xi}) = 0. \tag{25}$$

Let $X$ be a vector field, which belongs to $D$, then from (20) we get

$$\tilde{\eta}(X) = 0. \tag{26}$$

Using (26) we have from above definitions (19), (20), (21) and (22)

$$\tilde{\phi}(X) = \phi(X), \tag{27}$$

$$\tilde{\eta} = (1 - \eta(\mu))^{-1}\eta, \tag{28}$$

$$\tilde{\xi} = \xi + \mu \tag{29}$$

and

$$\tilde{g}(X, Y) = (1 - \eta(\mu))^{-1} g(X, Y). \tag{30}$$

Now using (27) and (28), we get

$$\tilde{\eta}(\tilde{\phi}(X)) = 0, \tag{31}$$

using (1), (27) and $\eta = 0$ we have
$\tilde{\phi}^2(X) = \phi^2(X) = X$.
Thus using (26) we get

$$\tilde{\phi}^2(X) = X + \tilde{\eta}(X)\tilde{\xi}. \tag{32}$$

Replacing $X$ by $\phi(X)$, $Y$ by $\phi(Y)$ in (22) and using (26) and (27) we have
$\tilde{g}(\tilde{\phi}X, \tilde{\phi}Y) = (1 - \eta(\mu))^{-1} g(\phi X, \phi Y)$.
Since $\eta = 0$, using (3) and (26), we get

$$\tilde{g}(\tilde{\phi}X, \tilde{\phi}Y) = \tilde{g}(X, Y) + \tilde{\eta}(X)\tilde{\eta}(Y). \tag{33}$$

Replacing $Y$ by $\tilde{\xi}$ in (22) and using (24) we obtain

$$\tilde{g}(X, \tilde{\xi}) = \tilde{\eta}(X). \tag{34}$$

Now, $\tilde{g}(\tilde{\phi}X, Y) = (1 - \eta(\mu))^{-1} g(\tilde{\phi}X + \tilde{\eta}(\tilde{\phi}X)\tilde{\xi}, Y + \tilde{\eta}(Y)\tilde{\xi})$.
From (27), (28) and (32) it can be easily seen that,
$\tilde{g}(\tilde{\phi}X, Y) = \tilde{g}(X, \tilde{\phi}Y)$.
Since $\eta = 0$, from (6), (23) and (26), we have

$$\tilde{F}(X, Y) = (1 - \eta(\mu))^{-1} g(X, Y). \tag{35}$$

By (26), (30) and (35) we have

$$\tilde{F}(X, Y) = \tilde{g}(X, Y) + \tilde{\eta}(X)\tilde{\eta}(Y). \tag{36}$$

For $X,\ Y \in D$ in (5) we get,

$$(\nabla_X \phi)Y = g(X,Y)\xi. \tag{37}$$

$$\begin{aligned}(\nabla_X \tilde{\phi})Y &= \nabla_X \tilde{\phi}(Y) - \tilde{\phi}(\nabla_X Y),\\ &= \nabla_X \phi(Y) - \phi(\nabla_X Y),\end{aligned}$$

$$(\nabla_X \tilde{\phi})Y = (\nabla_X \phi)Y. \tag{38}$$

Now, $\quad \tilde{g}(X,Y)\tilde{\xi} + \tilde{\eta}(Y)X + 2\tilde{\eta}(X)\tilde{\eta}(Y)\tilde{\xi}$
$= \tilde{g}(X,Y)\tilde{\xi}$
$= g(X,Y)\xi$, using (26), (29) and (30),

or,

$$(\nabla_X \tilde{\phi})Y = \tilde{g}(X,Y)\tilde{\xi} + \tilde{\eta}(Y)X + 2\tilde{\eta}(X)\tilde{\eta}(Y)\tilde{\xi}. \tag{39}$$

Hence (24), (25), (26), (31), (32), (33), (34),(36) and (39) shows that $(M, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g}, \tilde{F})$ is an LSP-Sasakian manifold.

## 3. The Nijenhuis tensor in the LSP-Sasakian manifold with respect to the $D$-deformation

The Nijenhuis tensor in the LSP-Sasakian manifold is defined by

$$N(X,Y) = [X,Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + [\phi X, \phi Y] + \{X\eta(Y) - Y\eta(X)\}\xi, \tag{40}$$

where $X, Y \in \chi(M)$ [5].

**Theorem 3.** *The Nijenhuis tensor in LSP-Sasakian manifold is invariant with respect to $D$ deformation.*

*Proof.* Let $X$ and $Y$ be the vector fields in $D$.
Then $\eta(X) = \eta(Y) = 0$. Also $\tilde{\eta}(X) = \tilde{\eta}(Y) = 0$.
For $X, Y \in D$ in (40), we get,
$N(X,Y) = [X,Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + [\phi X, \phi Y]$.
Under the $D$-deformation the Nijenhuis tensor becomes
$\tilde{N}(X,Y) = [X,Y] - \tilde{\phi}[\tilde{\phi}X, Y] - \tilde{\phi}[X, \tilde{\phi}Y] + [\tilde{\phi}X, \tilde{\phi}Y]$
$= [X,Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + [\phi X, \phi Y]$, using (27)
or, $\tilde{N}(X,Y) = N(X,Y)$.

## 4. Example of a 4-dimensional LSP-Sasakian manifold which remains invariant under $D$-deformation

We consider a 4−dimensional real manifold $M = \{(w, x, y, z) : w, x, y, z \in \mathbf{R}\}$. Let $\{E_1, E_2, E_3, E_4\}$ be a basis of $M$, where
$E_1 = e^{-z}\frac{\partial}{\partial w}$ $\quad E_2 = e^{-z}\frac{\partial}{\partial x}$ $\quad E_3 = e^{-z}\frac{\partial}{\partial y}$ $\quad E_4 = \frac{\partial}{\partial z}$

The Lorentzian metric is defined by
$g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_4) = g(E_2, E_4) = g(E_3, E_4) = g(E_1, E_2) = 0$,
$g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1$ and $g(E_4, E_4) = -1$
Let $\eta$ be the 1-form defined by $\eta(U) = g(U, E_4)$ for any $U \in \chi(M)$. Let $\phi$ be the $(1, 1)$ tensor field defined by
$\phi(E_1) = E_1, \phi(E_2) = E_2, \phi(E_3) = E_3, \phi(E_4) = 0$.
Then using the definition of $\phi$ and $g$ we have
$\eta(E_4) = -1, \phi^2(X) = X + \eta(X)E_4$ and
$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$ for any $X, Y \in \chi(M)$. Thus for $E_4 = \xi$, $(\phi, \xi, \eta, g)$ defines a Lorentzian paracontact structure on $M$.
Let $\nabla$ be the Levi-Civita connection with respect to the Lorentzian metric $g$ and $R$ be the curvature tensor of $g$. Then we have
$[E_i, E_4] = E_i, \forall i = 1, 2, 3,$ $\quad [E_i, E_j] = 0, \forall i, j = 1, 2, 3,$ and $[E_4, E_4] = 0$.
Let $X = a_1E_1 + a_2E_2 + a_3E_3 + a_4E_4$ and
$Y = b_1E_1 + b_2E_2 + b_3E_3 + b_4E_4$.
Then
$\phi(X) = a_1E_1 + a_2E_2 + a_3E_3$, $\phi(Y) = b_1E_1 + b_2E_2 + b_3E_3$, $\eta(X) = -a_4$ and $\eta(Y) = -b_4$.
Now $g(\phi X, \phi Y) = a_1b_1 + a_2b_2 + a_3b_3 = g(\phi X, Y) = g(X, \phi Y)$.
Also $g(X, Y) + \eta(X)\eta(Y) = (a_1b_1 + a_2b_2 + a_3b_3 - a_4b_4) + a_4b_4 = a_1b_1 + a_2b_2 + a_3b_3$.
Thus $F(X, Y) = g(X, Y) + \eta(X)\eta(Y)$, where $F(X, Y) = g(\phi X, Y)$.
From the above it is easy to see that $(\phi, \xi, \eta, g)$ is an LSP-Sasakian structure on $M$. Consequently $M^4(\phi, \xi, \eta, g)$ is an LSP-Sasakian manifold.

Now under the deformation $D$,
$\tilde{\phi}E_1 = \phi(E_1 + \tilde{\eta}(E_1)\tilde{\xi})$
using (14), (15) and since $1 + a > 0$ we obtain
$\tilde{\phi}E_1 = \phi(E_1 + \eta(E_1)\xi)$
and as $\eta(U) = g(U, E_4)$ we get
$\tilde{\phi}E_1 = \phi(E_1) = E_1$, [since $g(E_1, E_4) = 0$].
Similarly we have
$\tilde{\phi}E_2 = E_2$, $\tilde{\phi}E_3 = E_3$ and $\tilde{\phi}E_4 = 0$.

So under $D$-deformation LSP-Sasakian structure on $M$ remains invariant.

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