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$\mathcal{I}$-ASYMPTOTICALLY LACUNARY EQUIVALENT SET SEQUENCES DEFINED BY A MODULUS FUNCTION

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Abstract. Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be a non-trivial ideal, $\theta = (k_r)$ be a lacunary sequence and f be a modulus function. Our aim in this study is to introduce some new notions such that $\mathcal{I}_W(f)-$asymptotic equivalence, $\mathcal{I}_W(w_f)-$asymptotic equivalence and $\mathcal{I}_W(N_\theta^f)-$asymptotic equivalence for set sequences. We also prove some inclusion theorems.

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1. Introduction

Asymptotic equivalence was introduced by Pobyvanets [24] and Marouf extended Pobyvanets's work [20]. Patterson, Savaş and some other authors studied on this concept and they extended asymptotic equivalence to asymptotic statistical equivalence and asymptotic lacunary statistical equivalence. [21, 22]

Das, Savaş and Ghosal in [7] introduced $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence with ideal. Also in [25], $\mathcal{I}$-asymptotically statistical equivalent and $\mathcal{I}$-asymptotically lacunary statistical equivalent sequences were studied.

Wijsman statistical convergence which is implementation of the concept of statistical convergence to sequences of sets presented by Nuray and Rhoades [18]. After this definition, Ulusu and Nuray [28] introduced Wijsman lacunary statistical convergence of set sequences. In [29] they also defined asymptotically lacunary statistical equivalent set sequences and presented theorems about asymptotic equivalence Wijsman sense. In addition, they also presented asymptotically equivalent (Wijsman sense) analogs of theorems in [29].

Recently, Kişi, Savaş and Nuray [12] introduced $\mathcal{I}$-asymptotically statistical equivalent and $\mathcal{I}-$asymptotically lacunary statistical equivalent set sequences.

In this paper we introduce the concepts of $\mathcal{I}_W(f)$-asymptotically equivalent, $\mathcal{I}_W(w_f)$-asymptotically equivalent and $\mathcal{I}_W(N_\theta^f)$-asymptotically equivalent set sequences and we present some natural inclusion theorems.

2. Definitions and Notations

First we recall the basic definitions and concepts (see [20],[30]). Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \sim y$).

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$

(ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$

(iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if

(i) $\emptyset \notin \mathcal{F}$

(ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$

(iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$

If $\mathcal{I}$ is a non-trivial ideal of $\mathbb{N}$, then the family of sets

$$\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$$

is a filter of $\mathbb{N}$ and it is called the filter associated with the ideal.

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal in $\mathbb{N}$. The sequence (x_n) of elements of $\mathbb{R}$ is said to be $\mathcal{I}-$convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$ the set

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \mathcal{I}.$$

Now we have some easy but important examples about $\mathcal{I}-$convergence.

Example 1. *Take for $\mathcal{I}$ class the $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is an admissible ideal and $\mathcal{I}_f$ $-$convergence coincides with the usual convergence.*

Example 2. *Denote by $\mathcal{I}_d$ the class of all $A \subset \mathbb{N}$ which has natural density zero. Then $\mathcal{I}_d$ is an admissible ideal and $\mathcal{I}_d$-convergence coincides with the statistical convergence.*

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, A).$$

Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k\to\infty} d(x, A_k) = d(x, A)$$

for each $x \in X$. In this case we write $W - \lim A_k = A$.

Let (X, ρ) a metric space. For any non-empty closed subsets A_k of X, we say that the sequence $\{A_k\}$ is bounded if

$$\sup_k d(x, A_k) < \infty$$

for each $x \in X$. In this case we write $\{A_k\} \in L_\infty$.

Let (X, ρ) a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman statistical convergent to A if $\{d(x, A_k)\}$ is statistically convergent to $d(x, A)$; i.e., for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case we write $st - \lim_W A_k = A$ or $A_k \to A(WS)$.
In [17] Nakano introduced the notion of a modulus function as follows: By a modulus function, we mean a function f from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if x=0;

(ii) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$;

(iii) f is increasing;

(iv) f is continuous from the right at 0.

It follows from that f must be continuous on $[0, 1)$. A modulus may be bounded or unbounded. Başarır [3], Maddox [19], Pehlivan [23] and many others used a modulus function f to define some new sequence spaces.

3. Main Results

For non-empty closed subsets A_k and B_k of X, define $d(x; A_k, B_k)$ as follows:

$$d(x; A_k, B_k) = \begin{cases} \dfrac{d(x, A_k)}{d(x, B_k)} & , \quad \text{if, } x \notin A_k \cup B_k \\[2ex] L & , \quad \text{if, } x \in A_k \cup B_k. \end{cases}$$

We begin with the following definitions.

Definition 1. *Let (X, ρ) be a metric space, $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non-trivial in $\mathbb{N}$. For any non-empty closed subsets A_k, $B_k \subseteq X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are said to be strongly asymptotically equivalent of multiple L (Wijsman sense) with respect to the ideal $\mathcal{I}$ provided that every $\varepsilon > 0$, for each $x \in X$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} |d(x; A_k, B_k) - L| \geq \varepsilon \right\} \in \mathcal{I},$$

$\left(\text{denoted by } A_k \overset{\mathcal{I}_W(w)}{\sim} B_k \right)$ *and simply strongly asymptotically equivalent with respect to the ideal $\mathcal{I}$, if $L = 1$.*

Definition 2. *Let (X, d) be a metric space, $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non-trivial in $\mathbb{N}$ and f be a modulus function. For non-empty closed subsets $A_k, B_k \subset X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are said to be f-asymptotically equivalent of multiple L (Wijsman sense) with respect to the ideal $\mathcal{I}$ provided that for each $\varepsilon > 0$ and for each $x \in X$,*

$$\{k \in \mathbb{N} : f(|d(x; A_k, B_k) - L|) \geq \varepsilon\} \in \mathcal{I}.$$

$\left(\text{denoted by } A_k \overset{\mathcal{I}_W(f)}{\sim} B_k \right)$ *and simply f-asymptotically equivalent(Wijsman sense) with respect to the ideal I, if $L = 1$.*

Definition 3. *Let (X, d) be a metric space, $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non-trivial in $\mathbb{N}$ and f be a modulus function. For any non-empty closed subsets A_k, $B_k \subseteq X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are said to be strongly f-asymptotically equivalent of multiple L (Wijsman sense) with respect to the ideal I provided that every $\varepsilon > 0$, for each $x \in X$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} f(|d(x; A_k, B_k) - L|) \geq \varepsilon \right\} \in \mathcal{I},$$

$\left(\text{denoted by } A_k \overset{\mathcal{I}_W(w_f)}{\sim} B_k \right)$ *and simply strongly f-asymptotically equivalent with respect to the ideal $\mathcal{I}$, if $L = 1$.*

Definition 4. *Let (X, d) be a metric space. For any non-empty closed subsets A_k, $B_k \subseteq X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are said to be strongly f-asymptotically lacunary equivalent of multiple L (Wijsman sense) with respect to the ideal $\mathcal{I}$ provided that for each $\varepsilon > 0$ and for each $x \in X$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f(|d(x; A_k, B_k) - L|) \geq \varepsilon \right\} \in \mathcal{I},$$

$\left(\text{denoted by } A_k \overset{N_\theta^f(I_W)}{\sim} B_k\right)$ and simply strongly f-asymptotically lacunary equivalent with respect to the ideal $\mathcal{I}$, if $L = 1$.

Lemma 1. *Let f be a modulus function and let $0 < \delta < 1$. Then for $y \neq 0$ and each $\left(\frac{x}{y}\right) > \delta$, we have $f\left(\frac{x}{y}\right) \leq \frac{2f(1)}{\delta}\left(\frac{x}{y}\right)$.*

Theorem 2. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non-trivial in $\mathbb{N}$ and f be a modulus function. Then,*

(i) If $A_k \overset{\mathcal{I}_W(w)}{\sim} B_k$ then $A_k \overset{\mathcal{I}_W\left(w_f\right)}{\sim} B_k$ and

(ii) $\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$, then $A_k \overset{\mathcal{I}_W(w)}{\sim} B_k \Leftrightarrow A_k \overset{\mathcal{I}_W\left(w_f\right)}{\sim} B_k$.

Proof. $(i)-$Let $A_k \overset{\mathcal{I}_W(w)}{\sim} B_k$ and $\varepsilon > 0$ be given. Choose $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$. Then we can write

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} f\left(\left|d\left(x;A_k,B_k\right)-L\right|\right) &= \frac{1}{n}\sum_{\substack{k=1\\ \left|d\left(x;A_k,B_k\right)-L\right|\leq\delta}}^{n} f\left(\left|d\left(x;A_k,B_k\right)-L\right|\right)\\ &+ \frac{1}{n}\sum_{\substack{k=1\\ \left|d\left(x;A_k,B_k\right)-L\right|>\delta}}^{n} f\left(\left|d\left(x;A_k,B_k\right)-L\right|\right).\end{aligned}$$

Moreover, using the definiton of the modulus function f, we have

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|d\left(x;A_k,B_k\right)-L\right|\right) < \varepsilon + \left(\frac{2f(1)}{\delta}\right)\frac{1}{n}\sum_{k=1}^{n}\left|d\left(x;A_k,B_k\right)-L\right|.$$

Thus, for any $\gamma > 0$,

$$\begin{aligned}&\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f\left(\left|d\left(x;A_k,B_k\right)-L\right|\right) \geq \gamma\right\}\\ &\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left(\left|d\left(x;A_k,B_k\right)-L\right|\right) \geq \frac{(\gamma-\varepsilon)\,\delta}{2f(1)}\right\}.\end{aligned}$$

Since $A_k \overset{\mathcal{I}_W(w)}{\sim} B_k$, it follows the later set, and hence , the first set in above expression to $\mathcal{I}$. This proves that $A_k \overset{\mathcal{I}_W\left(w_f\right)}{\sim} B_k$.

(ii) If $\lim_{t\to\infty}\dfrac{f(t)}{t}=\alpha>0$, then we have $f(t)\geq\alpha t$ for all $t\geq 0$. Suppose that $A_k \overset{\mathcal{I}_W(w_f)}{\sim} B_k$. Since

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} f\left(\left|d\left(x;A_k,B_k\right)-L\right|\right) &\geq \frac{1}{n}\sum_{k=1}^{n} \alpha\left(\left|d\left(x;A_k,B_k\right)-L\right|\right)\\ &= \alpha\left(\frac{1}{n}\sum_{k=1}^{n}\left|d\left(x;A_k,B_k\right)-L\right|\right).\end{aligned}$$

It follows that for each $\varepsilon>0$,we have

$$\begin{aligned}&\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}\left|d\left(x;A_k,B_k\right)-L\right|\geq\varepsilon\right\}\\ &\subseteq\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n} f\left(\left|d\left(x;A_k,B_k\right)-L\right|\right)\geq\alpha\varepsilon\right\}.\end{aligned}$$

Since $A_k \overset{\mathcal{I}_W(w_f)}{\sim} B_k$, it follows that the later set belongs to $\mathcal{I}$, and therefore, the theorem is proved.

Theorem 3. *Let $\mathcal{I}\subset 2^{\mathbb{N}}$ be a non-trivial in $\mathbb{N}$ and f be a modulus function. Then,*

(i) If $A_k \overset{\mathcal{I}_W(w_f)}{\sim} B_k$ then $A_k \overset{\mathcal{I}_W(S)}{\sim} B_k$ and

(ii) If f is bounded, then $A_k \overset{\mathcal{I}_W(w_f)}{\sim} B_k \Leftrightarrow A_k \overset{\mathcal{I}_W(S)}{\sim} B_k$.

Proof. $(i)-$Suppose $A_k \overset{I_W(w_f)}{\sim} B_k$ and $\varepsilon>0$ be given. Then we can write

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} f\left(\left|d\left(x;A_k,B_k\right)-L\right|\right) &\geq \frac{1}{n}\sum_{\substack{k=1\\ |d(x;A_k,B_k)-L|\geq\varepsilon}}^{n} f\left(\left|d\left(x;A_k,B_k\right)-L\right|\right)\\ &\geq \frac{f(\varepsilon)}{n}.\left|\left\{k\leq n:\left|d\left(x;A_k,B_k\right)-L\right|\geq\varepsilon\right\}\right|.\end{aligned}$$

Therefore, for any $\gamma>0$, we have

$$\begin{aligned}&\left\{n\in\mathbb{N}:\frac{1}{n}\left\{\left|k\leq n:\left|d\left(x;A_k,B_k\right)-L\right|\geq\varepsilon\right|\right\}\geq\frac{\gamma}{f(\varepsilon)}\right\}\\ &\subseteq\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n} f\left(\left|d\left(x;A_k,B_k\right)-L\right|\right)\geq\gamma\right\}.\end{aligned}$$

Since $A_k \overset{\mathcal{I}_W(w_f)}{\sim} B_k$, it follows that the later set belongs to $\mathcal{I}$, and therefore $A_k \overset{\mathcal{I}_W(S)}{\sim} B_k$.

(ii) Suppose that f is bounded and $A_k \overset{\mathcal{I}_W(S)}{\sim} B_k$. Since f is bounded there exists a real number M such that $\sup f(t) \leq M$. And for $\varepsilon > 0$, we can write

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} f\left(\left|d\left(x;A_k,B_k\right)-L\right|\right) &= \frac{1}{n}\left[\sum_{\substack{k=1\\ |d(x;A_k,B_k)-L|\geq\varepsilon}}^{n} f\left(\left|d\left(x;A_k,B_k\right)-L\right|\right)\right. \\ &+ \left.\sum_{\substack{k=1\\ |d(x;A_k,B_k)-L|<\varepsilon}}^{n} f\left(\left|d\left(x;A_k,B_k\right)-L\right|\right)\right] \\ &\leq \frac{M}{n}\left|\left\{k\leq n : \left|d\left(x;A_k,B_k\right)-L\right|\geq\varepsilon\right\}\right| + f\left(\varepsilon\right). \end{aligned}$$

Now if $\varepsilon \to 0$, the theorem is proved. Since $A_k \overset{\mathcal{I}_W(S)}{\sim} B_k$, it follows that the later set belongs to $\mathcal{I}$, and therefore $A_k \overset{\mathcal{I}_W(w_f)}{\sim} B_k$.

Theorem 4. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non-trivial in $\mathbb{N}$, $\theta = \{k_r\}$ be a lacunary sequence and f be a modulus function. If $\liminf_r q_r > 1$, then $A_k \overset{\mathcal{I}_W(w_f)}{\sim} B_k \Longrightarrow A_k \overset{\mathcal{I}_W(N_\theta^f)}{\sim} B_k$.*

Proof. Suppose that $\liminf_r q_r > 1$, then there exists a $\delta > 0$ such that $q_r = \frac{k_r}{k_{r-1}} \geq 1+\delta$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}.$$

Let $A_k \overset{\mathcal{I}_W(w_f)}{\sim} B_k$. For a sufficiently large r, we obtain the following;

$$\begin{aligned} \frac{1}{k_r}\sum_{k=1}^{k_r} f\left(\left|d\left(x;A_k,B_k\right)-L\right|\right) &\geq \frac{1}{k_r}\sum_{k\in I_r} f\left(\left|d\left(x;A_k,B_k\right)-L\right|\right) \\ &= \left(\frac{h_r}{k_r}\right)\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|d\left(x;A_k,B_k\right)-L\right|\right) \\ &\geq \left(\frac{\delta}{1+\delta}\right)\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|d\left(x;A_k,B_k\right)-L\right|\right). \end{aligned}$$

which gives for any $\varepsilon > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f\left(\left|d\left(x; A_k, B_k\right) - L\right|\right) \geq \varepsilon \right\}$$

$$\subseteq \left\{ r \in \mathbb{N} : \frac{1}{k_r} \sum_{k=1}^{k_r} f\left(\left|d\left(x; A_k, B_k\right) - L\right|\right) \geq \frac{\varepsilon.\delta}{1+\delta} \right\}.$$

Since $A_k \overset{\mathcal{I}_W(w_f)}{\sim} B_k$, it follows that the later set belongs to I, and therefore $A_k \overset{\mathcal{I}_W(N_\theta^f)}{\sim} B_k$.

Theorem 5. *Let (X, ρ) be a metric space. Let $I \subset P(\mathbb{N})$ be a non-trivial in $\mathbb{N}$, $\theta = \{k_r\}$ be a lacunary sequence, A_k, B_k be non-empty closed subsets of X; and f be a modulus function. Then,*

(i) If $A_k \overset{\mathcal{I}_W(N_\theta)}{\sim} B_k$, then $A_k \overset{\mathcal{I}_W(N_\theta^f)}{\sim} B_k$; and

(ii) $\lim_{t\to\infty} \dfrac{f(t)}{t} = \alpha > 0$, then $A_k \overset{\mathcal{I}_W(N_\theta)}{\sim} B_k \Longleftrightarrow A_k \overset{\mathcal{I}_W(N_\theta^f)}{\sim} B_k$.

Proof. The proof is similar to the proof of theorem 3.1.

Theorem 6. *Let (X, ρ) be a metric space, $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non-trivial in $\mathbb{N}$, $\theta = \{k_r\}$ be a lacunary sequence, A_k, B_k be non-empty closed subsets of X; and f be a modulus function. Then,*

(i) If $A_k \overset{\mathcal{I}_W(N_\theta^f)}{\sim} B_k$, then $A_k \overset{\mathcal{I}_W(S_\theta)}{\sim} B_k$; and

(ii) If f is bounded, then $A_k \overset{\mathcal{I}_W(N_\theta^f)}{\sim} B_k \Longleftrightarrow A_k \overset{\mathcal{I}_W(S_\theta)}{\sim} B_k$.

Proof. (i) Suppose that $A_k \overset{\mathcal{I}_W(N_\theta^f)}{\sim} B_k$ and $\varepsilon > 0$ be given. Since

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} f\left(\left|d\left(x; A_k, B_k\right) - L\right|\right) &\geq \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \left|d\left(x; A_k, B_k\right) - L\right| \geq \varepsilon}} f\left(\left|d\left(x; A_k, B_k\right) - L\right|\right) \\ &\geq f(\varepsilon) \frac{1}{h_r} \left|\left\{k \in I_r : \left|d\left(x; A_k, B_k\right) - L\right| \geq \varepsilon\right\}\right| \end{aligned}$$

If follows that for any $\gamma > 0$, if we denote sets

$$A(\varepsilon,\gamma) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \{ k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon \} \right| \geq \gamma \right\}$$

$$B(\varepsilon,\gamma) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f(|d(x; A_k, B_k) - L|) \geq \gamma f(\varepsilon) \right\}.$$

Then $A(\varepsilon,\gamma) \subset B(\varepsilon,\gamma)$. Since $A_k \overset{\mathcal{I}_W(N_\theta^f)}{\sim} B_k$, so $B(\varepsilon,\gamma) \in \mathcal{I}$. But then, by the definition of an ideal, $A(\varepsilon,\gamma) \in \mathcal{I}$, and therefore, $A_k \overset{\mathcal{I}_W(S_\theta)}{\sim} B_k$.

(ii) Suppose that f is bounded and let $A_k \overset{\mathcal{I}_W(S_\theta)}{\sim} B_k$. Since f is bounded there exists a positive real number M such that $|f(x)| \leq M$ for all $x \geq 0$. Further, using the fact

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} f(|d(x; A_k, B_k) - L|) &= \frac{1}{h_r} \Bigg[\sum_{\substack{k \in I_r \\ |d(x;A_k,B_k)-L| \geq \varepsilon}} f(|d(x; A_k, B_k) - L|) \\ &+ \sum_{\substack{k \in I_r \\ |d(x;A_k,B_k)-L| < \varepsilon}} f(|d(x; A_k, B_k) - L|) \Bigg] \\ &\leq \frac{M}{h_r} \left| \{ k \in I_r : |d(x; A_k, B_k) - L| \geq \varepsilon \} \right| + f(\varepsilon). \end{aligned}$$

Now if $\varepsilon \to 0$, the theorem is proved. Since $A_k \overset{\mathcal{I}_W(S_\theta)}{\sim} B_k$, it follows the later set, and hence, the first set in above expression to $\mathcal{I}$. This proves that $A_k \overset{\mathcal{I}_W(N_\theta^f)}{\sim} B_k$.

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