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CERTAIN CLASS OF ANALYTIC FUNCTIONS WITH VARYING ARGUMENTS DEFINED BY SĂLĂGEAN DERIVATIVE

Á.O. Páll-Szabó, O. Engel

Abstract. In this paper we derive some results for certain new class of analytic functions with varying arguments defined by using Sălăgean derivative.

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Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic and univalent in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

We define the differential operator $\mathscr{D}^n : A \to A$, n positive integer, by

$$\mathscr{D}^0 f(z) = f(z),$$

$$\mathscr{D}^1 f(z) = \mathscr{D} f(z) = z f'(z),$$

$$\mathscr{D}^n f(z) = \mathscr{D}\left(\mathscr{D}^{n-1} f(z)\right). \tag{2}$$

We note that the differential operator $\mathscr{D}^n$ was introduced by Sălăgean, [6].

$$\mathscr{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k. \tag{3}$$

Definition 1. Let f and g be analytic functions in U. We say that the function f is subordinate to the function g, if there exist a function w, which is analytic in U and $w(0) = 0; |w(z)| < 1; z \in U$, such that $f(z) = g(w(z)); \forall z \in U$. We denote by $\prec$ the subordination relation.

Definition 2. For $\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}_0$ let $S(n, \lambda, A, B)$ denote the subclass of $\mathcal{A}$ which contain functions $f(z)$ of the form (1) such that

$$(1-\lambda)(\mathscr{D}^n f(z))' + \lambda(\mathscr{D}^{n+1} f(z))' \prec \frac{1+Az}{1+Bz}. \tag{4}$$

Attiya and Aouf defined in [2] the class $\mathscr{R}(n, \lambda, A, B)$ with a condition like (4), but there instead of the operator $\mathscr{D}$ they used the Ruscheweyh operator $\mathscr{R}$, where

$$\mathscr{R}^n f(z) = z + \sum_{k=2}^{\infty} \binom{n+k-1}{n} a_k z^k.$$

Definition 3. [3][8]A function $f(z)$ of the form (1) is said to be in the class $V(\theta_k)$ if $f \in A$ and $arg(a_k) = \theta_k$,$\forall k \geq 2$. If $\exists \delta \in \mathbb{R}$ such that
$\theta_k + (k-1)\delta \equiv \pi (mod\ 2\pi), \forall k \geq 2$ then $f(z)$ is said to be in the class $V(\theta_k, \delta)$. The union of $V(\theta_k, \delta)$ taken over all possible sequences $\{\theta_k\}$ and all possible real numbers δ is denoted by V.

Let $VS(n, \lambda, A, B)$ denote the subclass of V consisting of functions
$f(z) \in S(n, \lambda, A, B)$.

Coefficient estimates

Theorem 1. Let the function $f(z)$ defined by (1) be in V. Then
$f(z) \in VS(n, \lambda, A, B)$, if and only if

$$\sum_{k=2}^{\infty} k^{n+1} C_k \left| a_k \right| \leq (B-A)(n+1) \tag{5}$$

where

$$C_k = (1+B)[n+1+\lambda(k-1)].$$

The extremal functions are:

$$f(z) = z + \frac{(B-A)(n+1)}{k^{n+1} C_k} e^{i\theta_k} z^k, (k \geq 2).$$

Proof. We work with the technique used in [3].
Suppose that $f(z) \in VS(n, \lambda, A, B)$. Then

$$h(z) = (1-\lambda)(\mathscr{D}^n f(z))' + \lambda(\mathscr{D}^{n+1} f(z))' = \frac{1+Aw(z)}{1+Bw(z)}, \tag{6}$$

where

$$w \in H = \{w \text{ analytic}, w(0) = 0 \text{ and } |w(z)| < 1, z \in U\}.$$

From this we have

$$w(z) = \frac{1 - h(z)}{Bh(z) - A}.$$

Therefore

$$h(z) = 1 + \sum_{k=2}^{\infty} \frac{k^{n+1}\left[n + 1 + \lambda(k-1)\right]}{n+1} a_k z^{k-1}$$

and $|w(z)| < 1$ implies

$$\left| \frac{\sum\limits_{k=2}^{\infty} \frac{k^{n+1}[n+1+\lambda(k-1)]}{n+1} a_k z^{k-1}}{(B-A) + B \sum\limits_{k=2}^{\infty} \frac{k^{n+1}[n+1+\lambda(k-1)]}{n+1} a_k z^{k-1}} \right| < 1. \tag{7}$$

Since $f(z) \in V$, $f(z)$ lies in the $V(\theta_k, \delta)$ for some $\{\theta_k\}$ sequence and a real number δ such that $\theta_k + (k-1)\delta \equiv \pi(mod\ 2\pi), \forall k \geq 2$.
Set $z = re^{i\delta}$ in (7), then

$$\left| \frac{\sum\limits_{k=2}^{\infty} \frac{k^{n+1}[n+1+\lambda(k-1)]}{n+1} |a_k| r^{k-1}}{(B-A) - B \sum\limits_{k=2}^{\infty} \frac{k^{n+1}[n+1+\lambda(k-1)]}{n+1} |a_k| r^{k-1}} \right| < 1. \tag{8}$$

Since $Re\{w(z)\} < |w(z)| < 1$ we have

$$Re \left\{ \frac{\sum\limits_{k=2}^{\infty} \frac{k^{n+1}[n+1+\lambda(k-1)]}{n+1} |a_k| r^{k-1}}{(B-A) - B \sum\limits_{k=2}^{\infty} \frac{k^{n+1}[n+1+\lambda(k-1)]}{n+1} |a_k| r^{k-1}} \right\} < 1. \tag{9}$$

So

$$\sum_{k=2}^{\infty} k^{n+1} C_k |a_k| r^{k-1} \leq (B-A)(n+1). \tag{10}$$

$r \to 1$

$$\sum_{k=2}^{\infty} k^{n+1} C_k |a_k| \leq (B-A)(n+1).$$

Conversely, $f(z) \in V$ and satisfies (5). Since $r^{k-1} < 1$, we have

$$\left| \sum_{k=2}^{\infty} \frac{k^{n+1}\left[n + 1 + \lambda(k-1)\right]}{n+1} |a_k| z^{k-1} \right| \leq \sum_{k=2}^{\infty} \frac{k^{n+1}\left[n + 1 + \lambda(k-1)\right]}{n+1} |a_k| r^{k-1}$$

$$\leq (B-A) - B\sum_{k=2}^{\infty}\frac{k^{n+1}\left[n+1+\lambda\left(k-1\right)\right]}{n+1}\left|a_k\right|r^{k-1}$$

$$\leq \left|(B-A) + B\sum_{k=2}^{\infty}\frac{k^{n+1}\left[n+1+\lambda\left(k-1\right)\right]}{n+1}a_k z^{k-1}\right|$$

which gives (7) and hence follows that

$$(1-\lambda)(\mathscr{D}^n f(z))' + \lambda(\mathscr{D}^{n+1} f(z))' = \frac{1+Aw(z)}{1+Bw(z)}$$

that is $f(z) \in VS(n, \lambda, A, B)$.

Corollary 1. Let the function $f(z)$ define by (1) be in the class $VS(n, \lambda, A, B)$. Then

$$|a_k| \leq \frac{(B-A)(n+1)}{k^{n+1}C_k}, (k \geq 2).$$

The result (5) is sharp for the functions

$$f(z) = z + \frac{(B-A)(n+1)}{k^{n+1}C_k}e^{i\theta_k}z^k, (k \geq 2).$$

Distortion theorems

Theorem 2. Let the function $f(z)$ defined by (1) be in the class $VS(n, \lambda, A, B)$. Then

$$|z| - \frac{\left(B-A\right)\left(n+1\right)}{2^{n+1}C_2}\left|z\right|^2 \leq |f(z)| \leq |z| + \frac{\left(B-A\right)\left(n+1\right)}{2^{n+1}C_2}\left|z\right|^2. \tag{11}$$

Proof. We work with the technique used by Silverman [8]. Let

$$\Phi\left(k\right) = k^n C_k. \tag{12}$$

It is an increasing function of $k\,(k \geq 2)$,so

$$\Phi(2)\sum_{k=2}^{\infty}|a_k| \leq \sum_{k=2}^{\infty}\Phi(k)\,|a_k| \leq \left(B-A\right)\left(n+1\right)$$

or equivalently

$$\sum_{k=2}^{\infty}|a_k| \leq \frac{\left(B-A\right)\left(n+1\right)}{2\Phi(2)} = \frac{\left(B-A\right)\left(n+1\right)}{2^{n+1}C_2}. \tag{13}$$

This way we have

$$|f(z)| \le |z| + \sum_{k=2}^{\infty} |a_k| \, |z|^k \le |z| + |z|^2 \sum_{k=2}^{\infty} |a_k| \, ,$$

so

$$|f(z)| \le |z| + \frac{(B-A)\,(n+1)}{2^{n+1}C_2} \, |z|^2 \, .$$

Also, we have

$$|f(z)| \ge |z| - \sum_{k=2}^{\infty} |a_k| \, |z|^k \ge |z| - |z|^2 \sum_{k=2}^{\infty} |a_k| \, .$$

So

$$|f(z)| \ge |z| - \frac{(B-A)\,(n+1)}{2^{n+1}C_2} \, |z|^2 \, .$$

The result is sharp for the function

$$f(z) = z + \frac{(B-A)(n+1)}{2^{n+1}C_2} e^{i\theta_2} z^2,$$

at $z = \pm |z| \, e^{-i\theta_2}$.

Corollary 2. $f(z) \in U(0, r_1)$, where $r_1 = 1 + \dfrac{(B-A)(n+1)}{2^{n+1}C_2}$.

Theorem 3. Let the function $f(z)$ defined by (1) be in the class $VS(n, \lambda, A, B)$. Then

$$1 - \frac{(B-A)\,(n+1)}{2^n C_2} \, |z| \le \left| f'(z) \right| \le 1 + \frac{(B-A)\,(n+1)}{2^n C_2} \, |z| \, . \tag{14}$$

The result is sharp.

Proof. Let $\dfrac{\Phi\,(k)}{k} = k^{n-1}C_k$. It is an increasing function of $k (k \ge 2)$. According to *Theorem 1*, we have

$$\frac{\Phi(2)}{2} \sum_{k=2}^{\infty} k \, |a_k| \le \sum_{k=2}^{\infty} \Phi(k) \, |a_k| \le (B-A)\,(n+1) \, ,$$

or equivalently

$$\sum_{k=2}^{\infty} k \, |a_k| \le \frac{(B-A)\,(n+1)}{\Phi(2)} = \frac{(B-A)\,(n+1)}{2^n C_2} .$$

This way we have

$$\left|f'(z)\right| \leq 1 + |z| \sum_{k=2}^{\infty} k\,|a_k| \leq 1 + \frac{(B-A)\,(n+1)}{2^n C_2}\,|z|\,.$$

So

$$\left|f'(z)\right| \geq 1 - |z| \sum_{k=2}^{\infty} k\,|a_k| \geq 1 - \frac{(B-A)\,(n+1)}{2^n C_2}\,|z|\,.$$

Corollary 3. $f'(z) \in U(0, r_2)$, where $r_2 = 1 + \dfrac{(B-A)(n+1)}{2^n C_2}$.

Extreme points

Theorem 4. Let the function $f(z)$ defined by (1) be in the class $VS(n, \lambda, A, B)$, with $arg(a_k) = \theta_k$ where $\theta_k + (k-1)\delta \equiv \pi(mod\ 2\pi), \forall k \geq 2$. Define

$$f_1(z) = z$$

and

$$f_k(z) = z + \frac{(B-A)(n+1)}{k^{n+1} C_k} e^{i\theta_k} z^k, (k \geq 2; z \in U).$$

Then $f(z) \in VS(n, \lambda, A, B)$ if and only if $f(z)$ can expressed by $f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$, where $\mu_k \geq 0$ and $\sum_{k=1}^{\infty} \mu_k = 1$.

Proof. If $f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$, $\mu_k \geq 0$ and $\sum_{k=1}^{\infty} \mu_k = 1$, then

$$\sum_{k=2}^{\infty} k^{n+1} C_k \frac{(B-A)(n+1)}{k^{n+1} C_k} \mu_k = \sum_{k=2}^{\infty} (B-A)(n+1)\mu_k =$$

$$= (1-\mu_1)(B-A)(n+1) \leq (B-A)(n+1).$$

Hence $f(z) \in VS(n, \lambda, A, B)$. Conversly, let the function $f(z)$ defined by (1) be in the class $VS(n, \lambda, A, B)$, define

$$\mu_k = \frac{k^{n+1} C_k}{(B-A)(n+1)}\,|a_k|\,, (k \geq 2)$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k.$$

From *Theorem 1* , $\sum_{k=2}^{\infty} \mu_k \leq 1$ and so $\mu_1 \geq 0$. Since $\mu_k f_k(z) = \mu_k z + a_k z^k$, then

$$\sum_{k=1}^{\infty} \mu_k f_k(z) = z + \sum_{k=2}^{\infty} a_k z^k = f(z).$$

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Ágnes Orsolya Páll-Szabó,
Babeş-Bolyai University,
Cluj-Napoca, Romania
email: *kicsim21@yahoo.com*

Olga Engel
Babeş-Bolyai University,
Cluj-Napoca, Romania,
email: *engel_ olga@hotmail.com*