Acta Universitatis Apulensis No. 43/2015
ISSN: 1582-5329 pp. 19-31
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2015.43.02

GENERALIZED OSBORN LOOPS OF ORDER $4N$

A.O. Isere, J.O. Adéníran, T.G. Jaiyéọlá

Abstract. The smallest non-associative Osborn loop is of order 16. Attempts in the past to construct higher orders have been very difficult. In this work, we develop a new method of constructing examples of generalized Osborn loops of order $4n$. Two of such examples are presented. They are shown to be non-associative Osborn loops. These are further classified up to isomorphism to establish their existence as distinct Osborn loops of order $4n$.

2010 *Mathematics Subject Classification*: 20N05, 08A05.

Keywords: Osborn loops, classification, isomorphism.

1. Introduction

By a loop $G(\cdot)$ we shall mean a non-empty set G together with a binary operation $(\cdot)$ such that the following properties hold: (i) given $a, b \in G$ the equations $a \cdot x = b, y \cdot a = b$ have unique solutions x, y respectively, in G; (ii) $G(\cdot)$ possesses an identity element, i.e. there exists $e \in G$ such that $e \cdot x = x \cdot e = x$ for all $x \in G$ [27]. An overview of loop theory can be found in Jaiyéọlá [13].

A loop is called an Osborn loop [26, 3] if it obeys any of the following:

$$(x^{\lambda} \backslash y) \cdot zx = x(yz \cdot x) \quad (1)$$

or

$$x(yz \cdot x) = (x \cdot yE_x) \cdot zx \; \forall \; x, y, z \in G \quad (2)$$

where $E_x = R_x R_{x^{\rho}} = (L_x L_x^{\lambda})^{-1} = R_x L_x R_x^{-1} L_x^{-1}$

Among the class of Bol-Moufang type of loops is the Bol loop. A loop L is called a Bol loop if and only if:

$$(xy \cdot z)y = x(yz \cdot y) \qquad \text{for all } x, y, z \in L \tag{3}$$

Strictly speaking, (3) defines a right Bol loop. A left Bol-loop $(L, \cdot)$ is defined as:

$$x(y \cdot xz) = (x \cdot yx)z \qquad \text{for all } x, y, z \in L \tag{4}$$

A Bol loop refers to a left or right Bol loop. The loop that satisfies both (3) and (4) is called a Moufang loop. Therefore, the necessary and sufficient condition for a loop to be a Moufang loop is that the loop is both a left Bol loop and right Bol loop. The smallest order for which a non-associative finite Bol loop exist is 8. There are exactly six Bol loops of order 8 that are not associative These loops were classified by Burn [4]. Solarin and Sharma [30] determined and classified all Bol loops of order 12 that are not associative.

Every Moufang loop is a Bol loop. Therefore, a good knowledge of the classes of Moufang loops becomes indispensable in the classification of Bol loops. Chein [5, 6] found that Moufang loops of orders p, p^2, p^3 or pq (where p and q are primes) must be groups and by using combinatorial type methods discovered 13 Moufang loops of order ≤ 31. Purtill [28] has shown that Moufang loops of orders pqr and p^2q where p, q and r are distinct odd primes with $p \leq q \leq r$ are groups. See- [1] for detail.

It is to be noted that a Moufang loop is a variety of Osborn loops. Some of the earliest examples of infinite Osborn loops were constructed by Huthnance [8] in 1968. Other examples of Osborn loops can be found in Isere et al[9, 10]. Thus, examples of Osborn loops are still very few. These examples are presented in Huthnance [8], Isere et. al. [9, 10]. Some recent studies on this class of loops are by Adeniran and Isere [2], Jaiyéọlá [15, 17, 18], Jaiyéọlá and Adéníran [22, 19, 20], Jaiyéọlá et. al. [23]. The application of some identities in universal Osborn loops to cryptography were reported in Jaiyéọlá [14, 16], Jaiyéọlá and Adéníran [21].

The generalized Osborn loops of order $4n$ are a "k-construction" Osborn loops of order $4n$, where k is any integer. Given an integer k, we have a distinct Osborn loop of order $4n$ constructed in this way. Thus, a generalized Osborn loop gives k number of Osborn loops of order $4n$ constructed this way. In constructing Osborn loops in this way (as it is done in this work), 'a' and 'b' are non-negative variable integers while 'c' is a fixed integer. However, the combination $b+c$ is peculiar to Osborn loops constructed in this way. These loops are found to be non-universal Osborn loops except when $k = 1$. This work is aimed at developing a new method of constructing non-associative, generalized Osborn loops of order $4n$. Two of such examples are presented in the next section. They are shown to be non-associative Osborn loops. Furthermore, the constructed examples are classified upto isomorphism.

2. Main Results

2.1. Generalized Osborn Loops

Example 1. *Let* $I(\cdot) = C_{2n} \times C_2$, $I = \{(x^\alpha, y^\beta), 0 \leq \alpha \leq 2n-1, 0 \leq \beta \leq 1\}$ *such that the binary operation* $I(\cdot)$ *is defined as follows:*

$$(x^a, e) \cdot (x^b, y^\beta) = (x^{a+b}, y^\beta) \tag{5}$$

$$(x^a, y^\alpha) \cdot (x^b, e) = (x^{a+b}, y^\alpha) \tag{6}$$

$$(x^a, y^\alpha) \cdot (x^b, y^\beta) = (x^{a+b}, y^{\alpha+\beta}) \ if \ a \equiv 0(\text{mod } 2), b \equiv 0(\text{mod } 2) \tag{7}$$

$$= (x^{a+kb}, y^{\alpha+\beta}) \ if \ a \equiv 0(\text{mod } 2), b \equiv 1(\text{mod } 2) \tag{8}$$

$$(x^a, y^\alpha) \cdot (x^b, y^\beta) = (x^{a+kb}, y^{\alpha+k\beta}) \ if \ a \equiv 1(\text{mod } 2), b \equiv 1(\text{mod } 2) \tag{9}$$

$$(x^{b+c}, y^\delta) \cdot (x^a, y^\alpha) = (x^{a+b+c}, y^{\alpha+\delta}) \ if \ a \equiv 0(\text{mod } 2), b \equiv 0(\text{mod } 2) \tag{10}$$

$$(x^{b+c}, y^\delta) \cdot (x^a, y^\alpha) = (x^{a+kb+c}, y^{\alpha+\delta}) \ if \ a \equiv 0(\text{mod } 2), b \equiv 1(\text{mod } 2) \tag{11}$$

$$(x^{b+c}, y^{\beta+\gamma}) \cdot (x^a, y^\alpha) = (x^{b+c+ka}, y^{\beta+\gamma+k\alpha}) \ if \ a \equiv 1(\text{mod } 2), b \equiv 0(\text{mod } 2) \tag{12}$$

$$(x^{b+c}, y^{\beta+\gamma}) \cdot (x^a, y^\alpha) = (x^{c+ka+kb}, y^{\alpha+k\beta+\gamma}) \ if \ a \equiv 1(\text{mod } 2), b \equiv 1(\text{mod } 2) \tag{13}$$

where k *is any integer. Then* $I(\cdot)$ *is an Osborn loop of order* $4n$*, where* $n = 2, 3, 4, 6, 9, 12$ *and* 18.

Proof. We first show that $I(\cdot)$ satisfies Osborn identity below:

$$(X^\lambda \backslash Y) \cdot ZX = X(YZ \cdot X)$$

Now, we begin:

1. Let $X = (x^a, e)$; $Y = (x^b, e)$; $Z = (x^c, e)$, then by direct computations, we have
$$(X^\lambda \backslash Y) \cdot ZX = [(x^a, e)^\lambda \backslash (x^b, e)] \cdot [(x^c, e)(x^a, e)].$$
Let $[(x^a, e)^\lambda \backslash (x^b, e)] = (x^d e)$, then $(x^b, e) = (x^a, e)^\lambda (x^d, e) = (x^{d-a}, e)$ implies $b = d - a$; $d = a + b$.
$$\therefore \ [(x^a, e)^\lambda \backslash (x^b, e)] = (x^{a+b}, e) \Rightarrow (x^{a+b}, e) \cdot (x^{a+c}, e) = (x^{2a+b+c}, e).$$
Next, $X(YZ \cdot X) = (x^a, e)[(x^b, e)(x^c, e) \cdot (x^a, e)] = (x^a, e)[(x^{b+c}, e) \cdot (x^a, e)] =$
$$(x^a, e)(x^{a+b+c}, e) = (x^{2a+b+c}, e).$$

2. Let $X = (x^a, e)$; $Y = (x^b, e)$; $Z = (x^c, y^\gamma)$.

$$(X^\lambda \backslash Y) \cdot ZX = [(x^a, e)^\lambda \backslash (x^b, e)] \cdot [(x^c, y^\gamma)(x^a, e)].$$

Let $[(x^a, e)^\lambda \backslash (x^b, e)] = (x^d e)$, then $(x^b, e) = (x^a, e)^\lambda (x^d, e) = (x^{d-a}, e)$ implies $b = d - a; d = a + b$.

$$\therefore\ [(x^a, e)^\lambda \backslash (x^b, e)] = (x^{a+b}, e) \Rightarrow (x^{a+b}, e) \cdot (x^{a+c}, y^\gamma) = (x^{2a+b+c}, y^\gamma).$$

Next, $X(YZ \cdot X) = (x^a, e)[(x^b, e)(x^c, y^\gamma) \cdot (x^a, e)] = (x^a, e)[(x^{b+c}, y^\gamma) \cdot (x^a, e)] =$
$$(x^a, e)(x^{a+b+c}, y^\gamma) = (x^{2a+b+c}, y^\gamma).$$

3. Let $X = (x^a, e)$; $Y = (x^b, y^\beta)$; $Z = (x^c, e)$.

First Case: when b is even

$$(X^\lambda \backslash Y) \cdot ZX = [(x^a, e)^\lambda \backslash (x^b, y^\beta)] \cdot [(x^c, e)(x^a, e)].$$

Let $[(x^a, e)^\lambda \backslash (x^b, y^\beta)] = (x^d, y^\delta)$, then $(x^b, y^\beta) = (x^a, e)^\lambda (x^d, y^\delta) = (x^{d-a}, y^\delta)$ implies $b = d - a$; $d = a + b$ and $\delta = \beta$.

$$\therefore\ [(x^a, e)^\lambda \backslash (x^b, y^\beta)] = (x^{a+b}, y^\beta) \Rightarrow (x^{a+b}, y^\beta) \cdot (x^{a+c}, e) = (x^{2a+b+c}, y^\beta).$$

$$\text{Next, } X(YZ \cdot X) = (x^a, e)[(x^b, y^\beta)(x^c, e) \cdot (x^a, e)] =$$
$$(x^a, e)[(x^{b+c}, y^\beta) \cdot (x^a, e)] = (x^a, e)(x^{a+b+c}, y^\beta) = (x^{2a+b+c}, y^\beta).$$

The results of the remaining cases are established in the same way as above.

Second Case: when b is odd $(X^\lambda \backslash Y) \cdot ZX = (x^{2a+kb+c}, y^\beta)$ and $X(YZ \cdot X) = (x^{2a+kb+c}, y^\beta)$.

Let $X = (x^a, y^\alpha); Y = (x^b, e); Z = (x^c, e)$.

First Case: when a is even

$$(X^\lambda \backslash Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \backslash (x^b, e)] \cdot [(x^c, e)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \backslash (x^b, e)] = (x^d, y^\delta)$, the $(x^b, e) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{d-a}, y^{\delta-\alpha})$ implies $b = d - a$ and $0 = \delta - \alpha$; implies that $d = a + b, \delta = \alpha$.

$$\therefore\ [(x^a, y^\alpha)^\lambda \backslash (x^b, e)] = (x^{a+b}, y^\alpha) \Rightarrow (x^{a+b}, y^\alpha) \cdot (x^{a+c}, y^\alpha) = (x^{2a+b+c}, e).$$

$$\text{Next, } X(YZ \cdot X) = (x^a, y^\alpha)[(x^b, e)(x^c, e) \cdot (x^a, y^\alpha)] =$$
$$(x^a, y^\alpha)[(x^{b+c}, e) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)(x^{a+b+c}, y^\alpha). = (x^{2a+b+c}, e).$$

Second Case: when a is odd

$$(X^\lambda \backslash Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \backslash (x^b, e)] \cdot [(x^c, e)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \backslash (x^b, e)] = (x^d, y^\delta)$, then $(x^b, e) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{-(a+ka)}, y^{-\alpha})(x^d, y^\delta) = (x^{d-(ka)}, y^{\delta-\alpha})$ implies $b = d - (ka)$ and $0 = \delta - \alpha$; implies that $d = ka + b, \delta = \alpha$.

$$\therefore\ [(x^a, y^\alpha)^\lambda \backslash (x^b, e)] = (x^{ka+b}, y^\alpha)(x^{ka+b}, y^\alpha) \cdot (x^{a+c}, y^\alpha) = (x^{a+ka+b+c}, e).$$

$$\text{Next, } X(YZ \cdot X) = (x^a, y^\alpha)[(x^b, e)(x^c, e) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)[(x^{b+c}, e) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)(x^{ka+b+c}, y^\alpha). = (x^{a+ka+b+c}, e).$$

Let $X = (x^\alpha, e); Y = (x^b, y^\beta); Z = (x^c, y^\gamma)$.

First Case: when b is even

$$(X^\lambda \backslash Y) \cdot ZX = [(x^a, e)^\lambda \backslash (x^b, y^\beta)] \cdot [(x^c, y^\gamma)(x^a, e)].$$

Let $[(x^a, e)^\lambda \backslash (x^b, y^\beta)] = (x^d, y^\delta)$, then $(x^b, y^\beta) = (x^a, e)^\lambda (x^d, y^\delta) = (x^{d-a}, y^\delta)$ implies $b = d - a;\ d = a + b$ and $\delta = \beta$.

$$\therefore\ [(x^a, e)^\lambda \backslash (x^b, y^\beta)] = (x^{a+b}, y^\beta) \Rightarrow (x^{a+b}, y^\beta) \cdot (x^{a+c}, y^\gamma) = (x^{2a+b+c}, y^{\beta+\gamma}).$$

$$\text{Next, } X(YZ \cdot X) = (x^a, e)[(x^b, y^\beta)(x^c, y^\gamma) \cdot (x^a, e)] = (x^a, e)[(x^{b+c}, y^{\beta+\gamma}) \cdot (x^a, e)] = (x^a, e)(x^{a+b+c}, y^{\beta+\gamma}) = (x^{2a+b+c}, y^{\beta+\gamma}).$$

Second Case: when b is odd

$$(X^\lambda \backslash Y) \cdot ZX = [(x^a, e)^\lambda \backslash (x^b, y^\beta)] \cdot [(x^c, y^\gamma)(x^a, e)].$$

Let $[(x^a, e)^\lambda \backslash (x^b, y^\beta)] = (x^d y^\delta)$, then $(x^b, y^\beta) = (x^a, e)^\lambda (x^d, y^\delta) = (x^{d-a}, y^\delta)$ implies $b = d - a;\ d = a + b$ and $\delta = \beta$.

$$\therefore\ [(x^a, e)^\lambda \backslash (x^b, y^\beta)] = (x^{a+b}, y^\beta)(x^{a+b}, y^\beta) \cdot (x^{a+c}, y^\gamma) = (x^{2a+kb+c}, y^{\beta+\gamma}).$$

$$\text{Next, } X(YZ\cdot X) = (x^a,e)[(x^b,y^\beta)(x^c,y^\gamma)\cdot(x^a,e)] = (x^a,e)[(x^{b+c},y^{\beta+\gamma})\cdot(x^a,e)] = (x^a,e)(x^{a+kb+c},y^{\beta+\gamma}) = (x^{2a+kb+c},y^{\beta+\gamma}).$$

Let $X=(x^a,y^\alpha); Y=(x^b,e); Z=(x^c,y^\gamma)$.

First Case: when a is even

$$(X^\lambda\backslash Y)\cdot ZX = [(x^a,y^\alpha)^\lambda\backslash(x^b,e)]\cdot[(x^c,y^\gamma)(x^a,y^\alpha)].$$

Let $[(x^a,y^\alpha)^\lambda\backslash(x^b,e)] = (x^d,y^\delta)$, then $(x^b,e) = (x^a,y^\alpha)^\lambda(x^d,y^\delta) = (x^{d-a},y^{\delta-\alpha})$ implies $b=d-a$ and $0=\delta-\alpha$; implies that $d=a+b, \delta=\alpha$.

$$\therefore\ [(x^a,y^\alpha)^\lambda\backslash(x^b,e)] = (x^{a+b},y^\alpha) \Rightarrow (x^{a+b},y^\alpha)\cdot(x^{a+c},y^{\alpha+\gamma}) = (x^{2a+b+c},y^\gamma).$$

$$\text{Next, } X(YZ\cdot X) = (x^a,y^\alpha)[(x^b,e)(x^c,y^\gamma)\cdot(x^a,y^\alpha)] = (x^a,y^\alpha)[(x^{b+c},y^\gamma)\cdot(x^a,y^\alpha)] = (x^a,y^\alpha)(x^{a+b+c},y^{\alpha+\gamma}) = (x^{2a+b+c},y^\gamma).$$

Second Case: when a is odd

$$(X^\lambda\backslash Y)\cdot ZX = [(x^a,y^\alpha)^\lambda\backslash(x^b,e)]\cdot[(x^c,y^\gamma)(x^a,y^\alpha)].$$

Let $[(x^a,y^\alpha)^\lambda\backslash(x^b,e)] = (x^d,y^\delta)$, the $(x^b,e) = (x^a,y^\alpha)^\lambda(x^d,y^\delta) = (x^{-(ka)},y^{-\alpha})(x^d,y^\delta) = (x^{d-(ka)},y^{\delta-\alpha})$ implies $b=d-(ka)$ and $0=\delta-\alpha$; implies that $d=ka+b, \delta=\alpha$.

$$\therefore\ [(x^a,y^\alpha)^\lambda\backslash(x^b,e)] = (x^{ka+b},y^\alpha) \Rightarrow (x^{ka+b},y^\alpha)\cdot(x^{a+c},y^{\alpha+\gamma}) = (x^{a+ka+b+c},y^\gamma).$$

$$\text{Next, } X(YZ\cdot X) = (x^a,y^\alpha)[(x^b,e)(x^c,y^\gamma)\cdot(x^a,y^\alpha)] = (x^a,y^\alpha)[(x^{b+c},e)\cdot(x^a,y^\alpha)] = (x^a,y^\alpha)(x^{ka+b+c},y^{\alpha+\gamma}) = (x^{a+ka+b+c},y^\gamma).$$

Let $X=(x^a,y^\alpha)$; $Y=(x^b,y^\beta)$; $Z=(x^c,e)$.

First Case: when a and b are even

$$(X^\lambda\backslash Y)\cdot ZX = [(x^a,y^\alpha)^\lambda\backslash(x^b,y^\beta)]\cdot[(x^c,e)(x^a,y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \backslash (x^b, y^\beta)] = (x^d, y^\delta)$, then $(x^b, y^\beta) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{d-a}, y^\delta)$ implies $b = d - a; d = a + b$ and $\delta = \beta$.

$$\therefore\ [(x^a, y^\alpha)^\lambda \backslash (x^b, y^\beta)] = (x^{a+b}, y^{\alpha+\beta}) \Rightarrow (x^{a+b}, y^{\alpha+\beta}) \cdot (x^{a+c}, y^\alpha) = (x^{2a+b+c}, y^\beta)$$

$$\text{Next, } X(YZ \cdot X) = (x^a, y^\alpha)[(x^b, y^\beta)(x^c, e) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)[(x^{b+c}, y^\beta) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)(x^{a+b+c}, y^{\alpha+\beta}) = (x^{2a+b+c}, y^\beta).$$

Second Case: when a is odd and b is even

$$(X^\lambda \backslash Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \backslash (x^b, y^\beta)] \cdot [(x^c, e)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \backslash (x^b, y^\beta)] = (x^d y^\delta)$, then $(x^b, y^\beta) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{d-(ka)}, y^\delta)$ implies $b = d - (a + ka); d = ka + b$ and $\delta = \beta$.

$$\therefore\ [(x^a, y^\alpha)^\lambda \backslash (x^b, y^\beta)] = (x^{ka+b}, y^{\alpha+\beta}) \Rightarrow (x^{ka+b}, y^\beta) \cdot (x^{a+c}, e) = (x^{a+ka+b+c}, y^\beta).$$

$$\text{Next, } X(YZ \cdot X) = (x^a, y^\alpha)[(x^b, y^\beta)(x^c, e) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)[(x^{b+c}, y^\beta) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)(x^{ka+b+c}, y^\beta) = (x^{a+ka+b+c}, y^\beta).$$

Third Case: when a is even and b is odd

$$(X^\lambda \backslash Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \backslash (x^b, y^\beta)] \cdot [(x^c, e)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \backslash (x^b, y^\beta)] = (x^d, y^\delta)$, then $(x^b, y^\beta) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{d-a}, y^\delta)$ implies $b = d - a; d = a + b$ and $\delta = \beta$.

$$\therefore\ [(x^a, y^\alpha)^\lambda \backslash (x^b, y^\beta)] = (x^{a+b}, y^{\alpha+\beta}) \Rightarrow (x^{a+b}, y^{\alpha+\beta}) \cdot (x^{a+c}, y^\alpha) = (x^{2a+kb+c}, y^\beta).$$

$$\text{Next, } X(YZ \cdot X) = (x^a, y^\alpha)[(x^b, y^\beta)(x^c, e) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)[(x^{b+c}, y^\beta) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)(x^{a+kb+c}, y^{\alpha+\beta}) = (x^{2a+kb+c}, y^\beta).$$

Fourth Case: when a and b are odd

$$(X^\lambda \backslash Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \backslash (x^b, y^\beta)] \cdot [(x^c, e)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \backslash (x^b, y^\beta)] = (x^d y^\delta)$, then $(x^b, y^\beta) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{d-(ka)}, y^{\delta-\alpha})$ implies $b = d - (ka); d = ka + b$ and $\delta = \alpha + \beta$.

$$\therefore\ [(x^a, y^\alpha)^\lambda \backslash (x^b, y^\beta)] = (x^{ka+b}, y^{\alpha+\beta}) \Rightarrow (x^{ka+b}, y^{\alpha+\beta}) \cdot (x^{a+c}, y^\alpha) = (x^{a+ka+kb+c}, y^\beta).$$

$$\text{Next, } X(YZ \cdot X) = (x^a, y^\alpha)[(x^b, y^\beta)(x^c, e) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)[(x^{b+c}, y^\beta) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)(x^{ka+kb+c}, y^{\alpha+k\beta}) = (x^{a+ka+kb+c}, y^\beta).$$

Let $X = (x^a, y^\alpha)$; $Y = (x^b, y^\beta)$; $Z = (x^c, y^\gamma)$.

First Case: when a and b are even

$$(X^\lambda \backslash Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \backslash (x^b, y^\beta)] \cdot [(x^c, y^\gamma)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \backslash (x^b, y^\beta)] = (x^d, y^\delta)$, then $(x^b, y^\beta) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{d-a}, y^\delta)$ implies $b = d - a; d = a + b$ and $\delta = \beta$.

$$\therefore\ [(x^a, y^\alpha)^\lambda \backslash (x^b, y^\beta)] = (x^{a+b}, y^{\alpha+\beta}) \Rightarrow (x^{a+b}, y^{\alpha+\beta}) \cdot (x^{a+c}, y^{\alpha+\gamma}) = (x^{2a+b+c}, y^{\beta+\gamma}).$$

$$\text{Next, } X(YZ \cdot X) = (x^a, y^\alpha)[(x^b, y^\beta)(x^c, y^\gamma) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)[(x^{b+c}, y^{\beta+\gamma}) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)(x^{a+b+c}, y^{\alpha+\beta}) = (x^{2a+b+c}, y^{\beta+\gamma}).$$

Second Case: when a is odd and b is even

$$(X^\lambda \backslash Y) \cdot ZX = [(x^a, y^\alpha)^\lambda \backslash (x^b, y^\beta)] \cdot [(x^c, y^\gamma)(x^a, y^\alpha)].$$

Let $[(x^a, y^\alpha)^\lambda \backslash (x^b, y^\beta)] = (x^d y^\delta)$, then $(x^b, y^\beta) = (x^a, y^\alpha)^\lambda (x^d, y^\delta) = (x^{d-(a+ka)}, y^\delta)$ implies $b = d - (a + ka); d = a + ka + b$ and $\delta = \beta$.

$$\therefore\ [(x^a, y^\alpha)^\lambda \backslash (x^b, y^\beta)] = (x^{ka+b}, y^{\alpha+\beta}) \Rightarrow (x^{ka+b}, y^\beta) \cdot (x^{a+c}, y\alpha + \gamma) = (x^{a+ka+b+c}, y^\beta).$$

$$\text{Next, } X(YZ \cdot X) = (x^a, y^\alpha)[(x^b, y^\beta)(x^c, y\gamma) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)[(x^{b+c}, y^{\beta+\gamma}) \cdot (x^a, y^\alpha)] = (x^a, y^\alpha)(x^{ka+b+c}, y^{\beta+\gamma}) = (x^{a+ka+b+c}, y^{\beta+\gamma}).$$

Third Case: when a is even and b is odd

$$(X^\lambda \backslash Y)\cdot ZX = [(x^a,y^\alpha)^\lambda \backslash (x^b,y^\beta)]\cdot[(x^c,y^\gamma)(x^a,y^\alpha)].$$

Let $[(x^a,y^\alpha)^\lambda \backslash (x^b,y^\beta)] = (x^d,y^\delta)$, then $(x^b,y^\beta)=(x^a,y^\alpha)^\lambda(x^d,y^\delta)=(x^{d-a},y^\delta)$ implies $b=d-a;\ d=a+b$ and $\delta=\beta$.

$$\therefore\ [(x^a,y^\alpha)^\lambda \backslash (x^b,y^\beta)] = (x^{a+b},y^{\alpha+\beta}) \Rightarrow (x^{a+b},y^{\alpha+\beta})\cdot(x^{a+c},y^{\alpha+\gamma}) = (x^{2a+kb+c},y^{\beta+\gamma}).$$

$$\text{Next, } X(YZ\cdot X) = (x^a,y^\alpha)[(x^b,y^\beta)(x^c,y^\gamma)\cdot(x^a,y^\alpha)] = (x^a,y^\alpha)[(x^{b+c},y^{\beta+\gamma})\cdot(x^a,y^\alpha)] = (x^a,y^\alpha)(x^{a+kb+c},y^{\alpha+k\beta+\gamma}) = (x^{2a+kb+c},y^{\beta+\gamma}).$$

Fourth Case: when a and b is odd

$$(X^\lambda \backslash Y)\cdot ZX = [(x^a,y^\alpha)^\lambda \backslash (x^b,y^\beta)]\cdot[(x^c,y\gamma)(x^a,y^\alpha)].$$

Let $[(x^a,y^\alpha)^\lambda \backslash (x^b,y^\beta)] = (x^d y^\delta)$, then $(x^b,y^\beta)=(x^a,y^\alpha)^\lambda(x^d,y^\delta)=(x^{d-(ka)},y^{\delta-\alpha})$ implies $b=d-(ka);\ d=ka+b$ and $\delta=\alpha+\beta$.

$$\therefore\ [(x^a,y^\alpha)^\lambda \backslash (x^b,y^\beta)] = (x^{ka+b},y^{\alpha+\beta}) \Rightarrow (x^{ka+b},y^{\alpha+\beta})\cdot(x^{a+c},y^{\alpha+\gamma}) = (x^{a+ka+kb+c},y^{\beta+\gamma}).$$

$$\text{Next, } X(YZ\cdot X) = (x^a,y^\alpha)[(x^b,y^\beta)(x^c,e)\cdot(x^a,y^\alpha)] = (x^a,y^\alpha)[(x^{b+c},y^{\beta+\gamma})\cdot(x^a,y^\alpha)] = (x^a,y^\alpha)(x^{ka+kb+c},y^{\alpha+k\beta+\gamma}) = (x^{a+ka+kb+c},y^{\beta+\gamma}).$$

Since $(X^\lambda \backslash Y)\cdot ZX = X(YZ\cdot X)$ in all the 36 cases considered i.e. whenever $37\equiv 1(\mathrm{mod}\ 2n)$, then $n=2,3,4,6,9,12,18$. Also, (e,e) is the two sided identity.

Moreover, if $X=(x^a,e)$, then $X^{-1}=(x^{-a},e)$. If $X=(x^a,y^a)$ then

$$X^{-1}=(x^{-a},y^{-\alpha}) \text{ if } a=\text{even and } X^{-1}=(x^{-(ka)},y^{-\alpha}) \text{ if } a=\text{odd}.$$

Therefore, the inverses are defined. Also for non-associativity, let $X=(x^a,y^\alpha); Y=(x^b,y^\beta); Z=(x^c,y^\gamma)$ where a is an even integer and b an odd integer, then

$$(XY)Z = [(x^a,y^\alpha)(x^b,y^\beta)](x^c,y^\gamma) = (x^{a+kb},y^{\alpha+\beta})(x^c,y^\gamma) = (x^{a+kb+c},y^{\alpha+\beta+\gamma}) \text{ and}$$

$$X(YZ) = (x^a,y^\alpha)[(x^b,y^\beta)(x^c,y^\gamma)] = (x^a,y^\alpha)(x^{b+c},y^{\beta+\gamma}) = (x^{a+b+c},y^{\alpha+\beta+\gamma})$$

Therefore, $(XY)Z \neq X(YZ)$.

Remark 1. *Thus, the Example 1 is non-associative except when $n = 2$ which gives the group $C_4 \times C_2$. Generally, whenever $k = 1$ the examples become associative Osborn loops. Hence, they are non-associative Osborn loops of order $4n$, $n = 4, 6, 9, 12, 18$.*

Example 2. *Let $I(\cdot) = C_{2n} \times C_2$, $I = \{(x^\alpha, y^\beta), 0 \leq \alpha \leq 2n-1, 0 \leq \beta \leq 1\}$ such that the binary operation $(\cdot)$ is defined as follows:*

$$(x^a, e) \cdot (x^b, y^\beta) = (x^{a+b}, y^\beta) \tag{14}$$

$$(x^a, y^\alpha) \cdot (x^b, e) = (x^{a+b}, y^\alpha) \tag{15}$$

$$(x^a, y^\alpha) \cdot (x^b, y^\beta) = (x^{a+b}, y^{\alpha+\beta}) \ if \ a \equiv 0 (\text{mod } 2), b \equiv 0 (\text{mod } 2) \tag{16}$$

$$= (x^{a+b+kb}, y^{\alpha+\beta}) \ if \ a \equiv 0 (\text{mod } 2), b \equiv 1 (\text{mod } 2) \tag{17}$$

$$(x^a, y^\alpha) \cdot (x^b, y^\beta) = (x^{a+b+kb}, y^{\alpha+k\beta}) \ if \ a \equiv 1 (\text{mod } 2), b \equiv 1 (\text{mod } 2) \tag{18}$$

$$(x^{b+c}, y^\delta) \cdot (x^a, y^\alpha) = (x^{a+b+c}, y^{\alpha+\delta}) \ if \ a \equiv 0 (\text{mod } 2), b \equiv 0 (\text{mod } 2) \tag{19}$$

$$(x^{b+c}, y^\delta) \cdot (x^a, y^\alpha) = (x^{a+b+kb+c}, y^{\alpha+\delta}) \ if \ a \equiv 0 (\text{mod } 2), b \equiv 1 (\text{mod } 2) \tag{20}$$

$$(x^{b+c}, y^{\beta+\gamma}) \cdot (x^a, y^\alpha) = (x^{a+ka+b+kb+c}, y^{\alpha+k\beta+\gamma}) \ if \ a \equiv 1 (\text{mod } 2), b \equiv 1 (\text{mod } 2) \tag{21}$$

Then, $I(\cdot)$ is an Osborn loop of order $4n$, where $n = 4, 6, 9, 12$ and 18, and k any integer.

Proof. The proof is similar to that in Example 1 above.

2.2. Classification up to Isomorphism

Two loops shall be considered non-isomorphic if they contain different number of elements of the same order. Whenever, two loops contain the same number of elements we shall go further to consider the order of elements in their nuclei. If these coincide in both cases, we shall consider commutative patterns of both loops.

Theorem 1. *The Osborn loops in Examples 1 and 2 are non-isomorphic.*

Proof. **(i)** Example 1

$$(x^a, y^\alpha) \cdot (x^a, y^\alpha) = ((x^{2a}, y^{2\alpha})$$

$$(x^a, y^\alpha) \cdot (x^a, y^\alpha) = ((x^{2a}, y^{2\alpha}) = (e, e) \text{ if } a \equiv 0 (\text{mod } 2) \tag{22}$$

$$= (x^{a+ka}, y^{2\alpha}) = (e, e) \text{ if } a \equiv 1 (\text{mod } 2)$$

Obviously, the only possible solution to the equation (22) are $a = 0$ and $a = n$ i.e.$(x^{2n}, e) = (e, e)$ and $(e, y^\alpha) = (e, e)$. Therefore, Example 1 has 3 elements of order 2 whenever k is a positive odd number and 2 elements of order 2 whenever k is positive even number and $k = -2$ and above; $n + 3$ elements of order 2 whenever $k = -1$ and 2 elements whenever k is any negative number except $k = -1$.

(ii) Example 2

$$(x^a, y^\alpha) \cdot (x^a, y^\alpha) = ((x^{2a}, y^{2\alpha})$$

$$\begin{aligned} (x^a, y^\alpha) \cdot (x^a, y^\alpha) &= ((x^{2a}, y^{2\alpha}) = (e, e) \text{ if } a \equiv 0 (\text{mod } 2) \\ &= (x^{2a+ka}, y^{2\alpha}) = (e, e) \text{ if } a \equiv 1 (\text{mod } 2) \end{aligned} \tag{23}$$

Obviously, the only possible solution to the equation (23) are $a = 0$ and $a = n$ i.e.$(x^{2n}, e) = (e, e)$ and $(e, y^\alpha) = (e, e)$. When k is both positive and odd even integers,we have (x^n, e), (x^n, y^α) and (e, y^α) as elements of order 2. When $k = 0$, it has 3 elements of order 2. And when $k < 0$, it has 2 elements of order 2. Therefore, Examples 1 and 2 are non-isomorphic since they contain different number elements of the same order.

Acknowledgements. The first author wishes to express his profound gratitude and appreciation to the Management of Education Trust Found Academic Staff Training and Development-2009(ETF AST and D)for the grant given him to carry out this Research, as well as, to the management of Ambrose Alli University, Nigeria for their joint support of the grant.

References

[1] A.M. Asiru, *A study of the classification of finite Bol loops*, Ph.D thesis university of Agriculture, Abeokuta (2008).

[2] J.O. Ademiran, A. O. Isere, *Nuclear Automorphisms of a class of Osborn Loops*, Journal of the Nigerian Association of Mathematical Physics. 22 (2012), 5–8.

[3] A. S. Basarab, A.I. Belioglo, *UAI Osborn loops*, Quasigroups and loops. Mat. Issled. 51 (1979), 8–16.

[4] R. P. Burn, *Finite Bol loops*, Math. Camb. Phil. Soc. 84 (1978), 377-385.

[5] O. Chein, E.G. Goodaire, *Moufang loops with a unique non-identity commutator (associator, square)*, J. Alg. 130 (1990), 369–384.

[6] O. Chein, E.G. Goodaire, *Code loops are RA*2 *loops*, J. Alg. 130 (1990), 385–387.

[7] O. Chein, H.O. Pflugfelder, *The smallest Moufang Loop* Archiv Der Mathematik. 22 (1971) 573-576.

[8] E. D. Huthnance Jr., *A theory of generalised Moufang loops*, Ph.D. thesis, Georgia Institute of Technology (1968).

[9] A. O. Isere, J. O. Adeniran and A. R. T. Solarin, *Somes Examples Of Finite Osborn Loops*, Journal of Nigerian Mathematical Society, 31 (2012), 91-106.

[10] A. O. Isere, S.A. Akinleye and J. O. Adeniran, *On Osborn loops of order* $4n$, Acta Universitatis Apulensis. 37 (2014), 31-44.

[11] T.G. Jaiyeola, *The study of the universality of Osborn loops*, Ph.D. thesis, University of Agriculture, Abeokuta (2008).

[12] T.G. Jaiyeola, J.O. Adeniran, *Not Every Osborn loop is Universal*, Acta Math. Acad. Paed. Nviregvhaziensis 25, (2009), 189-190.

[13] T. G. Jaiyeola, *A study of new concepts in smarandache quasigroups and loops*, ProQuest Information and Learning(ILQ), Ann Arbor, USA, (2009) 127.

[14] T. G. Jaiyéọlá, *On three cryptographic identities in left universal Osborn loops*, Journal of Discrete Mathematical Sciences and Cryptography. 14, 1 (2011), 33–50. (DOI:10.1080/09720529.2011.10698322).

[15] T. G. Jaiyéọlá, *Osborn loops and their universality*, Scientific Annals of "Al.I. Cuza" University of Iasi. 58, 2 (2012), 437–452.

[16] T. G. Jaiyéọlá, *On two cryptographic identities in universal Osborn loops*, Journal of Discrete Mathematical Sciences and Cryptography. 16, 2-3 (2013), 95-116. (DOI-10.1080/09720529.2013.821371).

[17] T. G. Jaiyéọlá, *New identities in universal Osborn loops II*, Algebras, Groups and Geometries. 30, 1 (2013), 111-126.

[18] T. G. Jaiyéọlá, *On some simplicial complexes of universal Osborn loops*, Analele Universitatii De Vest Din Timisoara, Seria Matematica-Informatica. 52, 1 (2014), 65–79. DOI: 10.2478/awutm-2014-0005.

[19] T. G. Jaiyéọlá, J. O. Adéníran, *Not every Osborn loop is universal*, Acta Mathematica Academiae Paedagogiace Nyregyhziensis. 25, 2 (2009), 189–190.

[20] T. G. Jaiyéọlá, J. O. Adéníran, *Loops that are isomorphic to their Osborn loop isotopes(G-Osborn loops)*, Octogon Mathematical Magazine. 19, 2 (2011), 328–348.

[21] T. G. Jaiyéọlá and J. O. Adéníran, *On another two cryptographic identities in universal Osborn loops*, Surveys in Mathematics and its Applications. 5 (2010), 17–34.

[22] T. G. Jaiyéọlá, J. O. Adéníran, *A new characterization of Osborn-Buchsteiner loops*, Quasigroups And Related Systems. 20, 2 (2012), 233–238.

[23] T. G. Jaiyéọlá , J. O. Adéníran and A. R. T. Sòlárìn, *Some necessary conditions for the existence of a finite Osborn loop with trivial nucleus*, Algebras, Groups and Geometries. 28, 4 (2011), 363–380.

[24] T. G. Jaiyeola, *some simplicial complexes of universal Osborn loops*, Analele Universitatii De Vest Din Timisoara, Seria Matematica-Informatica. 52, 1 (2014), 65-79. (DOI: 10.2478/awutm-2014-0005).

[25] M. K. Kinyon, *A survey of Osborn loops*, Milehigh conference on loops, quasigroups and non-associative systems, University of Denver, Denver, Colorado (2005).

[26] J. M. Osborn, *Loops with the weak inverse property*, Pac. J. Math. 10 (1961), 295–304.

[27] H. O. Pflugfelder, *Quasigroups and loops : Introduction*, Sigma series in Pure Math. Heldermann Verlag, Berlin, 7 (1990), 147.

[28] M. Purtill, *On Moufang loops of order the product of three primes*, J. Algebra. 112 (1988), 122-128.

[29] A.R.T. Solarin, B.L. Sharma, *Some Examples of Bol Loops*, Acta Universitatis Carolinae-Mathematica et Physics. 25, 1 (1983), 59-68

[30] A.R.T. Solarin, B.L. Sharma, *Bol Loop of order* 12, An, Smitt. Univ. "‘Al. I ciza"’,Iasi Sect I a Mat (NS) 29 (1983), 69-80.

Abednego Orobosa Isere
Department of Mathematics,
Ambrose Alli University,
Ekpoma 310001, Nigeria
abednis@yahoo.co.uk
isereao@aauekpoma.edu.ng

John Olusola Adéníran
Mathematics Programme
National Mathematical Centre,
Abuja, Nigeria
ekenedilichineke@yahoo.com
adeniranoj@nmcabuja.org
Permanent Address:Department of Mathematics,
Federal University of Agriculture,
Abeokuta 110101, Nigeria.

Temitope Gbolahan Jaiyéọlá
Department of Mathematics,
Obafemi Awolowo University,
Ile Ife 220005, Nigeria.
jaiyeolatemitope@yahoo.com
tjayeola@oauife.edu.ng