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THE APPROXIMATE SOLUTIONS OF TIME-FRACTIONAL DIFFUSION EQUATION BY USING CRANK NICHOLSON METHOD

H. Bulut, S. Tuluce Demiray, M. Kayhan

Abstract. In this study, we consider approximate solution of Time-Fractional Diffusion Equation (TFDE) by using Crank-Nicholson Method. Besides, we utilize property of Riemann-Liouville derivative to obtain this solution. Then, we draw three dimensional graphics of this solution by means of programming language Mapple. Finally, we show table including error analysis for some values of α, x, t, M and N. Numerical results ensure to illustrate the effectiveness and reliability of this method.

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Keywords: Crank-Nicholson Method, Time-fractional Diffusion Equation, Riemann-Liouville.

1. Introduction

The exploration of solutions of nonlinear fractional differantial equations has a very important role in several sciences such as biology, system identification, physics, viscoelasticity, signal processing, probability and statistics, mechanical engineering, hydrodynamics, chemistry, solid state physics, finance, optical fibers, fluid mechanics, electric control theory, thermodynamics, heat transfer and fractional dynamics [1, 2]. In recent years, most authors have improved a lot of methods to find solutions of fractional differantial equations such as variational iteration method [3], homotopy decomposition method (HDM) [4], generalized Kudryashov method [5, 6], the modified Gauss elimination method [7], the Sinc-Legendre collocation method [9].

Time-fractional diffusion equation recently takes attention because it is a highly beneficial tool to identify problems involving non-Markovian random walks. This type of equation is procured from standard diffusion equation by substituting the first-order time derivative with a fractional derivative of α. The diffusion equation defines the propagation of particles from a region of higher concentration to a region of lower concentration due to collisions of the molecules and Brownian motion. While time-fractional diffusion equation is a generalization of the classical diffusion equation, which is procured from standard diffusion equation by substituting the first-order time derivative with a fractional derivative of α. It can be utilized to treat sub-diffusive flow process, in which the net motion of the particles happens more slowly than Brownian motion [12].

The development of numerical methods seems to be very substantial and necessary for solving fractional differential equations. Many authors have used to find solutions a lot of methods of time-fractional diffusion equations such as the modified Gauss elimination method [7], the Sinc-Legendre collocation method [9], Kansa method [10], Galerkin spectral method and Legendre collocation method [11], Von Neumann method [12], AOR method [13], Chebyshev collocation method [14], optimal homotpy analysis method [15], implicit finite difference approximation [16], regularization technique [17], the iterated Brownian motion [18], Green functions [19], semi-discrete finite element method [20], the backward problem [21], probability distributions [22], Wright functions [23], the methods of seperation of variable and Laplace transform [24], variational iteration method [25], and many more [26, 27].

In this paper, our aim is to obtain approximate solutions time-fractional diffusion equations by using Crank Nicholson method and compare analytical and approximate solutions. In Sec. 2, we give discrete approximation of fractional derivative. In Sec. 3, we present the fundamentals of Crank-Nicholson method for fractional order diffusion equation. In Sec. 4, as an application, we introduce numerical analysis of time-fractional diffusion equation by using Crank-Nicholson method. Also, we draw three dimensional graphics of approximate solutions that is obtained in this paper and give error analysis for different values of α.

2. Discrete Approximation of Fractional Derivative

For positive integers M and N, the grid magnitudes in space and time for finite difference algorithm are described by $h = 1/M$ and $k = 1/N$, consecutively. The grid points in the space interval [0,1] are the numbers $x_i = ih,\ \ i = 0, 1, 2, ..., M$, and the grid points in the time interval [0,1] are demonstrated $t_n = nk,\ \ i = 0, 1, 2, ..., N$. The values of the functions U and f at the grid points are indicated $U_i^n = U(t_n, x_i)$ and $f_i^n = f(t_n, x_i)$,consecutively.

Such as Crank Nicholson difference scheme, we will take from Ref.[8] a discrete approximation to the fractional derivative $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ at $(t_{n+\frac{1}{2}}, x_i)$. If we take $R(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u(s,x)}{(t-s)^\alpha} ds$, we obtain

$$\frac{\partial^\alpha U(t_{n+\frac{1}{2}}, x_i)}{\partial t^\alpha} = \frac{\partial}{\partial t} R\Big(t_{n+\frac{1}{2}}, x_i\Big) = \frac{R(t_{n+1}, x_i) - R(t_n, x_i)}{k} + O(k^2). \tag{1}$$

From here, approximations for $R(t_{n+1}, x_i)$ and $R(t_n, x_i)$ are obtained as following

$$\begin{aligned} R(t_{n+1}, x_i) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{n+1}} \frac{u(s,x)}{(t_{n+1}-s)^\alpha} ds \\ &= k \sum_{j=0}^{n} (a_j - j b_j) U_i^{n-j} - k \sum_{j=0}^{n} (a_j - (j+1) b_j) U_i^{n-j+1}, \end{aligned} \tag{2}$$

$$\begin{aligned} R(t_n, x_i) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{u(s,x)}{(t_n-s)^\alpha} ds \\ &= k \sum_{j=1}^{n} (a_{j-1} - (j-1) b_{j-1}) U_i^{n-j} - k \sum_{j=1}^{n} (a_{j-1} - (j) b_{j-1}) U_i^{n-j+1}, \end{aligned} \tag{3}$$

where $a_j = \frac{k^{-\alpha}}{(2-\alpha)\Gamma(1-\alpha)}[(j+1)^{2-\alpha} - j^{2-\alpha}]$ and $b_j = \frac{k^{-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}[(j+1)^{1-\alpha} - j^{1-\alpha}]$.

Consequently, we have attained the following approximation

$$\frac{\partial^\alpha U(t_{n+\frac{1}{2}}, x_i)}{\partial t^\alpha} \cong \sum_{j=0}^{n+1} w_{n,j} U_i^{n-j+1} \tag{4}$$

where

$$w_{n,0} = b_0 - a_0,$$

$$w_{n,1} = \begin{cases} 3a_0 - a_1 + 2b_1 - b_0, & \text{if } n = 0 \\ 2a_0 - a_1 + 2b_1 - b_0, & \text{if } n > 0 \end{cases},$$

$$w_{n,j} = \begin{cases} -a_{j-2} + 2a_{j-1} - a_j + (j-2)b_{j-2} - (2j-1)b_{j-1} + (j+1)b_j, & \text{if } j = 2, ..., n \\ a_n - a_{n-1} + (n-1)b_{n-1} - n b_n, & \text{if } j = n+1 \end{cases}.$$

Additionally, we get from Ref. [8]

$$\frac{\partial^2 U(t_{n+\frac{1}{2}}, x_i)}{\partial x^2} = \frac{1}{2}\Big[\frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{h^2} + \frac{U_{i+1}^{n} - 2U_i^{n} + U_{i-1}^{n}}{h^2}\Big] + O(h^2). \tag{5}$$

3. Crank-Nicholson Method for Fractional Order Diffusion Equation

We consider the following diffusion equation,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t). \tag{6}$$

Taking into consideration $0 \leq n \leq N-1, 1 \leq i \leq M-1$, we obtain the following equation by substituting Eqs. (4) and (5) into Eq. (6),

$$\sum_{j=0}^{n+1} w_{n,j} U_i^{n+1-j} - \Big[\frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{2h^2} + \frac{U_{i+1}^{n} - 2U_i^{n} + U_{i-1}^{n}}{2h^2}\Big] = f\Big(t_n + \frac{k}{2}, x_i\Big), \tag{7}$$

which are

$$U_i^0 = r(x_i),\ 1 \leq i \leq M-1 \quad , \quad U_0^n = U_M^n = 0,\ 0 \leq n \leq N.$$

Then regulating this system, we get

$$f\Big(t_n + \frac{k}{2}, x_i\Big) = \Big(-\frac{1}{2h^2}\Big) U_{i+1}^{n+1} + U_{i+1}^{n} + \sum_{j=0}^{n+1} w_{n,j} U_i^{n+1-j} + \frac{1}{h^2} U_i^{n+1} + \frac{1}{h^2} U_i^{n} + \Big(-\frac{1}{2h^2}\Big) U_{i-1}^{n+1} + U_{i-1}^{n}, \tag{8}$$

which are

$$U_i^0 = r(x_i),\ \ 1 \leq i \leq M-1 \quad , \quad U_0^n = U_M^n = 0,\ \ 0 \leq n \leq N.$$

It can be shown matrix form as following

$$DU_{i+1} + EU_i + FU_{i-1} = \varphi_i \tag{9}$$

where $\varphi_i = \Big[\varphi_i^0, \varphi_i^1, \varphi_i^2, ..., \varphi_i^N\Big]^T, \varphi_i^0 = r(x_i), \varphi_i^n = f\Big(t_{n+\frac{1}{2}}, x_i\Big),\ 1 \leq n \leq N,\ 1 \leq i \leq M$ and $U_i = \Big[U_i^0 U_i^1, U_i^2, ..., U_i^N\Big]^T$.

Dimension of D,E and F matrices is $(N+1) \times (N+1)$ and these matrices can be shown as following

$$D = \Big(-\tfrac{1}{2h^2}\Big)\begin{pmatrix} 0 & & & & \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \end{pmatrix}, F = \Big(-\tfrac{1}{2h^2}\Big)\begin{pmatrix} 0 & & & & \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \end{pmatrix},$$

$$E = \begin{pmatrix} 1 & & & & & \\ w_{0,1}+\frac{1}{h^2} & w_{.,0}+\frac{1}{h^2} & & & & \\ w_{1,2} & w_{.,1}+\frac{1}{h^2} & w_{.,0}+\frac{1}{h^2} & & & \\ w_{2,3} & w_{.,2} & w_{.,1}+\frac{1}{h^2} & w_{.,0}+\frac{1}{h^2} & & \\ \vdots & \ddots & \ddots & \ddots & & \\ & & & & & \\ w_{N-1,N}+ & w_{.,N-1} & \cdots & w_{.,2} & w_{0,1}+\frac{1}{h^2} & w_{.,0}+\frac{1}{h^2} \end{pmatrix}.$$

Using Gauss-eliminate method, Eq. (9) transforms as following

$$U_i = \alpha_{i+1}U_{i+1} + \beta_{i+1} \quad , \quad i = M-1, ..., 2, 1, 0. \tag{10}$$

In an attempt to determine α_{i+1} and β_{i+1} matrices, we can choose $\alpha_1 = O_{(N+1)\times(N+1)}$ and $\beta_1 = O_{(N+1)\times(N+1)}$ for $U_0 = \alpha_1 U_1 + \beta_1$. Substituting $U_i = \alpha_{i+1}U_{i+1} + \beta_{i+1}$ and $U_{i-1} = \alpha_i U_i + \beta_i$ into Eq.(8), we obtain as following

$$(D + E\alpha_{i+1} + F\alpha_i\alpha_{i+1})U_{i+1} + (E\beta_{i+1} + F\alpha_i\beta_{i+1} + F\beta_i) = \varphi_i \tag{11}$$

$$D + E\alpha_{i+1} + F\alpha_i\alpha_{i+1} = 0, \quad E\beta_{i+1} + F\alpha_i\beta_{i+1} + F\beta_i = \varphi_i, \quad 1 \le i \le M-1. \tag{12}$$

Finally, from this system we find

$$\alpha_{i+1} = -(E + F\alpha_i)^{-1}D, \quad \beta_{i+1} = (E + F\alpha_i)^{-1}(\varphi - \beta_i), \quad 1 \le i \le M-1. \tag{13}$$

4. Numerical Analysis

4.1. Example

We consider time-fractional diffusion equation [10]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) = f(x,t), \ 0 \le x \le 2, \ t \ge 0, \ 0 < \alpha < 1 \tag{14}$$

where initial condition is $u(x,0) = 0, 0 \le x \le 2$, boundary conditions are $u(0,t) = u(2,t) = 0, \ t \ge 0$ and $f(x,t) = \frac{2}{\Gamma(3-\alpha)}x(2-x)t^{2-\alpha} + t^2x(2-x) + 2t^2$. The exact solution of Eq.(14) is $u(x,t) = t^2x(2-x)$ [10]. We draw graphics of numerical solution which be calculated by means of programming language Mapple and give error analysis for different values of α in Figure 1, Figure 2, Figure 3 and Table 1.

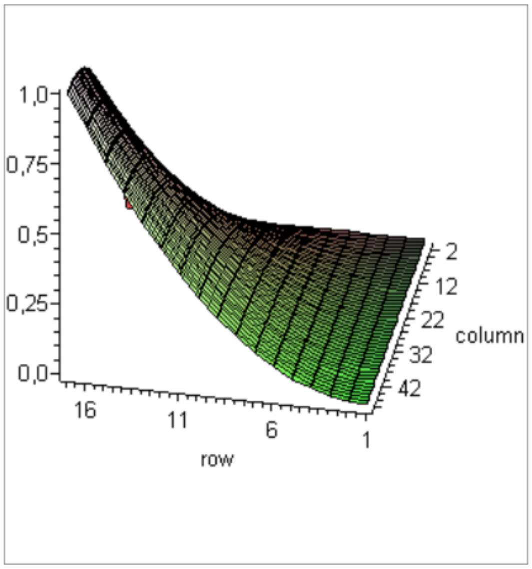

Figure 1: Three dimensional graphic of Eq. (14) for $M = 48, N = 16$, and $\alpha = 0.2$.

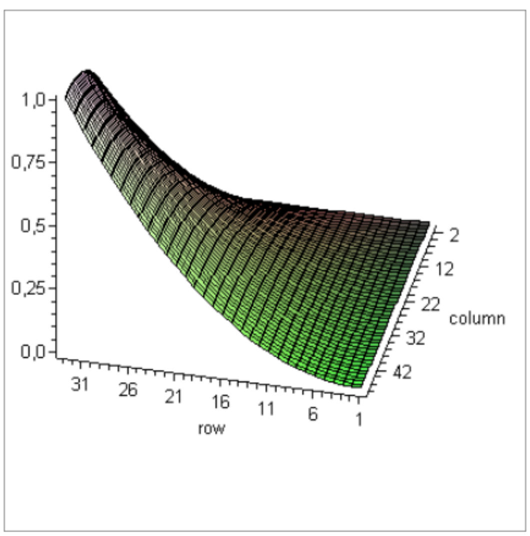

Figure 2: Three dimensional graphic of Eq. (14) for $M = 48, N = 32$, and $\alpha = 0.2$.

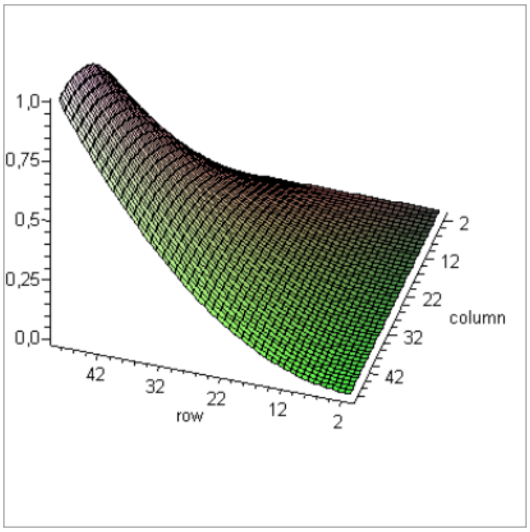

Figure 3: Three dimensional graphic of Eq. (14) for $M = 48, N = 64$, and $\alpha = 0.2$.

		$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.9$
M	N	Error	Error	Error
48	16	0.006171225937	0.004293378614	0.004318848857
48	32	0.003887392591	0.002222009029	0.002231505317
48	48	0.003715712439	0.001499041766	0.001503670513

Table 1: The error analysis of Eq. (14) for some values of α,M and N

5. Conclusions

In this paper,we implement Crank-Nicholson method to time-fractional diffusion equation. In course of this application, we find approximate solution and error analysis of this equation for some values of α, x, t, M and N.

According to these datas, it has been seen that Crank-Nicholson method has been influential for the approximate solutions of time-fractional diffusion equation and this method is highly influential and reliable in terms of finding approximate solutions and comparing with numerical and exact solutions. Thus, we can deduce that Crank-Nicholson method has an important role to obtain approximate solutions of fractional differential equations. We think that this method can also be implemented to other fractional differential equations.

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Hasan Bulut
Department of Mathematics, Faculty of Science,
Firat University,
Elazig, Turkey.
email: *hbulut@firat.edu.tr,*

Seyma Tuluce Demiray
Department of Mathematics, Faculty of Science,
Firat University,
Elazig, Turkey.
email: *seymatuluce@gmail.com,*

Mirac Kayhan
Department of Mathematics, Faculty of Science,
Firat University,
Elazig, Turkey.
e-mail: *mirackayhan@yandex.com,*