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THE NUMERICAL SOLUTION OF SIXTH ORDER BOUNDARY VALUE PROBLEMS BY THE LEGENDRE-GALERKIN METHOD

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Abstract. There are few techniques available to numerically solve sixth-order boundary-value problems with two-point boundary conditions. In this paper we show that the Legendre-Galerkin method is a very effective tool in numerically solving such problems. The method is then tested on examples with non-homogeneous boundary conditions and a comparison with other methods are made. It is shown that the Legendre-Galerkin method yields better results.

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1. Introduction

High-order differential equations often arise from mathematical modelling of a variety of physical phenomena. Sixth-order boundary value problems (BVPs) are known to arise in astrophysics; the narrow convecting layers bounded by stable layers, which are believed to surround A-type stars, may be modelled by sixth-order BVPs [4, 16]. Dynamo action in some stars may be modelled by such equations [9]. Moreover, when an infinite horizontal layer of fluid is heated from below and is subjected to the action of rotation, instability sets in. When this instability is of ordinary convection, then the governing ordinary differential equation is of sixth order [14]. Further discussion of the sixth-order BVPs are given in [3].

The literature of numerical analysis contains little on the solution of the sixth-order BVPs [4, 5, 14, 16]. Theorems that list conditions for the existence and uniqueness of solutions of such problems are thoroughly discussed in [1], but no numerical methods are contained therein.

Many numerical methods have been developed for solving sixth-order equation (1) numerically. Each of these methods has its inherent advantages and disadvantages and the search for alternative, more general, easier and more accurate methods is a continuous and ongoing process. Next, we present a selective review of the main and the most recent methods. These methods include: Finite difference methods [4, 5], modified decomposition method [18], homotopy perturbation method [10], non-polynomial splines approach [17], septic spline [13],variational iteration method [11] and sinc-Galerkin method [7]. The present work describes Legendre-Galerkin method for the solution of linear sixth-order ordinary differential equations of the form

$$Lu(x) = u^{(6)}(x) + \sum_{k=0}^{5} \mu_k(x)\, u^{(k)}(x) = f(x), \qquad -1 \leq x \leq 1, \tag{1}$$

subject to boundary conditions

$$u^{(i)}(-1) = u^{(i)}(1) = 0, \quad i = 0, 1, 2 \tag{2}$$

where $f(x)$ and $u(x)$ are continuous functions in $L^2(-1,1)$, and $\mu_k(x)$ is x^n.

To our best knowledge this is the first result on the application of Legendre-Galerkin for solving linear sixth-order boundary value problems.

The paper is organized into five sections. Section 2 contains notation, definitions, some results of Legendre polynomial, new lemmas and new theorems required for our subsequent development. In Section 3, we use the Legendre-Galerkin method to solve linear sixth-order and obtain the discrete system. Section 4 presents appropriate techniques to treat nonhomogenuous boundary condition and change interval. In Section 5, we give four numerical examples which will be tested to verify the reliability of the proposed algorithm.

2. Preliminaries and Fundamentals

Orthogonal polynomials are widely used in applications in mathematics, mathematical physics, engineering and computer science. One of the most common orthogonal polynomial set is the Legendre polynomials. The Legendre polynomials $P_n(x)$ satisfy the Legendre differential equation

$$(1-x^2)y'' - 2xy' + n(n-1)y = 0, -1 \leq x \leq 1, n \geq 0$$

with recurrence relations

$$\begin{aligned} P'_{n+1}(x) - P'_{n-1}(x) &= (2n+1)P_n(x), &(3)\\ xP'_n(x) - P'_{n-1}(x) &= nP_n(x) &(4) \end{aligned}$$

These polynomial are orthogonal on $[-1,1]$

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} \frac{2}{2n+1}, & if\ m = n, \\ 0, & if\ m \neq n \end{cases} \tag{5}$$

and

$$\int_{-1}^{1} P_n(x)dx = \begin{cases} 2, & n = 0, \\ 0, & n > 0 \end{cases} \tag{6}$$

Lemma 1. *Let n and l be any two positive integer numbers such that $n - l \leq N$ and $l > 0$, then*

$$\int_{-1}^{1} P_n(x)P''_{n-l}(x)dx = 0.$$

Proof. Integrating the left term by parts and using equation (6) can prove the above lemma.

Lemma 2. *Let n and m be any two integer numbers such that $n \geq m$, then*

$$\int_{-1}^{1} P_n(x)P'_m(x)dx = 0.$$

Proof. First, let $n = m$. Integrating the left hand side in Lemma 6 two times by parts yields

$$\int_{-1}^{1} P_n(x)P'_n(x)dx = \tfrac{1}{2}[(P_n(x))^2]_{-1}^{1} = 0 \Rightarrow \int_{-1}^{1} P_n(x)P'_m(x)dx = 0.$$

Second, let $n > m$. Using the recurrence equation (3) and (6), the left term in Lemma 6 can be written as

$$\begin{aligned} \int_{-1}^{1} P_n(x)P'_m(x)dx &= \int_{-1}^{1} P_n(x)[(2m+1)P_{m-1}(x) + P'_{m-2}(x)]dx \\ &= \int_{-1}^{1} P_n(x)P'_{m-2}(x)dx = \int_{-1}^{1} P_n(x)P'_{m-4}(x)dx \\ &= \cdots = \begin{cases} \int_{-1}^{1} P_n(x)P'_0(x)dx, & if\ m = even, \\ \int_{-1}^{1} P_n(x)P'_1(x)dx, & if\ m = odd \end{cases} \\ &= 0. \end{aligned}$$

Theorem 3. *Let n and m be any two integer numbers such that $n, m \leq N$, then*

$$\text{(i)} \int_{-1}^{1} P'_n(x)P_m(x)dx = \begin{cases} 2, & if\ n = m + i, \\ 0, & if\ n \neq m + i\ or\ m \geq n \end{cases}$$

$$\text{(ii)} \int_{-1}^{1} P''_n(x)P_m(x)dx = \begin{cases} n(n+1) - m(m+1), & if\ n \neq m + i, \\ 0, & if\ n = m + i\ or\ m \geq n \end{cases}$$

where $i = 1, 3, 5, \cdots, 2k + 1 \leq N - m$

Proof. (i) Integrating the left hand side for (i) by parts yields

$$\begin{aligned}\int_{-1}^{1} P_n'(x)P_m(x)dx &= [P_n(x)P_m(x)]_{-1}^{1} - \int_{-1}^{1} P_n(x)P_m'(x)dx \\ &= [1+(-1)^{n+m+1}] - \int_{-1}^{1} P_n(x)P_m'(x)dx \qquad (7)\end{aligned}$$

For $n = m+i, i = 1, 3, 5, \cdots \leq N-m$, by using Lemma (6), the integration (7) can be written as

$$\int_{-1}^{1} P_n'(x)P_m(x)dx = 2$$

As in the above case and Lemma (6) is considered but $n = m+i, i = 0, 2, 4, \cdots \leq N-m$, yields

$$\int_{-1}^{1} P_n'(x)P_m(x)dx = 0$$

For $m \geq n$, previous cases must be considered besides Lemma 6. So, the right hand side of equation (7) is equal to zero. (ii)To prove this point, the proof must be divided into four cases. First, let $n = m+i, i = 2, 4, 6, \cdots, 2k+1 \leq N-m$. Integrating the left hand side by parts two times and using Lemma 1 produce

$$\begin{aligned}\int_{-1}^{1} P_n''(x)P_m(x)dx &= [P_n'(x)P_{n-i}(x)]_{-1}^{1} - \int_{-1}^{1} P_n'(x)P_{n-i}'(x)dx \\ &= n(n+1) - [P_n(x)P_{n-i}'(x)]_{-1}^{1} + \int_{-1}^{1} P_n(x)P_{n-i}''(x)dx \\ &= n(n+1) - m(m+1).\end{aligned}$$

Second, let $n = m+i, i = 1, 3, 5, \cdots, 2k+1 \leq N-m$. As in the first case, but the value of i is odd numbers. So, the integration is zero. Third, let $m > n$. By using equation (6) and integrating the left side, the result is equal to zero. Finally, at $m = n$, the value of the integration also is equal to zero by using the integration by parts.

To solve the fourth-order equation, we need the following theorem

Theorem 4. *[**Gamel-Fathy formula (1)**] Let n and m be any two integer numbers such that $n, m \leq N$, then*

$$
\text{(i)} \int_{-1}^{1} P_n'''(x)P_m(x)dx = \begin{cases} \frac{1}{4}\prod_{i=0}^{3}(n-i+2) - \sum_{k=1}^{[\frac{m}{2}]}(4k-1)\Big[n(n+1) \\ \quad -2k(2k-1)\Big], \quad m = even, n = odd, 2k-1 < n, \\ \frac{1}{4}\prod_{i=0}^{3}(n-i+2) - \sum_{k=0}^{[\frac{m}{2}]}(4k+1)\Big[n(n+1) \\ \quad -2k(2k+1)\Big], \quad m = odd, n = even, 2k < n, \\ 0, \qquad otherwise \end{cases}
$$

$$
\text{(ii)} \int_{-1}^{1} P_n''''(x)P_m(x)dx = \begin{cases} \frac{1}{24}\prod_{i=0}^{5}(n-i+3) - \sum_{k=1}^{[\frac{m}{2}]}(4k-1)\Big[\frac{1}{4}\prod_{i=0}^{3}(n-i+2) \\ \quad - \sum_{r=0}^{[\frac{2k-1}{2}]}(4r+1)\big(n(n+1)-2r(2r+1)\big)\Big], \\ \qquad m \text{ and } n \text{ are even}, 2r < n, \\ \frac{1}{24}\prod_{i=0}^{5}(n-i+3) - \sum_{k=0}^{[\frac{m}{2}]}(4k+1)\Big[\frac{1}{4}\prod_{i=0}^{3}(n-i+2) \\ \quad - \sum_{r=1}^{k}(4r-1)\big(n(n+1)-2r(2r-1)\big)\Big], \\ \qquad m \text{ and } n \text{ are odd}, 2r-1 < n, \\ 0, \qquad otherwise \end{cases}
$$

Proof. At the beginning the first derivative of Legendre Polynomials can be written as

$$
P_m'(x) = \begin{cases} \sum_{k=1}^{[\frac{m}{2}]}(4k-1)P_{2k-1}(x), & m = \text{even} \\ \sum_{k=0}^{[\frac{m}{2}]}(4k+1)P_{2k}(x), & m = \text{odd} \end{cases} \tag{8}
$$

(i) Integrating the left hand side for (i) by parts yields

$$
\int_{-1}^{1} P_n'''(x)P_m(x)dx = [P_n''(x)P_m(x)]_{-1}^{1} - \int_{-1}^{1} P_n''(x)P_m'(x)dx \tag{9}
$$

By recalling (8), the value of the second term in right hand side in (9) has two cases.

$$\int_{-1}^{1} P_n''(x)P_m'(x)dx = \begin{cases} \sum_{k=1}^{[\frac{m}{2}]}(4k-1)\int_{-1}^{1} P_n''(x)P_{2k-1}(x)dx, & m = \text{even} \\ \\ \sum_{k=0}^{[\frac{m}{2}]}(4k+1)\int_{-1}^{1} P_n''(x)P_{2k}(x), & m = \text{odd} \end{cases} \tag{10}$$

By using theorem 3 (ii), (10) can be written as

$$\int_{-1}^{1} P_n''(x)P_m'(x)dx = \begin{cases} \sum_{k=1}^{[\frac{m}{2}]}(4k-1)\big(n(n+1)-2k(2k-1)\big), \\ \qquad\qquad m = \text{even}, n = \text{odd}, 2k-1 < n \\ \sum_{k=0}^{[\frac{m}{2}]}(4k+1)\big(n(n+1)-2k(2k+1)\big), \\ \qquad\qquad m = \text{odd}, n = \text{even}, 2k < n \end{cases} \tag{11}$$

The following table shows the second derivative of the polynomial at $x = -1$ and $x = 1$

$P_n''(x)$	$P_n''(1)$	$P_n''(-1)$
$P_0''(x) = 0$	$P_0''(1) = 0$	$P_0''(-1) = 0$
$P_1''(x) = 0$	$P_1''(1) = 0$	$P_1''(-1) = 0$
$P_2''(x) = 3P_0(x)$	$P_2''(1) = 3$	$P_2''(-1) = 3$
$P_3''(x) = 15P_1''(x)$	$P_3''(1) = 15$	$P_3''(-1) = -15$
$P_4''(x) = 35P_2''(x) + 10P_0''(x)$	$P_4''(1) = 45$	$P_4''(-1) = 45$
$P_5''(x) = 63P_3''(x) + 42P_1''(x)$	$P_5''(1) = 105$	$P_5''(-1) = -105$
$P_6''(x) = 99P_4''(x) + 90P_2''(x) + 21P_0''(x)$	$P_6''(1) = 210$	$P_6''(-1) = 210$
$P_7''(x) = 143P_5''(x) + 154P_3''(x) + 81P_1''(x)$	$P_7''(1) = 378$	$P_7''(-1) = -378$

By using Lagrange interpolation, $P_n''(1)$ can be written as $\frac{1}{8}n(n-1)(n+1)(n+2)$. So, The first term in right hand side in (9) is

$$\begin{aligned} [P_n''(x)P_m(x)]_{-1}^{1} &= P_n''(1)P_m(1) - P_n''(-1)P_m(-1) = P_n''(1) - (-1)^m P_n''(-1) \\ &= \frac{1}{8}n(n-1)(n+1)(n+2) - (-1)^{n+m}\frac{1}{8}n(n-1)(n+1)(n+2) \\ &= \frac{1}{8}(1-(-1)^{n+m})n(n-1)(n+1)(n+2) \end{aligned} \tag{12}$$

By recalling (11) beside (12), we can prove (i).

(ii) The integration in the left side in (ii) can be written as

$$\int_{-1}^{1} P_n''''(x)P_m(x)dx = \Lambda_{n,m} - \begin{cases} \sum\limits_{k=1}^{\left[\frac{m}{2}\right]}(4k-1)\int\limits_{-1}^{1} P_n'''(x)P_{2k-1}(x)dx, & m = \text{even} \\ \sum\limits_{k=0}^{\left[\frac{m}{2}\right]}(4k+1)\int\limits_{-1}^{1} P_n'''(x)P_{2k}(x), & m = \text{odd} \end{cases} \tag{13}$$

where

$$\Lambda_{n,m} = \frac{1}{48}(1+(-1)^{n+m})\prod_{i=0}^{5}(n-i+3) \tag{14}$$

$\Lambda_{n,m}$ results from the interpolation of the third derivative of Legendre polynomials at $x = 1$. By collecting theorem 4 (i), (13) beside (14), theorem 4 (ii) can be proved.

To solve the sixth order equation (1)-(2), we need the following theorem

Theorem 5. *[**Gamel-Fathy formula (2)**] Let n and m be any two integer numbers such that $n, m \leq N$, then*

$$\textit{(i)} \int_{-1}^{1} P_n''''''(x)P_m(x)dx = \begin{cases} \frac{1}{192}\prod\limits_{i=0}^{7}(n-i+4) - \sum\limits_{k=0}^{\left[\frac{m}{2}\right]}(4k+1)\Bigg[\frac{1}{24}\prod\limits_{i=0}^{5}(n-i+3) \\ \quad - \sum\limits_{r=1}^{k}(4r-1)\Bigg[\frac{1}{4}\prod\limits_{i=0}^{3}(n-i+2) \\ \quad - \sum\limits_{s=0}^{\left[\frac{2r-1}{2}\right]}(4s+1)\big(n(n+1)-2s(2s+1)\big)\Bigg]\Bigg], \\ \qquad\qquad m = \textit{even}, n = \textit{odd}, 2s < n, \\ \frac{1}{192}\prod\limits_{i=0}^{7}(n-i+4) - \sum\limits_{k=0}^{\left[\frac{m}{2}\right]}(4k+1)\Bigg[\frac{1}{24}\prod\limits_{i=0}^{5}(n-i+3) \\ \quad - \sum\limits_{r=1}^{\frac{2k-1}{2}}(4r-1)\Bigg[\frac{1}{4}\prod\limits_{i=0}^{3}(n-i+2) \\ \quad - \sum\limits_{s=0}^{r}(4s+1)\big(n(n+1)-2s(2s+1)\big)\Bigg]\Bigg], \\ \qquad\qquad m = \textit{odd}, n = \textit{even}, 2s-1 < n, \\ 0, \qquad\qquad \textit{otherwise} \end{cases}$$

$$\text{(ii)} \int_{-1}^{1} P_n''''''(x)P_m(x)dx = \begin{cases} \frac{1}{1920}\prod_{i=0}^{9}(n-i+5) - \sum_{k=1}^{\left[\frac{m}{2}\right]}(4k-1)\Bigg[\frac{1}{192}\prod_{i=0}^{7}(n-i+4) \\ \quad - \sum_{r=0}^{\left[\frac{2k-1}{2}\right]}(4r+1)\Bigg[\frac{1}{24}\prod_{i=0}^{5}(n-i+3) \\ \quad - \sum_{s=1}^{r}(4s-1)\Bigg[\frac{1}{4}\prod_{i=0}^{3}(n-i+2) \\ \quad - \sum_{v=0}^{\left[\frac{2s-1}{2}\right]}(4v+1)\big(n(n+1)-2v(2v+1)\big)\Bigg]\Bigg]\Bigg], \\ \qquad\qquad\qquad m \text{ and } n \text{ are even}, 2v < n, \\ \frac{1}{1920}\prod_{i=0}^{9}(n-i+5) - \sum_{k=0}^{\left[\frac{m}{2}\right]}(4k+1)\Bigg[\frac{1}{192}\prod_{i=0}^{7}(n-i+4) \\ \quad - \sum_{r=1}^{k}(4r-1)\Bigg[\frac{1}{24}\prod_{i=0}^{5}(n-i+3) \\ \quad - \sum_{s=0}^{\frac{2r-1}{2}}(4s+1)\Bigg[\frac{1}{4}\prod_{i=0}^{3}(n-i+2) \\ \quad - \sum_{v=1}^{v}(4v-1)\big(n(n+1)-2v(2v-1)\big)\Bigg]\Bigg]\Bigg], \\ \qquad\qquad\qquad m \text{ and } n \text{ are odd}, 2v-1 < n, \\ 0, \qquad\qquad\qquad \text{otherwise} \end{cases}$$

Proof. To prove (i), we transfer one derivative from P_n to P_m by using integration by parts to use (8). So, we can write the left-hand side of (i) as

$$\int_{-1}^{1} P_n'''''(x)P_m(x)dx = \Upsilon_{n,m} - \begin{cases} \sum_{k=1}^{\left[\frac{m}{2}\right]}(4k-1)\int_{-1}^{1} P_n''''(x)P_{2k-1}(x)dx, & m = \text{even} \\ \sum_{k=0}^{\left[\frac{m}{2}\right]}(4k+1)\int_{-1}^{1} P_n''''(x)P_{2k}(x), & m = \text{odd} \end{cases} \tag{15}$$

where

$$\Upsilon_{n,m} = \frac{1}{384}(1-(-1)^{n+m})\prod_{i=0}^{7}(n-i+4) \tag{16}$$

which comes from the intepolation of $P_n''''(1)$. By gathering theorem 4, (15) and (16), we can prove the required.

We can prove theorem 5(ii) in a similar manner of theorem 5(i).

3. Legendre-Galerkin method

This section presents how Legendre basis can be used beside Galekin method to solve the sixth-order differential equation (1) with the boundary conditions (2). First,we assume the solution of (1) is approximate by the finite expansion of Legendre basis function

$$u(x) = \sum_{j=0}^{n} c_j P_j(x) \tag{17}$$

The unknown coefficients c_j in equation (17) are determined by orthogonalizing the residual with respect to the basis functions

$$\langle u^{(6)}(x), P_r(x)\rangle + \sum_{k=0}^{5} \langle \mu_k(x)\, u^{(k)}(x), P_r(x)\rangle - \langle f(x), P_r(x)\rangle = 0 \tag{18}$$

where

$$\langle \zeta, \eta\rangle = \int_{-1}^{1} \zeta \cdot \eta \, dx$$

The method of approximating the integrals in (18) begins by integrating by parts to transfer all derivatives from u to P_r. The approximation of the last four inner products on the left-hand side of (18) has been thoroughly treated in [8]. We will list them for convenience

$$\big\langle \mu_2(x)\, u''(x), P_r(x)\big\rangle = \int_{-1}^{1} u(x)[\mu_2(x)P_r(x)]'' dx, \tag{19}$$

$$\big\langle \mu_1(x)\, u'(x), P_r(x)\big\rangle = -\int_{-1}^{1} u(x)[\mu_1(x)P_r(x)]' dx, \tag{20}$$

$$\langle \mu_0(x)\, u(x), P_r(x)\rangle = \int_{-1}^{1} \mu_0(x)\, u(x)\, P_r(x)\, d\,x, \tag{21}$$

and

$$\langle f(x), P_r(x)\rangle \simeq \sum_{i=0}^{m} f(x_i)\, \frac{2}{[(1-x_i^2)(P'_m(x_i))^2]}, \tag{22}$$

To solve the equation (1)-(2), we need the following lemma

Lemma 6. *The following relations hold*

$$\left\langle u^{(6)}(x), P_r(x)\right\rangle = \sum_{k=3}^{5}(-1)^{k+1}\left[u^{(k)}(x)P_r^{(5-k)}(x)\right]_{-1}^{1} + \int_{-1}^{1} u(x)P_r^{(6)}(x)dx, \quad (23)$$

$$\left\langle \mu_5(x)u^{(5)}(x), P_r(x)\right\rangle = \sum_{k=3}^{4}(-1)^{k}\left[u^{(k)}(x)(\mu_5(x)P_r(x))^{(4-k)}\right]_{-1}^{1} - \int_{-1}^{1} u(x)[\mu_5(x)P_r(x)]^{(5)}dx, \quad (24)$$

$$\left\langle \mu_4(x)u^{(4)}(x), P_r(x)\right\rangle = \left[u^{(3)}(x)\mu_4(x)P_r(x)\right]_{-1}^{1} + \int_{-1}^{1} u(x)[\mu_4(x)P_r(x)]^{(4)}dx, \quad (25)$$

$$\left\langle \mu_3(x)u^{(3)}(x), P_r(x)\right\rangle = -\int_{-1}^{1} u(x)[\mu_3(x)P_r(x)]'''dx \quad (26)$$

Proof. For $u^{(6)}$, the inner product with Legendre basis elements is given by

$$\left\langle u^{(6)}, P_r(x)\right\rangle = \int_{-1}^{1} u^{(6)}P_r(x)\,dx.$$

Integrating by parts to remove the sixth derivatives from the dependent variable u leads to the equality

$$\int_{-1}^{1} u^{(6)}(x)P_r(x)\,dx = B_{T,6} + \sum_{k=3}^{5}(-1)^{k+1}\left[u^{(k)}(x)P_r^{(5-k)}(x)\right]_{-1}^{1} + \int_{-1}^{1} u(x)P_r^{(6)}(x)dx \quad (27)$$

where the boundary term

$$B_{T,6} = \left[\sum_{k=0}^{2}(-1)^{k+1}u^{(k)}(x)P_r^{(5-k)}(x)\right]_{-1}^{1}$$

is zero because the three terms vanish due to the fact that u satisfies the boundary conditions (2) .

The inner product for $\left\{\mu_5(x)\, u^{(5)}(x)\right\}$ may be handled in a similar manner to yield

$$\int_{-1}^{1} \mu_5(x)u^{(5)}(x)P_r(x)\, d\, x = B_{T,5} + \sum_{k=3}^{4}(-1)^k \left[u^{(k)}(x)(\mu_5(x)P_r(x))^{(4-k)}\right]_{-1}^{1} - \int_{-1}^{1} u(x)(\mu_5(x)P_r(x))^{(5)}dx, \quad (28)$$

where the boundary term is

$$B_{T,5} = \left[\sum_{k=0}^{2}(-1)^k\, u^{(k)}(x)(\mu_5(x)P_r(x))^{(4-k)}\right]_{-1}^{1} = 0.$$

Similarly, for $\left\{\mu_4(x)u^{(4)}(x)\right\}$, after four integrations by parts to remove the four derivatives from the dependent variable u, we have the equality

$$\left\langle \mu_4(x)u^{(4)}(x), P_r(x)\right\rangle = B_{T,4} + \left[u^{(3)}(x)\mu_4(x)P_r(x)\right]_{-1}^{1} + \int_{-1}^{1} u(x)[\mu_4(x)P_r(x)]^{(4)}dx, \quad (29)$$

where the boundary term is

$$B_{T,4} = \left[\sum_{k=0}^{2}(-1)^{k+1}\, u^{(k)}(x)(\mu_4(x)P_r(x))^{(3-k)}\right]_{-1}^{1} = 0.$$

then (29) may be written as (25).

Replacing each term of (18) with the approximation defined in (19)-(26), we obtain the following theorem

Theorem 7. *If the assumed approximate solution of the boundary-value problem (1)-(2) is (17), then the discrete Legendre-Galerkin system for the determination of the unknown coefficients* $\{c_j\}_{j=0}^{n}$ *is given by*

$$\sum_{j=0}^{n}\Bigg[\sum_{k=3}^{5}(-1)^{k+1}\left[P_j^{(k)}(x)P_r^{(5-k)}(x)\right]_{-1}^{1} + \sum_{k=3}^{4}(-1)^{k}\left[P_j^{(k)}(x)(\mu_5(x)P_r(x))^{(4-k)}\right]_{-1}^{1} + \left[P_j^{(3)}\mu_4(x)P_r(x)\right]_{-1}^{1} + \int_{-1}^{1}\sum_{\sigma=0}^{6}(-1)^{\sigma}P_j(x)\left[\mu_\sigma(x)P_r(x)\right]^{(\sigma)}\, d\, x\Bigg]\, c_j = \sum_{q=0}^{m}\frac{2f(x_q)P_r(x_q)}{[(1-x_q^2)(P'_m(x_q))^2]}, \qquad \mu_6(x)=1, \quad 0 \le r \le n. \quad (30)$$

The system in (30)takes the matrix form

$$\mathbf{A}\ \mathbf{c} = \mathbf{b} \tag{31}$$

where

$$\mathbf{A} = \begin{pmatrix} e_{0,0}+v_{0,0} & e_{1,0}+v_{1,0} & \dots & e_{n,0}+v_{n,0} \\ e_{0,1}+v_{0,1} & e_{1,1}+v_{1,1} & \dots & e_{n,1}+v_{n,1} \\ e_{0,2}+v_{0,2} & e_{1,2}+v_{1,2} & \dots & e_{n,2}+v_{n,2} \\ e_{0,3}+v_{0,3} & e_{1,3}+v_{1,3} & \dots & e_{n,3}+v_{n,3} \\ \vdots & \vdots & \ddots & \vdots \\ e_{0,n}+v_{0,n} & e_{1,n}+v_{1,n} & \dots & e_{n,n}+v_{n,n} \end{pmatrix} \tag{32}$$

and

$$e_{j,r} = \sum_{k=0}^{6}(-1)^k \int_{-1}^{1} P_j(\mu_k(x)P_r(x))^{(k)}dx,\ \ \mu_6(x)=1,$$

$$v_{j,r} = \Bigg[\sum_{k=3}^{5}(-1)^{k+1}P_j^{(k)}(x)P_r^{(5-k)}(x) + \sum_{k=3}^{4}(-1)^k P_j^{(k)}(x)(\mu_5(x)P_r(x))^{(4-k)} + P_j^{(3)}(x)\mu_4(x)P_r(x)\Bigg]_{-1}^{1}$$

$e_{j,r}$ can be evaluated from theorems and lemmas in section 2 and the boundary term $v_{j,r}$ for case $\mu_i(x) = x^n$ can be calculated as

$$\left[P_j^{(3)}(x)\mu_4(x)P_r(x)\right]_{-1}^{1} = \frac{1}{48}\left[1+(-1)^{n+r+j}\right]\prod_{i=0}^{5}(j-i+3),$$

$$\sum_{k=3}^{4}(-1)^k\left[P_j^{(k)}(x)(\mu_5(x)P_r(x))^{(4-k)}\right]_{-1}^{1} = \frac{1}{384}\left[1-(-1)^{n+r+j}\right]\prod_{i=0}^{7}(j-i+4) - \frac{1}{96}\left[(2n+r(r+1))(1+(-1)^{n+r+j-1})\right]\prod_{i=0}^{5}(j-i+3),$$

and

$$\sum_{k=3}^{5}(-1)^{k+1}\left[P_j^{(k)}(x)P_r^{(5-k)}(x)\right]_{-1}^{1}=\frac{1}{3840}\left[1+(-1)^{r+j}\right]\prod_{i=0}^{9}(j-i+5)$$
$$-\frac{1}{768}\left[1-(-1)^{r+j-1}\right]r(r+1)\prod_{i=0}^{7}(j-i+4)$$
$$+\frac{1}{384}(1-(-1)^{r+j-1})\left(\prod_{i=0}^{3}(r-i+2)\right)\left(\prod_{i=0}^{5}(j-i+3)\right)$$

Now we have a linear system (30) of n equations for n unknown coefficients. We can obtain the coefficient of the approximate solution by solving this linear system by Q-R method. The solution $\mathbf{c}=(c_0,\ldots,c_n)^{\tau}$ gives the coefficients in the approximate Legendre-Galerkin solution $u(x)$.

4. Treatment of the Boundary Conditions and Solution Domain

In the previous section, we put our algorithm of the solution depends on the homogeneous boundary with solution domain $[-1,1]$. If the boundary conditions are nonhomogeneous or the solution domain is $[a,b]$, then these conditions need be converted to homogeneous conditions via an interpolation by a known function and the domain of solution must be convert to $[-1,1]$. For example, consider

$$Lu(y)=u^{(6)}(y)+\sum_{k=0}^{5}\mu_k(y)\,u^{(k)}(y)=f(y),\qquad a\le y\le b \tag{33}$$

subject to boundary conditions

$$u^{(i)}(a)=\theta_i,\qquad u^{(i)}(b)=\phi_i,\;\; i=0,1,2 \tag{34}$$

By using the linear transformation $y=\frac{b-a}{2}x+\frac{b+a}{2}$, the problem (33) can be written as

$$Lu(x)=\left(\frac{2}{b-a}\right)^6u^{(6)}(x)+\sum_{k=0}^{5}\mu_k(X)\left(\frac{2}{b-a}\right)^k u^{(k)}(x)=f(X),\quad -1\le x\le 1 \tag{35}$$

where

$$X=\frac{b-a}{2}x+\frac{b+a}{2}$$

subject to the boundary conditions

$$u^{(i)}(-1) = \left(\frac{2}{b-a}\right)^i \theta_i = \Theta_i, \quad u^{(i)}(1) = \left(\frac{2}{b-a}\right)^i \phi_i = \Phi_i, \; i = 0, 1, 2 \tag{36}$$

The nonhomogeneous boundary conditions in (36) can be transformed to homogeneous boundary conditions by the change of dependent variable

$$\Psi(x) = u(x) - \Omega(x) \tag{37}$$

where $\Psi(x)$ is the interpolating polynomial that satisfies $\Psi(-1) = \Theta_i$ and $\Psi(1) = \Phi_i, i = 0, 1, 2$. It is easy to see that

$$\Omega(x) = \sum_{i=0}^{5} \rho_i \, x^i$$

and

$$\begin{aligned}
\rho_0 &= \frac{1}{16}(8\Theta_0 + 5\Theta_1 + \Theta_2 + 8\Phi_0 - 5\Phi_1 + \Phi_2),\\
\rho_1 &= \frac{1}{16}(-15\Theta_0 - 7\Theta_1 - \Theta_2 + 15\Phi_0 - 7\Phi_1 + \Phi_2),\\
\rho_2 &= \frac{1}{8}(-3\Theta_1 - \Theta_2 + 3\Phi_1 - \Phi_2),\\
\rho_3 &= \frac{1}{8}(5\Theta_0 + 5\Theta_1 + \Theta_2 - 5\Phi_0 + 5\Phi_1 - \Phi_2),\\
\rho_4 &= \frac{1}{16}(\Theta_1 + \Theta_2 - \Phi_1 + \Phi_2),\\
\rho_5 &= \frac{1}{16}(-3\Theta_0 - 3\Theta_1 - \Theta_2 + 3\Phi_0 - 3\Phi_1 + \Phi_2)
\end{aligned}$$

The new problem with homogeneous boundary conditions is then

$$L\Psi(x) = \Psi^{(6)}(x) + \sum_{k=0}^{5} \mu_k(x)\, \Psi^{(k)}(x) = \tilde{f}(x), \qquad -1 \le x \le 1 \tag{38}$$

subject to boundary conditions

$$\Psi^{(i)}(-1) = 0, \qquad \Psi^{(i)}(1) = 0, \; i = 0, 1, 2 \tag{39}$$

where

$$\begin{aligned}
\tilde{f}(x) &= f(x) - L\Omega(x)\\
&= f(x) - \sum_{\kappa=0}^{5} \mu_\kappa(x) \Omega^{(\kappa)}(x)
\end{aligned}$$

Now apply the Legendre-Galerkin method to (38). We define an approximate solution of (38) via the formula

$$\Psi(x) = \sum_{j=0}^{n} c_j P_j(x) \tag{40}$$

Then, the approximate solution of (37) is

$$u(x) = \sum_{j=0}^{n} c_j P_j(x) + \Omega(x) \tag{41}$$

By using the inverse linear transformation $x = \frac{2}{b-a}y - \frac{b+a}{b-a}$, we can find $u(y)$ that is the approximate solution of (33).

Algorithm

- compute $\int_{-1}^{1} P_n^{(i)}(x)P_m(x)dx, \quad \text{for} \quad i = 0, 1, 2, \ldots, 6$
- if the domain is $[a, b]$, $y = \frac{b-a}{2}x + \frac{b+a}{2}$
- if nonhomogeneous boundary, $\Psi(x) = u(x) - \Omega(x)$
- assume $\Psi(x) = c_0P_0(x) + c_1P_1(x) + \ldots + c_nP_n(x) \quad \text{and} \quad n < N \in Z$
- apply $L\Psi(x)$
- evaluate $\mathbf{A}$ and $\mathbf{b}$
- solve $\mathbf{A\ c = b}$
- OUTPUT: the values of $c_0, c_1, c_2, ..., c_n$
- $u(x) = \Psi(x) + \Omega(x)$
- use the transformation $x = \frac{2}{b-a}y - \frac{b+a}{b-a}$
- end

5. Numerical Examples

The four examples included in this section were selected in order to illustrate the performance of the Legendre-Galerkin method in solving sixth-order boundary-value problems. In all example, demonstrate that the Legendre-Galerkin method can be applied to solve nonhomogeneous boundary conditions. Absolute error is calculated to show the accuracy by

$$||E_{LG}|| = |u_{\text{exact}}(x) - u_{\text{Legendre-Galerkin}}(x)|$$

We also compare our method with other method introduce in [12, 10, 11, 7, 18, 2, 17]. It is shown that the Legendre-Galerkin method yields better results.

Example 1. [10, 11] Consider the following special sixth-order boundary value problem involving a parameter c

$$u^{(6)} = (1+c)\,u^{(4)} - c\,u'' + c\,x, \qquad 0 \le x \le 1,$$

subject to the boundary conditions

$$\begin{aligned} &u(0) = u'(0) = u''(0) = 1 \\ &u(1) = \frac{7}{6} + \sinh(1), \qquad u'(1) = \frac{1}{2} + \cosh(1), \qquad u''(x) = 1 + \sinh(1) \end{aligned}$$

whose exact solution is

$$u(x) = 1 + \frac{x^3}{6} + \sinh(x)$$

Table 1 exhibit the numerical results for small and large values of $c = 1, 10, 100, 1000,$ 1000000. **Tables 2 and 3** exhibit the maximum absolute error for small and large values of c.

Table 1: Maximum absolute errors for example 1 at different n and c

n	$c=1$	$c=10$	$c=100$	$c=1000$	$c=1000000$
8	1.081E-02	3.960E-05	1.902E-07	8.731E-08	8.077E-08
10	3.726E-10	3.874E-10	2.152E-09	1.884E-11	1.496E-11
12	3.108E-14	3.241E-14	3.641E-14	1.376E-14	5.440E-15
14	6.661E-16	3.330E-15	1.776E-15	2.442E-15	1.998E-15

Example 2. [2, 17] Consider the following linear boundary value problem of sixth order

$$u^{(6)} + x\,u = -\left(24 + 11\,x + x^3\right)\, e^x, \qquad 0 \le x \le 1,$$

subject to the boundary conditions

$$\begin{aligned} u(0) &= u(1) = 0, \\ u'(0) &= 1, \qquad u'(1) = -e, \\ u''(0) &= 0, \qquad u''(1) = -4\,e \end{aligned}$$

Table 2: Comparison maximum absolute error for example 1

Method	$c = 1$	$c = 10$	$c = 100$
Legendre-Galerkin, $n = 14$	6.66E-16	3.33E-15	1.77E-15
Homotopy perturbation [11]	1.70E-08	1.7E-05	1.1E-03
Adomian decomposition [11]	1.7E-08	1.1E-05	1.1E-03
Differential transformation [11]	1.0E-04	1.5E-04	1.7E-03
Variational [11]	1.9E-08	1.1E-05	1.0E-03

Table 3: Comparison maximum absolute error for example 1

Method	$c = 1000$	$c = 1000000$
Legendre-Galerkin, $n = 14$	2.44E-15	1.99E-15
Homotopy perturbation [11]	1.0E-01	6.5E+04
Adomian decomposition [11]	1.0E-01	6.5E+04
Differential transformation [11]	1.0E-03	1.2E-03
Variational [11]	1.0E-01	6.5E+04

whose exact solution is

$$u(x) = x\ (1 - x)\ \exp(x)$$

The maximum errors are listed in **Table 4** for different values of n. **Table 5** shows the comparison between Legendre-Galerkin method and non-polynomial splines method in [2].

Table 4: Maximum absolute errors for example 2 for different n

n	Legendre-Galerkin $\|E_{LG}\|$
10	1.045E-07
12	1.780E-09
14	7.155E-13
16	9.436E-16

Table 5: Comparison maximum absolute error for example 2

Legendre-Galerkin,$n = 16$	non-polynomial splines
9.436E-16	4.410E-11

Example 3. [2, 17] Consider the following linear boundary value problem of sixth order

$$u^{(6)} + u = 6\ [2\,x\ \cos\ x + 5\ \sin\ x]\,, \qquad -1 \le x \le 1,$$

subject to the boundary conditions

$$\begin{aligned} u(-1) &= u(1) = 0, \\ u'(-1) &= 2\,\sin(1), \qquad u'(1) = 2\,\sin(1), \\ u''(-1) &= -4\,\cos(-1) + 2\,\sin(-1) \\ u''(1) &= 4\,\cos(1) + 2\,\sin(1) \end{aligned}$$

whose exact solution is

$$u(x) = \left(x^2 - 1\right)\,\sin(x)$$

The maximum errors are listed in **Table 6** for different values of n. **Table 7** shows the comparison between Legendre-Galerkin method , non-polynomial splines method in [17] and septic spline in [13].

Table 6: Maximum absolute errors for example 3 at different n

n	Legendre-Galerkin $\|E_{LG}\|$
8	4.298E-03
10	1.236E-06
12	7.329E-10
14	4.669E-13
16	1.831E-15

Table 7: Comparison maximum absolute error for example 3

Legendre-Galerkin,$n = 16$	non-polynomial splines	septic spline
1.831E-15	9.200E-09	7.17E-08

Example 4. [13] Consider the following linear boundary value problem of sixth order

$$u^{(6)} + (5x+1)\,u = \left[185\,x - 25x^2 + 10x^4\right]\,\cos\,x + \left(270 - 36\,x^2\right)\,\sin\,x, \qquad -1 \leq x \leq 1,$$

subject to the boundary conditions

$$\begin{aligned} u(-1) &= 4\,\cos(1), u(1) = -2\,\cos(1), \\ u'(-1) &= \cos(1) + 4\,\sin(1), \qquad u'(1) = \cos(1) + 2\,\sin(1), \\ u''(-1) &= -16\,\cos(1) + 2\,\sin(1) \\ u''(1) &= 14\,\cos(1) - 2\,\sin(1) \end{aligned}$$

whose exact solution is

$$u(x) = \left(2\,x^3 - 5x + 1\right)\,\cos(x)$$

The maximum errors are listed in **Table 8** for different values of n. **Table 9** shows the comparison between Legendre-Galerkin method and septic spline method in [13].

Table 8: Maximum absolute errors for example 4 at different n

n	Legendre-Galerkin $\|E_{LG}\|$
8	5.147E-01
10	2.309E-05
12	1.643E-08
14	1.231E-11
16	1.110E-14
18	9.769E-15

Table 9: Comparison maximum absolute error for example 4

Legendre-Galerkin,$n = 18$	Septic spline
9.769E-15	8.68E-07

6. Conclusion

In this paper, Legendre-Galerkin method has been successfully used for finding the solution of linear sixth-order boundary value problems. Comparison of the results obtained by the present method with those obtained by the septic splines method, non-polynomial splines method, modified decomposition method, homotopy perturbation method, variational iteration method and differential transformation method reveals that the present method is superior because of the lower error. It may be concluded that the method is very powerful and efficient in finding the numerical solutions for such problem.

It is seen that Legendre-Galerkin method can be an alternative way for the solution of sixth-order differential equations that have no exact solutions.

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