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INFINITELY MANY NONTRIVIAL SOLUTIONS FOR NONLINEAR PROBLEMS INVOLVING THE $(P_1(X), P_2(X))$-LAPLACE OPERATOR

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Abstract. In this paper, by using the mountain pass theorem and the Ekeland variational principle, we show the existence of infinitely many nontrivial weak solutions of a Dirichlet problem involving the $(p_1(x), p_2(x))$-Laplace problem.

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1. introduction

In this article, we study the existence and multiplicity of nontrivial weak solution for the following problem

$$\begin{cases} -\text{div}\left((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u\right) = f(x,u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N \,(N \geq 3)$ is a bounded smooth domain, $p_i(x) \in C\left(\overline{\Omega}\right)$ with $1 < p_i(x) < N$, for any $x \in C\left(\overline{\Omega}\right)$ and for $i = 1, 2$ and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying some certain conditions.

Recently, an increasing attention has been paid to the study of differential equations and variational problems with $p(x)$-growth condition. The main interest in studying such problems arises from the presence of the $p(x)$-Laplacian operator $p(x)-$Laplace operator $\Delta_{p(x)}(u) = -\text{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$. The $p(x)-$Laplace operator is a generalization of $p-$Laplace operator $\Delta_p(u) = -\text{div}\left(|\nabla u|^{p-2}\nabla u\right)$ obtained in the case when $p(x) \equiv p$ (a constant). The $p(x)$-Laplacian possesses more complicated structure than the p-Laplace operator; for example, it is not homogeneous. This fact implies some diffculties; for example, we can not use the Lagrange Multiplier Theorem and the theory of Sobolev spaces in many problems involving this operator.

Furthermore, some of the nonlinear problems including the $p(x)$-Laplace operator are very attractive because those problems can be used to model dynamical phenomenons that stem from the study of electrorheological fluids or elastic mechanics [1, 9, 15, 23, 26]. Problems with variable exponent growth conditions also appear in the mathematical modelling of stationary thermo-rheological viscous flows of non-Newtonian fluids, in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium and image processing [2, 3, 4, 7, 22]. In recent years, the similar problems of the form (**1**) have been studied by many authors using various methods [5, 8, 13, 14, 19, 21].

In [20], the author studied the problem (1) when the nonlinearity is $f(x,u) = \pm(-\lambda|u|^{m(x)-2} + |u|^{q(x)-2})$, by using the $\mathbb{Z}_2$-Symmetric Mountain Pass theorem and variational methods, and they showed existence of infinitely many weak solutions for any $\lambda > 0$ and nontrivial weak solutions for the cases when λ is large enough.

In [17], the authors dealt with the problem (**1**) and showed infinitely many weak solution by the help of the Mountain Pass theorem and Fountain theorem where the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition.

In [18], the authors studied the following nonlinear Neumann problem

$$\begin{cases} -\triangle_{p_{1(x)}} u - \triangle_{p_{2(x)}} u + |u|^{p_1(x)-2}u + |u|^{p_2(x)-2}u = \lambda f(u,v) & \text{in } \Omega, \\ |\nabla u|^{p_1(x)-2} \frac{\partial u}{\partial \eta} + |\nabla u|^{p_2(x)-2} \frac{\partial u}{\partial \eta} = \mu g(x,u) & \text{on } \partial\Omega, \end{cases} \qquad (\mathbf{2})$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded smooth domain, $p_i(x) \in C(\overline{\Omega})$ with $p_i(x) > 1$, for any $x \in C(\overline{\Omega})$ and for $i = 1, 2$; $\lambda, \mu \in \mathbb{R}$ such that $\lambda^2 + \mu^2 \neq 0$. By using Mountain Pass theorem, Fountain theorem and dual Fountain theorem, the authors proved the existence weak of solutions for problem (**2**).

This paper is organized as follows. In Section 2, we present some necessary and preliminary knowledge of the variable exponent Lebesgue-Sobolev spaces and the weighted variable exponent Lebesgue space. In Section 3, using the variational method, we show the existence of infinitely many weak solutions of problem (**1**).

2. PRELIMINARIES

We recall in what follows some definitions and basic properties of variable exponent Lebesgue-Sobolev spaces, $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, $W_0^{1,p(x)}(\Omega)$ and $L_{c(x)}^{p(x)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$. Furhermore, we give the properties of the $(p_1(x), p_2(x))$-Laplace operator. In this context, we refer to [12, 16, 25] for the fundamental properties of these spaces.

Set

$$C_{+}\left(\overline{\Omega}\right)=\{p:\ p\in C\left(\overline{\Omega}\right),\min p\left(x\right)>1,\forall x\in\overline{\Omega}\}.$$

For any $p\left(x\right)\in C_{+}\left(\overline{\Omega}\right)$, we denote

$$1<p^{-}:=\min_{x\in\overline{\Omega}}p\left(x\right)\leq p\left(x\right)\leq p^{+}:=\max_{x\in\overline{\Omega}}p\left(x\right)<\infty.$$

Let $p\left(x\right)\in C_{+}\left(\overline{\Omega}\right)$. We define the variable exponent Lebesgue space by

$$L^{p(x)}\left(\Omega\right)=\left\{u\mid u:\Omega\rightarrow\mathbb{R}\text{ is measurable, }\int_{\Omega}\left|u\left(x\right)\right|^{p(x)}dx<\infty\right\},$$

then $L^{p(x)}\left(\Omega\right)$ endowed with the norm

$$\left|u\right|_{p(x)}=\inf\left\{\lambda>0:\int_{\Omega}\left|\frac{u\left(x\right)}{\lambda}\right|^{p(x)}dx\leq1\right\},$$

becomes a Banach space.

Proposition 1. *[12, 16] If $p\in C\left(\overline{\Omega}\right)$, the conjugate space of $L^{p(x)}\left(\Omega\right)$ is $L^{p'(x)}\left(\Omega\right)$, where $\frac{1}{p'(x)}+\frac{1}{p(x)}=1$. For any $u\in L^{p(x)}\left(\Omega\right)$ and $v\in L^{p'(x)}\left(\Omega\right)$, we have*

$$\left|\int_{\Omega}uvdx\right|\leq\left(\frac{1}{p^{-}}+\frac{1}{\left(p^{-}\right)'}\right)\left|u\right|_{p(x)}\left|v\right|_{p'(x)}.$$

The modular of $L^{p(x)}\left(\Omega\right)$ space, which is the mapping $\rho_{p(x)}\left(u\right):L^{p(x)}\left(\Omega\right)\rightarrow\mathbb{R}$ is defined by

$$\rho_{p(x)}\left(u\right)=\int_{\Omega}\left|u\left(x\right)\right|^{p(x)}dx,\ \forall u\in L^{p(x)}\left(\Omega\right).$$

Proposition 2. *[12, 16] If $u,u_n\in L^{p(x)}\left(\Omega\right)$, $n=1,2,...$ then the following statements are equivalent:*

$$(i)\ \lim_{n\rightarrow\infty}\left|u_n-u\right|_{p(x)}=0,$$

$$(ii)\ \lim_{n\rightarrow\infty}\rho_{p(x)}(u_n-u)=0,$$

$$(iii)\ u_n\rightarrow u\text{ in measure in }\Omega\text{ and }\lim_{n\rightarrow\infty}\rho_{p(x)}(u_n)=\rho_{p(x)}\left(u\right).$$

Proposition 3. *[12, 16] If* $u, u_n \in L^{p(x)}(\Omega)$, $n = 1, 2, ...,$ *we have*

$$(i)\ \ |u|_{p(x)} < 1\,(=1; > 1) \Leftrightarrow \rho_{p(x)}(u) < 1\,(=1; > 1),$$

$$(ii)\ |u|_{p(x)} > 1 \implies |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+},$$

$$(iii)\ |u|_{p(x)} < 1 \implies |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-},$$

$$(iv)\ \ \lim_{n\to\infty} |u_n|_{p(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n) = 0;$$

$$(v)\ \ \lim_{n\to\infty} |u_n|_{p(x)} \to \infty \Leftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n) \to \infty.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \quad |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)},\ \forall u \in W^{1,p(x)}(\Omega).$$

The space $W_0^{1,p(x)}(\Omega)$ is denoted by the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. We can define an equivalent norm

$$\|u\|_{1,p(x)} = |\nabla u|_{p(x)},$$

for all $u \in W_0^{1,p(x)}(\Omega)$. Since Poincaré inequality holds [13], i.e. there exists a positive constant $C > 0$ such that

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)},\ \forall u \in W_0^{1,p(x)}(\Omega).$$

We also consider the weighted variable exponent Lebesgue space $L_{c(x)}^{p(x)}(\Omega)$. Let $c : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be a measurable real function such that $c(x) > 0$ a.e. $x \in \Omega$. We define

$$L_{c(x)}^{p(x)}(\Omega) = \left\{ u \mid u : \Omega \to \mathbb{R} \text{ is measurable}, \int_\Omega c(x)\,|u(x)|^{\,p(x)}\,dx < \infty, c(x) > 0 \right\},$$

with the norm norm

$$|u|_{L^{p(x)}_{c(x)}(\Omega)} := |u|_{c(x),p(x)} = \inf\left\{\lambda > 0 : \int_{\Omega} c(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\},$$

then $L^{p(x)}_{c(x)}(\Omega)$ is a Banach space which has similar properties with the usual variable exponent Lebesgue spaces. The modular of this space is $\rho_{(c(x),p(x))}(u) : L^{p(x)}_{c(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{(c(x),p(x))}(u) = \int_{\Omega} c(x)|u(x)|^{p(x)} dx.$$

Proposition 4. *[16] If $p^+ < \infty$ and $u_n \in L^{p(x)}_{c(x)}(\Omega), n = 1, 2, ...$ we have*

$$(i)\ \lim_{n\to\infty} |u_n|_{(c(x),p(x))} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{(c(x),p(x))}(u_n) = 0,$$

$$(ii)\ |u_n|_{(c(x),p(x))} \to \infty \Leftrightarrow \rho_{(c(x),p(x))}(u_n) \to \infty.$$

Proposition 5. *[12, 16] (i)If $p^- > 1$ and $p^+ < \infty$ then, the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, $W_0^{1,p(x)}(\Omega)$ and $L^{p(x)}_{c(x)}(\Omega)$ are separable and reflexive Banach spaces;*

(ii) Assume that the boundary of Ω possesses the cone property and $p(x) \in C_+(\overline{\Omega})$. If $q(x) \in C_+(\overline{\Omega})$ and $q(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$, for all $x \in \overline{\Omega}$, then there is compact and continuous embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega),$$

also there is a constant $c > 0$ such that

$$|u|_{q(x)} \leq c\|u\|,$$

for all $u \in W_0^{1,p(x)}(\Omega)$.

Proposition 6. *[10] Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\Omega)$ and $1 \leq p(x)q(x) \leq \infty$ for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega), u \neq 0$. Then*

$$(i)\ \ |u|_{p(x)q(x)} \leq 1 \implies |u|^{p^+}_{p(x)q(x)} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)},$$

$$(ii)\ \ |u|_{p(x)q(x)} \geq 1 \implies |u|^{p^-}_{p(x)q(x)} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)}.$$

In particular, if $p(x) = p$ *is constant then* $||u|^p|_{pq(x)} = |u|^p_{pq(x)}$.

We give the properties of the $(p_1(x), p_2(x))$-Laplace operator

$$-(\triangle_{p_1(x)} + \triangle_{p_2(x)})u := -\mathrm{div}\left(|\nabla u|^{p_1(x)-2}\nabla u\right) - \mathrm{div}\left(|\nabla u|^{p_1(x)-2}\nabla u\right)$$

We consider the following functional,

$$\Lambda(u) = \int_\Omega \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_\Omega \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx, \text{ for all } u \in X,$$

where $X := W_0^{1,p_1(x)}(\Omega) \cap W_0^{1,p_2(x)}(\Omega)$ with its norm given by

$$\|u\| := \|u\|_{p_1(x)} + \|u\|_{p_2(x)}$$

for all $u \in X$. We denote

$$p_M(x) = \max\{p_1(x), p_2(x)\}, \quad p_m(x) = \max\{p_1(x), p_2(x)\}, \text{ for all } u \in X,$$

It is easy to see that $p_M, p_m \in C_+\left(\overline{\Omega}\right)$.

Remark 1. *(i)* $(X, \|u\|)$ *is a separable and reflexive Banach spaces.*

(ii) If $q(x) \in C_+\left(\overline{\Omega}\right)$ *such that* $q(x) < p_M^*(x)$ *for any* $x \in \overline{\Omega}$, *we have* $X := W_0^{1,p_1(x)}(\Omega) \cap W_0^{1,p_2(x)}(\Omega) = W_0^{1,p_M(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ *the embedding is continuous and compact.*

3. The main results

In this paper, we consider the problem (1) in the particular case

$$f(x,u) = m(x)|u|^{r(x)-2} u + n(x)|u|^{s(x)-2} u$$

where $r(x), s(x), p_M(x), p_m(x) \in C_+\left(\overline{\Omega}\right)$ with

$$r^- \le r^+ < p_m^- < p_M^+ < s^- \le s^+ < s^*(x), \text{ for any } x \in \overline{\Omega} \qquad (\mathbf{3})$$

and the following conditions hold:

(M) $m \in L^{\beta(x)}(\Omega)$ and $\beta \in C_+\left(\overline{\Omega}\right)$ such that $\frac{Np(x)}{Np(x)-r(x)(N-p(x))} < \beta(x) < \frac{p(x)}{p(x)-r(x))}$ for all $x \in \Omega$.

(N) $n \in L^{\alpha(x)}(\Omega)$ and $\alpha \in C_+\left(\overline{\Omega}\right)$ such that $\frac{p(x)}{p(x)-s(x)} < \alpha(x) < \frac{Np(x)}{Np(x)-s(x)(N-p(x))}$ for all $x \in \Omega$.

Define the energy functional $I : X \to \mathbb{R}$ associated with (1) by

$$I(u)=\int_{\Omega}\frac{1}{p_1(x)}|\nabla u|^{p_1(x)}dx+\int_{\Omega}\frac{1}{p_2(x)}|\nabla u|^{p_2(x)}dx$$
$$-\int_{\Omega}\frac{m(x)}{r(x)}|\nabla u|^{r(x)}dx-\int_{\Omega}\frac{n(x)}{s(x)}|\nabla u|^{s(x)}dx,$$

where

$$\Lambda(u)=\int_{\Omega}\frac{1}{p_1(x)}|\nabla u|^{p_1(x)}dx+\int_{\Omega}\frac{1}{p_2(x)}|\nabla u|^{p_2(x)}dx,$$

and

$$J(u)=\int_{\Omega}\frac{m(x)}{r(x)}|\nabla u|^{r(x)}dx-\int_{\Omega}\frac{n(x)}{s(x)}|\nabla u|^{s(x)}dx.$$

It is obvious that $\Lambda\in C^1(X,\mathbb{R})$. Denote $L=\Lambda':X\to X^*$, then

$$\langle L(u),\upsilon\rangle=\int_{\Omega}|\nabla u|^{p_1(x)-2}\nabla u\nabla\upsilon dx+\int_{\Omega}|\nabla u|^{p_2(x)-2}\nabla u\nabla\upsilon dx,\ \text{for all } u,\upsilon\in X,$$

in which $\langle .,.\rangle$ is the dual pair between X and X^*. Furthermore, $J\in C^1(X,\mathbb{R})$ and

$$\langle J'(u),\upsilon\rangle=\int_{\Omega}m(x)|u|^{r(x)-2}u\upsilon dx+\int_{\Omega}n(x)|u|^{s(x)-2}u\upsilon dx,\ \text{for all } u,\upsilon\in X.$$

We say that $u\in X$ is a weak solution of (1) if

$$\int_{\Omega}|\nabla u|^{p_1(x)-2}\nabla u\nabla\upsilon dx+\int_{\Omega}|\nabla u|^{p_2(x)-2}\nabla u\nabla\upsilon dx$$
$$=\int_{\Omega}m(x)|u|^{r(x)-2}u\upsilon dx+\int_{\Omega}n(x)|u|^{s(x)-2}u\upsilon dx,$$

for all $\upsilon\in X$.

Theorem 1. *Suppose that conditions (**3**), (**M**) and (**N**) are satisfied. Then the problem (**1**) has at least two nontrivial weak solutions in X.*

Theorem 2. *Suppose that conditions (**3**), (**M**) and (**N**) are satisfied and the following condition;*

$$(\boldsymbol{M}_1) X \textit{is a Banach space}, I \in C^1(X, \mathbb{R}) \textit{ is an even functional},$$

*Then the problem (**1**) has infinitely many nontrivial weak solutions in X.*

Lemma 3. *(**Mountain-Pass lemma**) [24] Let X be a Banach space and $I \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition. Assume that $I(0) = 0$ and there exists a positive real number ρ aand $u \in X$ such that*

(i) $\|u\| > \rho, I(u) \leq I(0)$,

(ii) $\alpha = \inf\{I(u) : u \in X, \|u\| = \rho\} > 0.$

Put $G = \{\phi \in C([0,1], X) : \phi(0) = 0, \phi(1) = u\}$. Set $\beta = \inf\{\max I(\phi([0,1])) : \phi \in G\}$. Then, $\beta \geq \alpha$ and β is a critical value of I.

Lemma 4. *(**Symmetric Mountain-Pass lemma**) [24] Let X be an infinite dimensional real Banach space and let $I \in C^1(X, \mathbb{R})$ be even, satisfying the Palais-Smale condition and $I(0) = 0$. Suppose that*

(i) There exist two positive real numbers α and ρ such that

$$\inf_{u \in \partial B_\rho} I(u) \geq \alpha > 0,$$

where B_ρ is open ball in X of radius ρ centered at the origin and ∂B_ρ is its boundary.

(ii) For each finite dimensional linear subspace $X_1 \subset X$, the set

$$\{u \in X_1 : I(u) \geq 0\},$$

is bounded. Then, I has an unbounded sequence of critical values.

Lemma 5. *[13]*

(i) $L : X \to X^*$ is a continuous, bounded and strictly monotone operator,

(ii) L is a mapping of type (S_+), namely,

$$u_n \rightharpoonup u \quad \textit{in} \quad X \quad \textit{and} \quad \overline{\lim_{n \to \infty}} \langle L'(u_n), u_n - u \rangle \leq 0 \quad \textit{implies } u_n \to u \quad \textit{in} \quad X,$$

(iii) $L : X \to X^*$ is a homeomorphism.

Lemma 6. *Assume that the conditions (**3**), (**M**) and (**N**) are satisfied.*

(i) L is weakly lower semi-continuous from X to $\mathbb{R}$,

(ii) $I(u) = 0$,

(iii) There exist two positive real numbers ρ and α such that

$$\inf\{I(u) : u \in X, \|u\| = \rho\} > 0,$$

(iv) There exists $\psi \in X$ such that $\psi \geq 0, \psi \neq 0$ and $I(t\psi) < 0$, for $t > 0$ small enough,

(v) There exists $u \in X$ such that $\|u\| > \rho, I(u) \leq 0$,

(vi) The set

$$G = \{\varphi \in C([0,1], X) : \varphi(0) = 0, \varphi(1) = u\},$$

is not empty,

(vii) I satisfies Palais-Smale condition on X i.e. there exists a sequence $\{u_n\} \subset X$ which satisfies the properties;

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{in} \quad X^* \quad \text{as } n \to \infty,$$

possesses a convergent subsequence in X.

Proof. (i) Let $\{u_n\} \subset X$ be a sequence such that $u_n \rightharpoonup u$ in X. Using Lemma 5, we have

$$\Lambda \leq \lim_{n\to\infty} \inf \Lambda(u_n). \tag{4}$$

Furthermore, from Remark 1 (ii), the conditions (**3**), (**M**) and (**N**) the embedding

$$(a)\, X \hookrightarrow L^{r(x)}_{m(x)}(\Omega) \quad \text{and} \quad X \hookrightarrow L^{s(x)}_{n(x)}(\Omega),$$

are compact, see [[25],Theorems 2.7, 2.8]. Then, there exist two positive constants c_1 and c_2 such that

$$\int_\Omega m(x)\,|u|^{r(x)} \leq c_1 \left(\|u\|^{r^-} + \|u\|^{r^-}\right) \tag{5}$$

and

$$\int_\Omega n(x)\,|u|^{s(x)} \leq c_2 \left(\|u\|^{s^-} + \|u\|^{s^-}\right) \tag{6}$$

for all $x \in X$. On the other hand, there are follows that

$$(b)\, u_n \to u \;\; \text{in} \;\; L^{r(x)}_{m(x)}(\Omega) \quad \text{and} \quad u_n \to u \;\; \text{in} \;\; L^{s(x)}_{n(x)}(\Omega).$$

This fact together with the above inequality (6) implies

$$I(u) \leq \lim_{n\to\infty} \inf I(u_n).$$

Thus, I is weakly lower semi continuous.

(*ii*) This comes from the definition of I.

(*iii*) we can write the following inequality from the definition of I,

$$I(u) \geq \frac{1}{p_M^+}\int_\Omega (|\nabla u|^{p_1(x)}+|\nabla u|^{p_2(x)})dx - \frac{1}{r^-}\int_\Omega m(x)|\nabla u|^{r(x)}dx - \frac{1}{s^-}\int_\Omega n(x)|\nabla u|^{s(x)}dx.$$

Let $\|u\|<1$. Using together with inequality (5) and (6), we have

$$\begin{aligned} I(u) &\geq \frac{1}{p_M^+}\left(\|u\|_{p_1(x)}^{p_1^+}+\|u\|_{p_2(x)}^{p_2^+}\right) - \frac{c_3}{r^-}\|u\|^{r^-} - \frac{c_4}{s^-}\|u\|^{s^-} \\ &\geq \frac{c_5}{p_M^+}\|u\|^{p_M^+} - \frac{c_3}{r^-}\|u\|^{r^-} - \frac{c_4}{s^-}\|u\|^{s^-} \\ &= \left(\frac{c_5}{p_M^+} - \frac{c_3}{r^-}\|u\|^{r^- - p_M^+} - \frac{c_4}{s^-}\|u\|^{s^- - p_M^+}\right)\|u\|^{p_M^+}. \end{aligned}$$

Let us define the function $\gamma : [0,1] \to \mathbb{R}$ by

$$\gamma(t) = \frac{c_5}{p_M^+} - \frac{c_3}{r^-}t^{r^- - p_M^+} - \frac{c_4}{s^-}t^{r^- - p_M^+}.$$

Since γ is positive in a neighbourhood of the origin, for example, for a fixed

$$t_0 \in \left(0, \left(\frac{r^-}{c_3 p_M^+}\right)^{\frac{1}{r^- - p_M^+}}\right),$$

the conclusion of the lemma follows at once.

(*iv*) Let $\psi \in C_0^\infty(\Omega), \psi \geq 0, \psi \neq 0$ and t sufficiently small. Moreover, we obtain the following inequality

$$I(t\psi) \leq \frac{1}{p_m^-}t^{p_M^+}\int_\Omega (|\nabla\psi|^{p_1(x)}+|\nabla\psi|^{p_2(x)})dx$$

$$-\frac{t^{r^-}}{r^-}\int_\Omega m(x)|\psi|^{r(x)}dx - \frac{t^{s^-}}{s^-}\int_\Omega n(x)|\psi|^{s(x)}dx.$$

Since $r^- < p_M^+ < s^-$, we conclude that $I(t\psi) < 0$.

(v) If choose $t > 1$ sufficiently large and use assumptions of Lemma 6, we obtain from (iv) that $I(u) \leq 0$.

(vi) If we consider the function $\varphi \in C([0,1], X)$ defined by $\varphi(t) = tu$ for every $t \in [0,1]$, it is clear that $\varphi \in G$ and $G \neq \varnothing$.

(vii) Assume that $\{u_n\} \subset X$ is a sequence which satisfies the following properties:

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{in} \quad X^* \quad \text{as } n \to \infty, \tag{7}$$

where X^* is dual space of X and c is a positive constant. We prove that $\{u_n\}$ possesses a convergent subsequence. First, we show that $\{u_n\}$ is bounded in X. We assume by contradiction $\|u_n\| \to \infty$ as $n \to \infty$. By using relation (7), and considering $\|u_n\| > 1$, for n large enough, we obtain

$$\begin{aligned} c &\geq I(u_n) - \frac{1}{s^-}\langle I'(u_n), u_n\rangle \\ &= \Lambda(u_n) - J(u_n) - \frac{1}{s^-}\int_\Omega |\nabla u_n|^{p_1(x)}\,dx - \frac{1}{s^-}\int_\Omega |\nabla u_n|^{p_2(x)}\,dx \\ &\quad + \frac{1}{s^-}\int_\Omega m(x)|u_n|^{r(x)}\,dx + \frac{1}{s^-}\int_\Omega n(x)|u_n|^{s(x)}\,dx \\ &\geq c_6\left(\frac{1}{p_M^+} - \frac{1}{s^-}\right)\left(\|u_n\|_{p_1(x)}^{p_1^-} + \|u_n\|_{p_2(x)}^{p_2^-}\right) \\ &\quad - \left(\frac{1}{r^-} - \frac{1}{s^-}\right)\int_\Omega m(x)|u_n|^{r(x)}\,dx. \end{aligned}$$

Moreover, by (4), we can write

$$c + \left(\frac{1}{r^-} - \frac{1}{s^-}\right)\|u_n\|^{r^+} \geq c_6\left(\frac{1}{p_M^+} - \frac{1}{s^-}\right)\|u_n\|^{p_m^-}.$$

Since $r^+ < p_m^-$ and $p_M^+ < s^-$, the sequence $\{u_n\}$ is bounded in X. Therefore, there exists a subsequence, again denoted by $\{u_n\}$, and $u \in X$ such that $u_n \rightharpoonup u$ in X. By using relation (7), the embedding (a) and (b), we can write

$$\langle I'(u_n) - I'(u), u_n - u\rangle \to 0 \quad \text{as} \quad n \to \infty.$$

On the other hand, we have

$$\int_{\Omega}\left(|\nabla u_n|^{p_1(x)-2}\nabla u_n-|\nabla u|^{p_1(x)-2}\nabla u\right)(\nabla u_n-\nabla u)\,dx \tag{8}$$

$$+\int_{\Omega}\left(|\nabla u_n|^{p_2(x)-2}\nabla u_n-|\nabla u|^{p_2(x)-2}\nabla u\right)(\nabla u_n-\nabla u)\,dx$$

$$=\langle I'(u_n),u_n-u\rangle+\int_{\Omega}\left(m(x)|u_n|^{r(x)-2}u_n-m(x)|u|^{r(x)-2}u\right)(u_n-u)\,dx$$

$$+\int_{\Omega}\left(n(x)|u_n|^{s(x)-2}u_n-n(x)|u|^{s(x)-2}u\right)(u_n-u)\,dx,$$

Using $\{u_n\}$ converges strongly to u in $L^{r(x)}_{m(x)}(\Omega)$, Proposition 1, Proposition 5 and Proposition 6, we have

$$\left|\int_{\Omega}\left(m(x)|u_n|^{r(x)-2}u_n-m(x)|u|^{r(x)-2}u\right)(u_n-u)\,dx\right|$$

$$\leq\left|\int_{\Omega}m(x)|u_n|^{r(x)-2}u_n(u_n-u)\,dx\right|+\left|\int_{\Omega}m(x)|u|^{r(x)-2}u(u_n-u)\,dx\right|$$

$$\leq c_7|m|_{\beta(x)}||u_n|^{r(x)-1}|_{\theta(x)}|u_n-u|_{m(x),r(x)}$$

$$+c_8|m|_{\beta(x)}||u|^{r(x)-1}|_{\theta(x)}|u_n-u|_{m(x),r(x)}$$

$$\leq c_9\|u_n\||u_n-u|_{m(x),r(x)}+c_{10}\|u\||u_n-u|_{m(x),r(x)},$$

where $\theta\in C_+\left(\overline{\Omega}\right)$ such that $\frac{1}{\beta(x)}+\frac{1}{\theta(x)}+\frac{1}{r(x)}=1$. Since $|u_n-u|_{m(x),r(x)}\to 0$ as $n\to\infty$, using Proposition 4, it follows that

$$\lim_{n\to\infty}\int_{\Omega}\left(m(x)|u_n|^{r(x)-2}u_n-m(x)|u|^{r(x)-2}u\right)(u_n-u)\,dx=0. \tag{9}$$

With similar arguments we can obtain that

$$\lim_{n\to\infty}\int_\Omega \left(n(x)|u_n|^{s(x)-2}u_n - n(x)|u|^{s(x)-2}u\right)(u_n-u)\,dx = 0. \tag{10}$$

Moreover, by using relation (8), (9) and (10), we get

$$\int_\Omega \left(|\nabla u_n|^{p_1(x)-2}\nabla u_n - |\nabla u|^{p_1(x)-2}\nabla u\right)(\nabla u_n - \nabla u)\,dx$$
$$+\int_\Omega \left(|\nabla u_n|^{p_2(x)-2}\nabla u_n - |\nabla u|^{p_2(x)-2}\nabla u\right)(\nabla u_n - \nabla u)\,dx = 0.$$

On the other hand, by using Lemma 5, we have

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n - \nabla u|^{p_1(x)}\,dx + \lim_{n\to\infty}\int_\Omega |\nabla u_n - \nabla u|^{p_2(x)}\,dx = 0,$$

and using the following inequality

$$|\nabla u_n|^{p_1(x)} + |\nabla u_n|^{p_2(x)} \geq |\nabla u_n|^{p_M(x)}, \quad \text{for all } \ x\in\Omega,$$

We get $\lim_{n\to\infty}\int_\Omega |\nabla u_n - \nabla u|^{p_M(x)}\,dx = 0$. It follows that $\|u_n - u\|_{p_M(x)} \to \infty$ as $n\to\infty$. The proof of Lemma 6 is complete.

4. PROOFS

Proof of Theorem 1. By Lemma 3 and Lemma 6, we deduce the existence of $u_1\in X$ as a nontrivial weak solution of (1). Now, we will prove that there exists a second weak solution $u_2\in X$ such that $u_1\neq u_2$

By Lemma 6, it follows that there exists on the boundary of the ball centered at the origin and of the ρ in X, denoted $B_\rho\subset X$, such that

$$\inf_{\partial B_\rho(0)} I > 0.$$

On the other hand, from Lemma 6 (iv), there exists $\varphi\in X$ such that $I(t\varphi)<0$ for all $t>0$ small enough. Moreover, we write for all $u\in X$

$$I(u) \geq \frac{c_5}{p_M^+}\|u\|^{p_M^+} - \frac{c_3}{r^-}\|u\|^{r^-} - \frac{c_4}{s^-}\|u\|^{s^-},$$

it follows that

$$-\infty < \underline{c} := \inf_{B_\rho(0)} I < 0.$$

So, we get

$$0 < \varepsilon < \inf_{\partial B_\rho(0)} I - \inf_{B_\rho(0)} I.$$

Applying Ekelands variational principle [11] to the functional $I : \overline{B_\rho(0)} \to \mathbb{R}$, we can find $u_\varepsilon \in \overline{B_\rho(0)}$ such that $u_\varepsilon \in B_\rho(0)$. Now, we define $\Phi : \overline{B_\rho(0)} \to \mathbb{R}$ by $\Phi(u) = I(u) + \varepsilon \|u - u_\varepsilon\|$. It is clear that u_ε is a minimum point of Φ, and this implies that $\|I'(u_\varepsilon)\| \leq \varepsilon$. So, we deduce that there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that $I(u_n) \to \underline{c}$ and $I'(u_n) \to 0$. Using the fact that I satisfies Palais-Smale condition on X, we conclude that $\{u_n\}$ converges strongly to u_2 in X. Thus, u_2 is a weak solution for (1) and considering the relation $0 > \underline{c} = I(u_2)$, it follows that u_2 is nontrivial.

Moreover, it is clear that $u_1 \neq u_2$ since

$$I(u_1) = \overline{c} > 0 > \underline{c} = I(u_2)$$

The proof is complete.

Proof of Theorem 2. In view of $(\mathbf{M}_1)$, I is even. So, in order to apply Lemma 4, it is enough to deduce that condition of Lemma 4 (ii) is holds. By Proposition 3 (ii), we can write

$$\begin{aligned} I(u_n) &\leq \frac{c_{11}}{p_m^-} \int_\Omega \left(|\nabla u_n|^{p_1(x)} + |\nabla u_n|^{p_2(x)} \right) dx - \frac{1}{s^+} \int_\Omega n(x) |u_n|^{s(x)} dx \\ &\leq \frac{c_{11}}{p_m^-} \|u_n\|^{p_M^+} - \frac{1}{s^+} \int_\Omega n(x) |u_n|^{s(x)} dx \end{aligned}$$

Let $u \in X$ be arbitrary but fixed. Set $\Omega = \Omega_< \cup \Omega_\geq$, where $\Omega_< := \{x \in \Omega : |u(x)| < 1\}$ and $\Omega_\geq := \Omega \diagup \{x \in \Omega : |u(x)| < 1\}$. Then, we know

$$I(u_n) \leq \frac{c_{11}}{p_m^-} \|u_n\|^{p_M^+} - \frac{1}{s^+} \int_{\Omega \diagup \{x \in \Omega : |u(x)| < 1} n(x) |u_n|^{s(x)} dx$$

$$\leq \frac{c_{11}}{p_m^-} \|u\|^{p_M^+} - \frac{1}{s^+} \int_\Omega n(x) |u_n|^{s^-} dx + \frac{1}{s^+} \int_{\{x \in \Omega : |u(x)| < 1\}} n(x) |u_n|^{s^-} dx.$$

On the other hand, we can find a positive constant c_{12} such that

$$\int_{\{x\in\Omega:|u(x)|<1\}} n(x)\,|u_n|^{s^-}\,dx \le c_{12}.$$

Then

$$I(u_n) \le \frac{c_{11}}{p_m^-}\|u_n\|^{p_M^+} - \frac{1}{s^+}\int_\Omega n(x)\,|u_n|^{s^-}\,dx + c_{12},$$

for all $u \in X$. The functional $|.|_{n(x),s^-} : X \to \mathbb{R}$ defined by

$$|u|_{n(x),s^-} = \left(\int_\Omega n(x)\,|u|^{s^-}\,dx\right)^{\frac{1}{s^-}}$$

is a norm in X. Since X_1 is a finite dimensional subspace of X, the norms $|.|_{n(x),s^-}$ and $\|.\|$ are equivalent. Thus, there exists a positive constant c_{13} such that

$$\|u\| \le c_{13}\,|u|_{n(x),s^-}\,,$$

for all $u \in X_1$. So, we obtain

$$I(u_n) \le \frac{c_{11}}{p_m^-}\|u_n\|^{p_M^+} - \frac{c_{13}}{s^+}\|u_n\|^{s^-} + c_{12},$$

for all $u \in X_1$. Since $p_M^+ < s^-$, $\{u \in X : I(u) \ge 0$ is bounded. Therefore, I has an unbounded sequence of critical values in X. Consequently, I possesses infinitely many critical points in X.

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