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ON CURVE COUPLES WITH JOINT TIMELIKE FRENET PLANES IN MINKOWSKI 3-SPACE

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Abstract. In this study, we have investigated the possibility of whether any timelike Frenet plane of a given space curve in Minkowski 3-space $\mathbb{E}_1^3$ also becomes any timelike Frenet plane of another space curve in the same space. We study possible nine cases. As a result, we obtain some results for given curves by matching their Frenet planes with each other one by one.

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1. Introduction

In the theory of curves in Euclidean space, one of the important and interesting problem is characterization of a regular curve. In the solution of the problem, the curvature functions k_1 (or $\varkappa$) and k_2 (or τ) of a regular curve have an effective role. For example: if $k_1 = 0 = k_2$, then the curve is a geodesic or if $k_1 =$constant$\neq 0$ and $k_2 = 0$,then the curve is a circle with radius $(1/k_1)$, etc. Thus we can determine the shape and size of a regular curve by using its curvatures.

Another way in the solution of the problem is the relationship between the Frenet vectors of the curves (see [5]). For instance *Bertrand* curves and Mannheim curves (see [5] [7] and [6]).

The other way in the solution of the problem is the relationship between the Frenet planes of the curves. In ([8]), the authors asked the following question and investigated the possible answers of the question:

Is it possible that one of the Frenet planes of a given curve in $\mathbb{E}^3$ be a Frenet plane of another space curve in the same space? Then they give many interesting results. Also, in ([9]) and ([10]), the authors considered curve couples with joint spacelike Frenet planes and joint lightlike Frenet planes in Minkowski 3-space.

In this paper, we have investigated the possibility of whether any timelike Frenet plane of a given space curve in Minkowski 3-space $\mathbb{E}_1^3$ also becomes any timelike Frenet plane of another space curve in the same space. We have obtained some characterizations of a given space curve by considering nine possible case. Consequently, with the present paper, we complete the series of the papers on curve couples with joint Frenet planes in Minkowski 3-space ([9], [10]).

2. Preliminaries

The Minkowski space $\mathbb{E}_1^3$ is the Euclidean 3-space $\mathbb{E}^3$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}_1^3$. Recall that a vector $v \in \mathbb{E}_1^3 \backslash \{0\}$ can be *spacelike* if $g(v, v) > 0$, *timelike* if $g(v, v) < 0$ and *null (lightlike)* if $g(v, v) = 0$ and $v \neq 0$. In particular, the vector $v = 0$ is a spacelike . The norm of a vector v is given by $||v|| = \sqrt{|g(v, v)|}$, and two vectors v and w are said to be orthogonal, if $g(v, w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^3$, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null. Spacelike curve in $\mathbb{E}_1^3$ is called *pseudo null curve* if its principal normal vector N is null. A null curve α is parameterized by pseudo-arc s if $g(\alpha''(s), \alpha''(s)) = 1$. A spacelike or a timelike curve $\alpha(s)$ has unit speed, if $g(\alpha'(s), \alpha'(s)) = \pm 1$ ([11],[5],[1],[2]).

Let $\{T, N, B\}$ be the moving Frenet frame along a curve α in $\mathbb{E}_1^3$, consisting of the tangent, the principal normal and the binormal vector fields respectively. Depending on the causal character of α, the Frenet equations have the following forms.

Case I. If α is a non-null curve, the Frenet equations are given by ([**?**]):

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2 k_1 & 0 \\ -\epsilon_1 k_1 & 0 & \epsilon_3 k_2 \\ 0 & -\epsilon_2 k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{1}$$

where k_1 and k_2 are the first and the second curvature of the curve respectively. Moreover, the following conditions hold:

$g(T, T) = \epsilon_1 = \pm 1$, $g(N, N) = \epsilon_2 = \pm 1, g(B, B) = \epsilon_3 = \pm 1$ and $g(T, N) = g(T, B) = g(N, B) = 0$. Also, the following equations hold:

$$T \times N = \epsilon_1 \epsilon_2 B,\ N \times B = \epsilon_2 \epsilon_3 T,\ B \times T = \epsilon_1 \epsilon_3 B. \tag{2}$$

Case II. If α is a null curve, the Frenet equations are given by ([11],[1])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_2 & 0 & -k_1 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{3}$$

where the first curvature $k_1 = 0$ if α is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$g(T, T) = g(B, B) = g(T, N) = g(N, B) = 0, g(N, N) = g(T, B) = 1$. Also, the following equations hold:

$$T \times N = -T,\ B \times T = -N,\ N \times B = -B. \tag{4}$$

Case III. if α is pseudo null curve, the Frenet formulas have the form ([11],[2])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ 0 & k_2 & 0 \\ -k_1 & 0 & -k_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{5}$$

where the first curvature $k_1 = 0$ if α is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$g(N,N) = g(B,B) = g(T,N) = g(T,B) = 0, g(T,T) = g(N,B) = 1$. Also, the following equations hold:

$$T \times N = N,\ N \times B = T,\ B \times T = B. \tag{6}$$

3. On Curve Couples with Joint Timelike Frenet Planes in Minkowski 3-space

Let us consider the given two space curves C and $\overline{C}$, defined on the same open interval $I \subset \mathbb{R}$. Let us attach moving triads $\{C, T, N, B\}$ and $\{\overline{C}, \overline{T}, \overline{N}, \overline{B}\}$ to C and $\overline{C}$ at the corresponding points of C and $\overline{C}$. We denote the arcs, curvatures and torsions of C and $\overline{C}$ by s, k_1, k_2 and $\overline{s}$, $\overline{k_1}$, $\overline{k_2}$ respectively. At each point $C(s)$ of the curve C, the planes spanned by $\{T, N\}$, $\{N, B\}$, $\{T, B\}$ are known respectively as the osculating plane, the normal plane and the rectifying plane. We denote these planes by OP, NP and RP, respectively. Now, we assume that $\overline{C}$ be a arbitrary unit speed space curve with curvatures $\overline{k_1}$, $\overline{k_2}$ and Frenet vectors $\overline{T}$, $\overline{N}$, $\overline{B}$. At each point $\overline{C}(\overline{s})$ of the curve $\overline{C}$, the planes spanned by $\left\{\overline{T}, \overline{N}\right\}$, $\left\{\overline{N}, \overline{B}\right\}$, $\left\{\overline{T}, \overline{B}\right\}$ are known respectively as the osculating plane, the normal plane and the rectifying plane. We denote these planes by $\overline{OP}$, $\overline{NP}$ and $\overline{RP}$, respectively. Let $\frac{d\overline{s}}{ds} = f'$.

In this section we ask the following question:

"*Is it possible that one of the timelike Frenet planes of a given curve be a timelike Frenet plane of another space curve?*" and we investigate the answer of the question. For this, we consider the following possible cases:

Case	*Frenet plane of C*	*Frenet plane of $\overline{C}$*	*Condition*
1	$sp\{T, N\} = OP$	$sp\left\{\overline{T}, \overline{N}\right\} = \overline{OP}$	$OP = \overline{OP}$
2	$sp\{T, N\} = OP$	$sp\left\{\overline{N}, \overline{B}\right\} = \overline{NP}$	$OP = \overline{NP}$
3	$sp\{T, N\} = OP$	$sp\left\{\overline{T}, \overline{B}\right\} = \overline{RP}$	$OP = \overline{RP}$
4	$sp\{N, B\} = NP$	$sp\left\{\overline{T}, \overline{N}\right\} = \overline{OP}$	$NP = \overline{OP}$
5	$sp\{N, B\} = NP$	$sp\left\{\overline{N}, \overline{B}\right\} = \overline{NP}$	$NP = \overline{NP}$
6	$sp\{N, B\} = NP$	$sp\left\{\overline{T}, \overline{B}\right\} = \overline{RP}$	$NP = \overline{RP}$
7	$sp\{T, B\} = RP$	$sp\left\{\overline{T}, \overline{N}\right\} = \overline{OP}$	$RP = \overline{OP}$
8	$sp\{T, B\} = RP$	$sp\left\{\overline{N}, \overline{B}\right\} = \overline{NP}$	$RP = \overline{NP}$
9	$sp\{T, B\} = RP$	$sp\left\{\overline{T}, \overline{B}\right\} = \overline{RP}$	$RP = \overline{RP}$

Now we investigate these possible cases step by step.

If the $OP = sp\{T, N\}$ is timelike then T spacelike (timelike), N timelike (spacelike) and B spacelike. We have the following Frenet formulae:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2 k_1 & 0 \\ -\epsilon_1 k_1 & 0 & k_2 \\ 0 & -\epsilon_2 k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad \begin{matrix} g(T,T) = \epsilon_1 \\ g(N,N) = \epsilon_2 \\ g(B,B) = 1 \end{matrix} \tag{7}$$

$OP = \overline{OP}$

In this case, we investigate the answer of the following question:

"Is it possible that the timelike osculating plane of a given space curve be the timelike osculating plane of another space curve in $\mathbb{E}_1^3$?".

Now, we investigate the answer of the question. We assume that the timelike osculating plane of the given curve C is the timelike osculating plane of another space curve $\overline{C}$. Since the osculating plane of $\overline{C}$ is a timelike plane and spanned by spacelike (timelike) vector $\overline{T}$ and timelike(spacelike) vector $\overline{N}$, its binormal vector field $\overline{B}$ is a spacelike vector. Then $\overline{C}$ is spacelike(timelike) curve satisfying the Frenet formulae [1]. Since $B^{\perp} = Sp\{T, N\} = Sp\left\{\overline{T}, \overline{N}\right\} = \overline{B}^{\perp}$, B is parallel to $\overline{B}$. Thus we have the following relation

$$\overline{X} = X + aT + bN, \quad a \neq 0, b \neq 0 \tag{8}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (8) with respect to s and applying the Frenet formulas given in (7), we get

$$\overline{T}f' = (1 + a' - \epsilon_1 bk_1)T + (\epsilon_2 ak_1 + b')N + bk_2 B. \tag{9}$$

Since $\overline{T} \in Sp\{T, N\}$, we can write $\overline{T} = \lambda T + \mu N$, for some constant λ and μ. From (9) we have,

$$(\lambda T + \mu N)\, f' = (1 + a' - \epsilon_1 bk_1)T + (\epsilon_2 ak_1 + b')N + bk_2 B. \tag{10}$$

Multiplying the equation (10) by B, we obtain

$$bk_2 = 0.$$

Thus b or k_2 must be zero which is contradiction with our assumption. Thus we give the following theorem:

Theorem 1. *There exist no a pair of space curves $\left(C, \overline{C}\right)$ for which timelike osculating plane of C is timelike osculating plane of $\overline{C}$.*

$OP = \overline{NP}$

In this case, we investigate the answer of the following question: "Can the timelike osculating plane of a given space curve be the timelike normal plane of another space curve?"

Now, we investigate the answer of the question. We assume that the timelike osculating plane of the given curve C is the timelike normal plane of another space curve $\overline{C}$. Since the normal plane of $\overline{C}$ is a timelike plane, we have two subcases:

Case(2-1) $\overline{N}$ is spacelike(timelike) vector and $\overline{B}$ is timelike(spacelike) vector

Case(2-2) $\overline{N}$ and $\overline{B}$ are linearly independent null vectors

Case(2-1) Since $\overline{N}$ is spacelike(timelike) and $\overline{B}$ is timelike(spacelike), $\overline{T}$ is a spacelike vector. Therefore, $\overline{C}$ is a spacelike curve satisfying the Frenet formulae (1). Since $B^{\perp} = Sp\{T, N\} = Sp\left\{\overline{N}, \overline{B}\right\} = \overline{T}^{\perp}$, B is parallel to $\overline{T}$. Thus we have the following relation

$$\overline{X} = X + aT + bN, \quad a \neq 0, b \neq 0 \tag{11}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (11) with respect to s and applying the Frenet formulae given in (7), we get

$$\overline{T}f' = (1 + a' - \epsilon_1 bk_1)T + (\epsilon_2 ak_1 + b')N + bk_2B. \tag{12}$$

First, multiplying the equation (12) by B, we obtain

$$b = \frac{f'}{k_2}. \tag{13}$$

Next, multiplying the equation (13) by N, we have

$$a = \frac{-\epsilon_2}{k_1}\left(\frac{f'}{k_2}\right)'. \tag{14}$$

Substituting (13) and (14) in (11), we get

$$\overline{X} = X - \frac{\epsilon_2}{k_1}\left(\frac{f'}{k_2}\right)' T + \frac{f'}{k_2}N. \tag{15}$$

Case(2-2) Since $\overline{N}$ and $\overline{B}$ are linearly independent null(lightlike) vectors, $\overline{C}$ is a pseudo null curve satisfying the Frenet formulae (5). Since $\overline{T}^{\perp} = sp\{\overline{N}, \overline{B}\} = sp\{T, N\} = B^{\perp}$, B is parallel to $\overline{T}$. Thus we have the following relation

$$\overline{X} = X + aT + bN, \quad a \neq 0, b \neq 0 \tag{16}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. If we continue in the similar way in Case(2-1), we get

$$\overline{X} = X - \frac{\epsilon_2}{k_1}\left(\frac{f'}{k_2}\right)' T + \frac{f'}{k_2}N. \tag{17}$$

Thus we prove the following theorem.

Theorem 2. *Let C be a given unit speed curve with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the timelike osculating plane of the curve C is the timelike normal plane of another space curve $\overline{C}$,then $\overline{C}$ has the following form*

$$\overline{C} = C - \frac{\epsilon_2}{k_1}\left(\frac{1}{k_2}\frac{d\overline{s}}{ds}\right)' T + \frac{1}{k_2}\frac{d\overline{s}}{ds}N. \tag{18}$$

Without loss of generality, we assume that the curves C and $\overline{C}$ have the same parameter s and $k_1 = k_2$ =constant$\neq 0$. Then the curve C is a spacelike (timelike) spherical helix lying in S_1^2 (H_2^0).

$OP = \overline{RP}$

In this case, we investigate the answer of the following question: "Can the timelike osculating plane of a given space curve be the timelike rectifying plane of another space curve?"

Now, we investigate the answer of the question. We assume that the timelike osculating plane of the given curve C is the timelike rectifying plane of another space curve $\overline{C}$. Since the rectifying plane of $\overline{C}$ is a timelike plane, we have two subcases:

(3-1) $\overline{T}$ is spacelike(timelike) vector and $\overline{B}$ is timelike(spacelike) vector

(3-2) $\overline{T}$ and $\overline{B}$ are linearly independent null vectors

Case(3-1) Since $\overline{T}$ is spacelike(timelike) and $\overline{B}$ is timelike(spacelike), $\overline{N}$ is a spacelike vector. Therefore, $\overline{C}$ is a spacelike(timelike) curve satisfying the Frenet formulae (1). Since $\overline{N}^{\perp} = sp\{\overline{T}, \overline{B}\} = sp\{T, N\} = B^{\perp}$, B is parallel to $\overline{N}$. Thus we have the following relation

$$\overline{X} = X + aT + bN, \quad a \neq 0, b \neq 0 \tag{19}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (19) with respect to s and applying the Frenet formulae given in (7), we get

$$\overline{T}f' = (1 + a' - \epsilon_1 bk_1)T + (\epsilon_2 ak_1 + b')N + bk_2 B. \tag{20}$$

Since $\overline{T} \in Sp\{T, N\}$, we can write $\overline{T} = \lambda T + \mu N$, for some constant λ and μ. From (20) we have,

$$(\lambda T + \mu N)\, f' = (1 + a' - \epsilon_1 bk_1)T + (\epsilon_2 ak_1 + b')N + bk_2 B. \tag{21}$$

Multiplying the equation (21) by B, we obtain

$$bk_2 = 0.$$

Thus b or k_2 must be zero which is contradiction with our assumption.

Case(3-2) Since $\overline{T}$ and $\overline{B}$ are linearly independent null(lightlike) vectors, $\overline{C}$ is a Cartan null curve satisfying the Frenet formulae (5). Since $\overline{N}^{\perp} = sp\{\overline{T}, \overline{B}\} = sp\{T, N\} = B^{\perp}$, B is parallel to $\overline{N}$. Thus we have the following relation

$$\overline{X} = X + aT + bN, \quad a \neq 0, b \neq 0, \tag{22}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. If we continue in the similar way in Case(3-1), we have

$$bk_2 = 0.$$

Thus b or k_2 must be zero which is contradiction with our assumption. Thus we give the following theorem:

Theorem 3. *There exist no a pair of space curves $(C, \overline{C})$ for which timelike osculating plane of C is timelike rectifying plane of $\overline{C}$.*

If the normal plane of C ,that is $NP = sp\{N, B\}$, is a timelike plane, we have two subcases:

Case (A). N is spacelike(timelike) vector and B is timelike(spacelike) vector

Case (B). N and B are linearly independent null(lightlike) vectors.

Case (A). Since N is spacelike (timelike) vector and B is timelike (spacelike) vector, T is spacelike vector. Therefore C is a spacelike curve satisfying the following Frenet formulae:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2 k_1 & 0 \\ -k_1 & 0 & \epsilon_3 k_2 \\ 0 & -\epsilon_2 k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad \begin{array}{c} g(T,T) = 1 \\ g(N,N) = \epsilon_2 \\ g(B,B) = \epsilon_3 \end{array} \tag{23}$$

Here,we have three subcases:

$NP = \overline{OP}$

In this case, we investigate the answer of the following question: "Can the timelike normal plane of a given space curve be the timelike osculating plane of another space curve?".

Now, we investigate the answer of the question. Let assume that the timelike normal plane of the given curve C is a the timelike osculating plane of another space curve $\overline{C}$. Since the osculating plane of $\overline{C}$ is a timelike plane and spanned by spacelike(timelike) vector $\overline{T}$ and timelike(spacelike) vector $\overline{N}$, its binormal vector field $\overline{B}$ is a spacelike vector. Then $\overline{C}$ is a spacelike(timelike) curve satisfying the Frenet formulae [1]. Since $\overline{B}^{\perp} = sp\{\overline{T}, \overline{N}\} = sp\{N, B\} = T^{\perp}$, $\overline{B}$ is parallel to T. Thus we have the following relation

$$\overline{X} = X + aN + bB, \quad a \neq 0, b \neq 0, \tag{24}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (24) with respect to s and applying the Frenet formulae given in (23), we have

$$\overline{T} f' = (1 - ak_1)T + (a' - \epsilon_2 k_2 b)N + (b' + a\epsilon_3 k_2)B. \tag{25}$$

Multiplying the equation (25) by T, we obtain

$$a = \frac{1}{k_1}. \tag{26}$$

Substituting (26) in (25), we get

$$\overline{T} f' = (a' - \epsilon_2 k_2 b)N + (b' + a\epsilon_3 k_2)B. \tag{27}$$

If we take derivative of (27) with respect to s, we obtain

$$\begin{aligned} \overline{T} f'' + \overline{\epsilon_2}(f')^2 \overline{k_1 N} &= (-a'k_1 + \epsilon_2 b k_2 k_1)T \\ &+ (a'' - 2\epsilon_2 b' k_2 - \epsilon_2 b k_2' - \epsilon_2 \epsilon_3 a k_2^2)N \\ &+ (2\epsilon_3 a' k_2 + \epsilon_3 a k_2' + b'' - \epsilon_2 \epsilon_3 b k_2^2)B. \end{aligned} \tag{28}$$

Multiplying the equation (28) by T, we get

$$b = -\frac{\epsilon_2 k_1'}{k_2 k_1^2}. \tag{29}$$

If we put (26) and (29) in (24), we have

$$\overline{X} = X + \frac{1}{k_1}N - \frac{\epsilon_2 k_1'}{k_2 k_1^2}B. \tag{30}$$

Thus we prove the following theorem:

Theorem 4. *Let C be a given unit speed curve with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the timelike normal plane, $sp\{N, B\}$ with timelike(spacelike) vector N and spacelike(timelike) vector B,respectively, of the curve C is the timelike osculating plane of another space curve $\overline{C}$, then $\overline{C}$ has the following form*

$$\overline{C} = C + \frac{1}{k_1}N - \frac{\epsilon_2 k_1'}{k_2 k_1^2}B. \tag{31}$$

$NP = \overline{NP}$

In this case we investigate the answer of the following question: "Can the timelike normal plane of a given space curve be the timelike normal plane of another space curve?".

Now, we investigate the answer of the question. We assume that the timelike normal plane of the given curve C is the timelike normal plane of another space curve $\overline{C}$. Since the normal plane of $\overline{C}$ is a timelike plane, we have two cases:

Case (5-1)**.** $\overline{N}$ is spacelike(timelike) vector and $\overline{B}$ is timelike(spacelike) vector

Case (5-2)**.** $\overline{N}$ and $\overline{B}$ are linearly independent null(lightlike) vectors.

Case (5-1). Since $\overline{N}$ is a spacelike(timelike) vector and $\overline{B}$ is a timelike(spacelike) vector, $\overline{T}$ is a spacelike vector. Therefore, $\overline{C}$ is a spacelike curve satisfying the Frenet formulae [1]. Since $\overline{T}^{\perp} = sp\{\overline{N}, \overline{B}\} = sp\{N, B\} = T^{\perp}$, T is parallel to $\overline{T}$. Thus we have the following relation

$$\overline{X} = X + aN + bB, \quad a \neq 0, b \neq 0 \tag{32}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (32) with respect to s and applying the Frenet formulae given in (23), we get

$$\overline{T}f' = (1 - ak_1)T + (a' - \epsilon_2 k_2 b)N + (b' + a\epsilon_3 k_2)B. \tag{33}$$

Multiplying the equation (33) by T, we have

$$a = \frac{1 - f'}{k_1}. \tag{34}$$

Next multiplying the equation (33) by N,we get

$$b = \frac{\epsilon_2}{k_2}\left(\frac{1 - f'}{k_1}\right)'. \tag{35}$$

Substituting (34) and (35) in (32), we find

$$\overline{X} = X + \left(\frac{1 - f'}{k_1}\right)N + \frac{\epsilon_2}{k_2}\left(\frac{1 - f'}{k_1}\right)'B. \tag{36}$$

Case (5-2). Since $\overline{N}$ and $\overline{B}$ are linearly independent null(lightlike) vectors, $\overline{C}$ is a pseudo null curve satisfying the Frenet formulae (5). Since $\overline{T}^{\perp} = sp\{\overline{N}, \overline{B}\} = sp\{N, B\} = T^{\perp}$, T is parallel to $\overline{T}$. Thus we have the following relation

$$\overline{X} = X + aN + bB, \quad a \neq 0, b \neq 0 \tag{37}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. If we continue in the similar way in Case(5-1), we obtain

$$\overline{X} = X + \left(\frac{1-f'}{k_1}\right) N + \frac{\epsilon_2}{k_2}\left(\frac{1-f'}{k_1}\right)' B. \tag{38}$$

Thus we give the following theorem:

Theorem 5. *Let C be a given unit speed curve with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the timelike normal plane, $sp\{N, B\}$ with timelike(spacelike) vector N and spacelike(timelike) vector B, of the curve C is the timelike normal plane of another space curve $\overline{C}$,then $\overline{C}$ has the following form*

$$\overline{C} = C + \frac{1}{k_1}\left(1 - \frac{d\overline{s}}{ds}\right) N + \frac{\epsilon_2}{k_2}\left(\frac{1}{k_1}\left(1 - \frac{d\overline{s}}{ds}\right)\right)' B. \tag{39}$$

Without loss of generality, we assume that the curves C and $\overline{C}$ have the same parameter in theorem 5, then $C = \overline{C}$.

$NP = \overline{RP}$

In this case we investigate the answer of the following question: "Can the timelike normal plane of a given space curve be the timelike rectifying plane of another space curve?".

Now, we investigate the answer of the question. We assume that the timelike normal plane of the given curve C is the timelike rectifying plane of another space curve $\overline{C}$. Since the rectifying plane of $\overline{C}$ is a timelike plane, we have two cases:

Case (6-1)**.** $\overline{T}$ is spacelike(timelike) vector and $\overline{B}$ is timelike(spacelike) vector

Case (6-2)**.** $\overline{T}$ and $\overline{B}$ are linearly independent null(lightlike) vectors.

Case (6-1). Since $\overline{T}$ is a spacelike(timelike) vector and $\overline{B}$ is a timelike(spacelike) vector, $\overline{N}$ is a spacelike vector. Therefore, $\overline{C}$ is a spacelike(timelike) curve satisfying the Frenet formulae [1]. Since $\overline{N}^{\perp} = sp\{\overline{T}, \overline{B}\} = sp\{N, B\} = T^{\perp}$, T is parallel to $\overline{N}$.Thus we have the following relation

$$\overline{X} = X + aN + bB, \quad a \neq 0, b \neq 0 \tag{40}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (40) with respect to s and applying the Frenet formulae given in (23), we get

$$\overline{T} f' = (1 - ak_1)T + (a' - \epsilon_2 k_2 b)N + (b' + a\epsilon_3 k_2)B. \tag{41}$$

Multiplying Eq. (41) by T, we obtain

$$a = \frac{1}{k_1}. \tag{42}$$

Substituting (42) in (41), we get

$$\overline{T}f' = (a' - \epsilon_2 k_2 b)N + (b' + a\epsilon_3 k_2)B. \tag{43}$$

If we denote

$$\lambda = \frac{a' - \epsilon_2 k_2 b}{f'} \text{ and } \mu = \frac{b' + a\epsilon_3 k_2}{f'}, \tag{44}$$

we have

$$\overline{T} = \lambda N + \mu B. \tag{45}$$

Differentiating the equation (45), we find

$$(f')^2 \overline{k_1 N} = -k_1 \lambda T + (\lambda' - \epsilon_2 k_2 \mu)N + (\epsilon_2 k_2 \lambda + \mu')B. \tag{46}$$

Multiplying the equation (46) by N, we obtain

$$\lambda' - \epsilon_2 k_2 \mu = 0. \tag{47}$$

Substituting the equation (44) in (47), we get

$$b' + \delta b = \gamma \tag{48}$$

and

$$b(s) = e^{-\int \delta ds} \left\{ \int e^{\int \delta ds} \gamma \quad ds + t \right\}, \text{ where } t \in \mathbb{R} \tag{49}$$

where

$$\delta = \frac{(k_2' f' - f'' k_2)}{2k_2 f'} \tag{50}$$

and

$$\gamma = \frac{\left(\frac{1}{k_1}\right)'' f' - \left(\frac{1}{k_1}\right)' f'' + f' \frac{k_2^2}{k_1}}{2\epsilon_2 k_2 f'}. \tag{51}$$

Case (6-2). Since $\overline{T}$ and $\overline{B}$ are linearly independent null(lightlike) vectors, $\overline{C}$ is a Cartan null curve satisfying the Frenet formulae (3). Since $\overline{N}^{\perp} = sp\{\overline{T}, \overline{B}\} = sp\{N, B\} = T^{\perp}$, T is parallel to $\overline{N}$. Thus we have the following relation

$$\overline{X} = X + aN + bB, \quad a \neq 0, b \neq 0 \tag{52}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. If we continue in the similar way in Case(6-1), we find

$$a = \frac{1}{k_1}. \tag{53}$$

and

$$b(s) = e^{-\int \delta ds} \left\{ \int e^{\int \delta ds} \gamma \ \ ds + t \right\}, \text{ where } t \in \mathbb{R}. \tag{54}$$

where

$$\delta = \frac{(k_2' f' - f'' k_2)}{2k_2 f'} \tag{55}$$

and

$$\gamma = \frac{\left(\frac{1}{k_1}\right)'' f' - \left(\frac{1}{k_1}\right)' f'' + f' \frac{k_2^2}{k_1}}{2\epsilon_2 k_2 f'}. \tag{56}$$

Thus we give the following theorem:

Theorem 6. *Let C be a given unit speed curve with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the timelike normal plane, $sp\{N, B\}$ with timelike(spacelike) vector N and spacelike(timelike) vector B, of the curve C is the timelike rectifying plane of another space curve $\overline{C}$, then $\overline{C}$ has the following form*

$$\overline{C} = C + \frac{1}{k_1} N + e^{-\int \delta ds} \left\{ \int e^{\int \delta ds} \gamma \ \ ds + t \right\} B \tag{57}$$

where

$$\delta = \frac{f' k_2' - f'' k_2}{2k_2 f'} \tag{58}$$

and

$$\gamma = \frac{\left(\frac{1}{k_1}\right)'' f' - \left(\frac{1}{k_1}\right)' f'' + f' \frac{k_2^2}{k_1}}{2\epsilon_2 k_2 f'}. \tag{59}$$

Without loss of generality, we assume that the curves C and $\overline{C}$ have the same parameter s ($\overline{s} = s$) and if $k_1 = 1$ and $k_2 \neq 0$, k_2 is a non-constant function then we get

$$b = \frac{1}{2\epsilon_2 \sqrt{k_2}} \int (k_2)^{\frac{3}{2}} ds + \frac{1}{\sqrt{k_2}} t$$

for some $t \in \mathbb{R}$ and $\epsilon_2 = g(N, N)$. In this case, the curve C is a timelike Salkowski curve in $\mathbb{E}_1^3$.

Without loss of generality, we assume that the curves C and $\overline{C}$ have the same parameter s ($\overline{s} = s$) and if $k_2 = 1$ and $k_1 \neq 0$, k_1 is a non-constant function, then we get

$$b = \frac{1}{2\epsilon_2} \left(\frac{1}{k_1}\right)' + \frac{1}{2\epsilon_2} \int \frac{1}{k_1} ds + c$$

for some $c \in \mathbb{R}$ and $\epsilon_2 = g(N, N)$. In this case, the curve C is a timelike anti- Salkowski curve in $\mathbb{E}_1^3$.

Case B. Since N and B are linearly independent null(lightlike) vectors, C is a pseudo null curve with $k_1 = 1$ satisfying the following Frenet formulae:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ 0 & k_2 & 0 \\ -k_1 & 0 & -k_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad \begin{matrix} g(T,T)=1 \\ g(N,N)=0 \\ g(B,B)=1 \end{matrix} . \tag{60}$$

Here, we have three subcases:
$NP = \overline{OP}$

In this case we investigate the answer of the following question: "Can the timelike normal plane of a given space curve be the timelike osculating plane of another space curve?".

Now, we investigate the answer of the question. Let assume that the timelike normal plane of the given curve C is a the timelike osculating plane of another space curve $\overline{C}$. Since the osculating plane of $\overline{C}$ is a timelike plane and spanned by spacelike(timelike) vector $\overline{T}$ and timelike(spacelike) vector $\overline{N}$, its binormal vector field $\overline{B}$ is a spacelike vector. Then $\overline{C}$ is a spacelike(timelike) curve satisfying the Frenet formulae (1).Since $\overline{B}^{\perp} = sp\{\overline{T}, \overline{N}\} = sp\{N, B\} = T^{\perp}$, T is parallel to $\overline{B}$.Thus we have the following relation

$$\overline{X} = X + aN + bB, \quad a \neq 0, b \neq 0 \tag{61}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (61) with respect to s and applying the Frenet formulae given in (60), we have

$$\overline{T}f' = (1-b)T + (a' + ak_2)N + (b' - bk_2)B. \tag{62}$$

Multiplying the equation (62), by T, we obtain

$$b = 1. \tag{63}$$

Substituting (63) in (62), we get

$$\overline{T}f' = (a' + ak_2)N + (b' - bk_2)B. \tag{64}$$

Multiplying the equation (64) by itself, we get

$$a' + ak_2 = \frac{\overline{\epsilon_1}(f')^2}{2(b' - bk_2)}. \tag{65}$$

Solving the differential equation (65) and using (63), we get

$$a = e^{-\int k_2 ds}\left[t - \frac{1}{2}\int e^{\int k_2 ds}\frac{\overline{\epsilon_1}(f')^2}{k_2)}ds\right]. \tag{66}$$

If we put (63) and (66) in (61), we have

$$\overline{X} = X + e^{-\int k_2 ds}\left[t - \frac{1}{2}\int e^{\int k_2 ds}\frac{\overline{\epsilon_1}(f')^2}{k_2}ds\right]N + B. \tag{67}$$

Thus we prove the following theorem:

Theorem 7. *Let C be a given unit speed curve with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the timelike normal plane, $sp\{N, B\}$ with linearly independent null(lightlike) vectors N and B, of the curve C is the timelike osculating plane of another space curve $\overline{C}$, then $\overline{C}$ has the following form*

$$\overline{C} = C + e^{-\int k_2 ds}\left[t - \frac{1}{2}\int e^{\int k_2 ds}\frac{\overline{\epsilon_1}(f')^2}{k_2}ds\right]N + B \tag{68}$$

for some $t \in \mathbb{R}$ and $\overline{\epsilon_1} = g(\overline{T}, \overline{T})$.

$NP = \overline{NP}$

In this case we investigate the answer of the following question: "Can the timelike normal plane of a given space curve be the timelike normal plane of another space curve?".

Now, we investigate the answer of the question. We assume that the timelike normal plane of the given curve C is the timelike normal plane of another space curve $\overline{C}$. Since the normal plane of $\overline{C}$ is a timelike plane, we have two cases:

Case (8-1). $\overline{N}$ is spacelike(timelike) vector and $\overline{B}$ is timelike(spacelike) vector

Case (8-2). $\overline{N}$ and $\overline{B}$ are linearly independent null(lightlike) vectors.

Case (8-1). Since $\overline{N}$ is a spacelike(timelike) vector and $\overline{B}$ is a timelike(spacelike) vector, $\overline{T}$ is a spacelike vector. Therefore, $\overline{C}$ is a spacelike curve satisfying the Frenet formulae (1). Since $\overline{T}^{\perp} = sp\{\overline{N}, \overline{B}\} = sp\{N, B\} = T^{\perp}$, T is parallel to $\overline{T}$. Thus we have the following relation

$$\overline{X} = X + aN + bB, \quad a \neq 0, b \neq 0 \tag{69}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (69) with respect to s and applying the Frenet formulae given in (60), we get

$$\overline{T}f' = (1 - b)T + (a' + ak_2)N + (b' - bk_2)B. \tag{70}$$

Multiplying the equation (70) by T, we have

$$b = 1 - f'. \tag{71}$$

Next multiplying the equation (70) by B, we get

$$a' + ak_2 = 0. \tag{72}$$

Solving the differential equation (72), we get

$$a = e^{-\int k_2 ds} + t \tag{73}$$

for some constant $t \in \mathbb{R}$. Substituting (71) and (73) in (69) , we find

$$\overline{X} = X + (e^{-\int k_2 ds} + t)N + (1 - f')\,B. \tag{74}$$

Case (8-2). Since $\overline{N}$ and $\overline{B}$ are linearly independent null(lightlike) vectors, $\overline{C}$ is a pseudo null curve satisfying the Frenet formulae (5). Since $\overline{T}^{\perp} = sp\{\overline{N}, \overline{B}\} = sp\{N, B\} = T^{\perp}$, $\overline{T}$ is parallel to T. Thus we have the following relation

$$\overline{X} = X + aN + bB, \quad a \neq 0, b \neq 0 \tag{75}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. If we continue in the similar way in Case(8-1), we get

$$\overline{X} = X + (e^{-\int k_2 ds} + t)N + (1 - f')\, B \tag{76}$$

for some $t \in \mathbb{R}$. Thus we give the following theorem:

Theorem 8. *Let C be a given unit speed curve with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the timelike normal plane, $sp\{N, B\}$ with linearly independent null(lightlike) vectors N and B, of the curve C is the timelike normal plane of another space curve $\overline{C}$, then $\overline{C}$ has the following form*

$$\overline{C} = C + (e^{-\int k_2 ds} + t)N + (1 - f')\, B \tag{77}$$

for some $t \in \mathbb{R}$.

$NP = \overline{RP}$

In this case we investigate the answer of the following question: "Can the timelike normal plane of a given space curve be the timelike rectifying plane of another space curve?".

Now, we investigate the answer of the question. We assume that the timelike normal plane of the given curve C is the timelike rectifying plane of another space curve $\overline{C}$. Since the rectifying plane of $\overline{C}$ is a timelike plane, we have two cases:

Case (9-1)**.** $\overline{T}$ is spacelike(timelike) vector and $\overline{B}$ is timelike(spacelike) vector

Case (9-2)**.** $\overline{T}$ and $\overline{B}$ are linearly independent null(lightlike) vectors.

Case (9-1). Since $\overline{T}$ is a spacelike(timelike) vector and $\overline{B}$ is a timelike(spacelike) vector, $\overline{N}$ is a spacelike vector. Therefore, $\overline{C}$ is a spacelike(timelike) curve satisfying the Frenet formulae (1). Since $\overline{N}^{\perp} = sp\{\overline{T}, \overline{B}\} = sp\{N, B\} = T^{\perp}$, $\overline{N}$ is parallel to$T$. Thus we have the following relation

$$\overline{X} = X + aN + bB, \quad a \neq 0, b \neq 0 \tag{78}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (78) with respect to s and applying the Frenet formulae given in (60), we get

$$\overline{T} f' = (1 - bk_1)T + (a' + ak_2)N + (b' - bk_2)B. \tag{79}$$

Multiplying the equation (79) by T, we obtain

$$b = 1. \tag{80}$$

Substituting (80) in (79), we get

$$\overline{T} f' = (a' + ak_2)N + (b' - bk_2)B. \tag{81}$$

By taking the scalar product of (81) with itself, we have

$$a' + ak_2 = \frac{\overline{\epsilon_1}(f')^2}{2(b' - bk_2)}. \tag{82}$$

Solving the differential equation (82) and using (80), we get

$$a = e^{-\int k_2 ds}\left[t - \frac{1}{2}\int e^{\int k_2 ds}\frac{\overline{\epsilon_1}(f')}{k_2}ds\right] \tag{83}$$

for some $t \in \mathbb{R}$. Substituting (80) and (83) in (78), we get

$$\overline{X} = X + e^{-\int k_2 ds}\left[t - \frac{1}{2}\int e^{\int k_2 ds}\frac{\overline{\epsilon_1}(f')^2}{k_2}ds\right]N + B.$$

for some $t \in \mathbb{R}$. Thus, we give the following theorem:

Theorem 9. *Let C be a given unit speed curve with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the timelike normal plane, $sp\{N,B\}$ with linearly independent null(lightlike) vectors N and B, of the curve C is the timelike rectifying plane, $sp\{\overline{T},\overline{B}\}$ with spacelike(timelike) vector $\overline{T}$ and $\overline{B}$ timelike(spacelike) vector, of another space curve $\overline{C}$, then $\overline{C}$ has the following form*

$$\overline{C} = C + e^{-\int k_2 ds}\left[t - \frac{1}{2}\int e^{\int k_2 ds}\frac{\overline{\epsilon_1}(f')^2}{k_2}ds\right]N + B \tag{84}$$

for some $t \in \mathbb{R}$.

Case (9-2). Since $\overline{T}$ and $\overline{B}$ are linearly independent null(lightlike) vectors, $\overline{C}$ is a Cartan null curve satisfying the Frenet formulae (3). Since $\overline{N}^{\perp} = sp\{\overline{T},\overline{B}\} = sp\{N,B\} = T^{\perp}$, $\overline{N}$ is parallel to$T$. Thus, we have the following relation

$$\overline{X} = X + aN + bB, \quad a \neq 0, b \neq 0 \tag{85}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (85) with respect to s and applying the Frenet formulae given in (60), we get

$$\overline{T}f' = (1 - bk_1)T + (a' + ak_2)N + (b' - bk_2)B. \tag{86}$$

Multiplying the equation (86) by T, we obtain

$$b = 1. \tag{87}$$

Substituting (87) in (86), we get

$$\overline{T}f' = (a' + ak_2)N + (b' - bk_2)B. \tag{88}$$

By taking the scalar product of (88) with itself, we have

$$2(b' - bk_2)(a' + ak_2) = 0. \tag{89}$$

Then, from (89), we get $a' + ak_2 = 0$ or $b' - bk_2 = 0$. Assume that $b' - bk_2 = 0$.Then

$$\overline{T}f' = (a' + ak_2)N. \tag{90}$$

Differentiating (90), we get

$$\overline{T}f'' + (f')^2\overline{k_1 N} = (a'' + 2a'k_2 + ak_2' + ak_2^2)N. \tag{91}$$

By taking the scalar product of (91) with itself, we have $(f')^4(\overline{k_1})^2 = 0 \Rightarrow f' = 0$ or $\overline{k_1} = 0$ which is a contradiction. Therefore,

$$a' + ak_2 = 0. \tag{92}$$

Solving (92), we get

$$a = e^{-\int k_2 ds}. \tag{93}$$

Substituting (87) and (93) in (85), we get

$$\overline{X} = X + e^{-\int k_2 ds}N + B. \tag{94}$$

Thus, we give the following theorem:

Theorem 10. *Let C be a given unit speed curve with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the timelike normal plane, $sp\{N, B\}$ with linearly independent null(lightlike) vectors N and B, of the curve C is the timelike rectifying plane,$sp\{\overline{T}, \overline{B}\}$ linearly independent null(lightlike) vectors $\overline{T}$ and $\overline{B}$, of another space curve $\overline{C}$,then $\overline{C}$ has the following form*

$$\overline{C} = C + +e^{-\int k_2 ds}N + B \tag{95}$$

for some $t \in \mathbb{R}$.

If the rectifying plane of C,that is $RP = sp\{T, B\}$, is a timelike plane, we have two subcases:

Case A**.** T is spacelike(timelike) vector and B is timelike(spacelike) vector

Case B**.** T and B are linearly independent null(lightlike) vectors.

Case A. Since T is spacelike(timelike) vector and B is timelike(spacelike) vector, N is a spacelike vector. Therefore, C is a spacelike(timelike) curve satisfying the following Frenet formulae:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -\epsilon_1 k_1 & 0 & \epsilon_3 k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad \begin{array}{l} g(T,T) = \epsilon_1 \\ g(N,N) = 1 \\ g(B,B) = \epsilon_3 \end{array} \tag{96}$$

Here, we have three subcases:

$RP = \overline{OP}$

In this case, we investigate the answer of the following question: "Can the timelike rectifying plane of a given space curve be the timelike osculating plane of another space curve?".

Now, we investigate the answer of the question. Let assume that the timelike rectifying plane, $sp\{T, B\}$ with spacelike(timelike) vector T and timelike(spacelike) vector B, of the given curve C is a the timelike osculating plane of another space curve $\overline{C}$. Since the osculating plane of $\overline{C}$ is a timelike plane and spanned by spacelike(timelike) vector $\overline{T}$ and timelike (spacelike) vector $\overline{N}$, its binormal vector field $\overline{B}$ is a spacelike vector. Then $\overline{C}$ is a spacelike(timelike) curve satisfying the Frenet formulae (1). Since $N^{\perp} = sp\{T, B\} = sp\{\overline{T}, \overline{N}\} = \overline{B}^{\perp}$, $\overline{B}$ is parallel to N. Thus we have the following relation

$$\overline{X} = X + aT + bB, \quad a \neq 0, b \neq 0 \tag{97}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (97) with respect to s and applying the Frenet formulae (96), we have

$$\overline{T}f' = (1 + a')T + (ak_1 - bk_2)N + b'B. \tag{98}$$

Then multiplying the equation (98) by N, we obtain

$$ak_1 - bk_2 = 0. \tag{99}$$

Substituting (99) in (98), we have

$$\overline{T}f' = (1 + a')T + b'B. \tag{100}$$

Differentiating equation (100) with respect to s, we have

$$\overline{T}f'' + \overline{\epsilon_2}(f')^2\overline{k_1 N} = a''T + (k_1 + a'k_1 - b'k_2)N + b''B. \tag{101}$$

Multiplying the equation (101) by N, we obtain

$$k_1 + a'k_1 - b'k_2 = 0. \tag{102}$$

Differentiating (99) and substituting (102),we have

$$\begin{aligned} a &= \frac{k_1 k_2}{k_2 k_1' - k_1 k_2'} = \frac{k_1}{k_2}\frac{1}{(\frac{k_1}{k_2})'}, \\ b &= \frac{k_1^2}{k_2 k_1' - k_1 k_2'} = -\frac{1}{(\frac{k_2}{k_1})'}. \end{aligned} \tag{103}$$

Substituting (103) in (97), we have

$$\overline{X} = X + \frac{k_1 k_2}{k_2 k_1' - k_1 k_2'}T + \frac{k_1^2}{k_2 k_1' - k_1 k_2'}B. \tag{104}$$

In this case, the same result is obtained for Case(B). Thus we give the following theorem for both Case(A) and Case (B):

Theorem 11. *Let C be a given unit speed curve with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the timelike rectifying plane of the curve C is the timelike osculating plane of another space curve $\overline{C}$, then $\overline{C}$ has the following form*

$$\overline{C} = C + \frac{k_1 k_2}{k_2 k_1' - k_1 k_2'}T + \frac{k_1^2}{k_2 k_1' - k_1 k_2'}B. \tag{105}$$

Thus we give the following results:
If $a = \frac{k_1}{k_2}\frac{1}{\left(\frac{k_1}{k_2}\right)'}$ =constant then we get

$$\frac{k_1}{k_2} = c_2 e^{c_1 s}.$$

If $b = -\frac{1}{\left(\frac{k_2}{k_1}\right)'}$ =constant then we get

$$\frac{k_2}{k_1} = cs + d, \ c \in \mathbb{R}, \ d \in \mathbb{R},$$

which means that the curve C is a rectifying curve (see [3]).

We note that such curves also called canonical geodesics by Izumiya and Takeuchi in [4].

$RP = \overline{NP}$

In this case we investigate the answer of the following question: "Can the timelike rectifying plane of a given space curve be the timelike normal plane of another space curve?".

Now, we investigate the answer of the question. We assume that the timelike rectifying plane of the given curve C is the timelike normal plane of another space curve $\overline{C}$. Since the normal plane of $\overline{C}$ is a timelike plane,we have two cases:

Case (11-1). $\overline{N}$ is spacelike(timelike) vector and $\overline{B}$ is timelike(spacelike) vector

Case (11-2). $\overline{N}$ and $\overline{B}$ are linearly independent null(lightlike) vectors.

Case (11-1). Since $\overline{N}$ is a spacelike(timelike) vector and $\overline{B}$ is a timelike(spacelike) vector, $\overline{T}$ is a spacelike vector. Therefore, $\overline{C}$ is a spacelike curve satisfying the Frenet formulae (1). Since $N^{\perp} = sp\{T, B\} = sp\{\overline{N}, \overline{B}\} = \overline{T}^{\perp}$, N is parallel to $\overline{T}$. Thus we have the following relation

$$\overline{X} = X + aN + bB, \quad a \neq 0, b \neq 0 \tag{106}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (106) with respect to s and applying the Frenet formulae (96), we get

$$\overline{T}f' = (1 + a')T + (ak_1 - bk_2)N + b'B. \tag{107}$$

Multiplying the equation (107) by N ,T and B respectively, we obtain

$$ak_1 - bk_2 = f', \tag{108}$$

$$a = -s + c_1 \text{ and } b' = 0 \tag{109}$$

for some constant c_1. Substituting (109) in (108), we find

$$b = \frac{-f' + (-s + c_1)k_1}{k_2} = \text{constant} \tag{110}$$

for some constant c_1. Substituting (109) and (110) in (106), we have

$$\overline{X} = X + (-s + c_1)T + \left(\frac{-f' + (-s + c_1)k_1}{k_2}\right)B. \tag{111}$$

Case (11-2) Since $\overline{N}$ and $\overline{B}$ are linearly independent null(lightlike) vectors, $\overline{C}$ is a pseudo null curve satisfying the Frenet formulae (5).Since $N^{\perp} = sp\{T, B\} = sp\{\overline{N}, \overline{B}\} = \overline{T}^{\perp}$, N is parallel to $\overline{T}$.Thus we have the following relation

$$\overline{X} = X + aN + bB, \quad a \neq 0, b \neq 0 \tag{112}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. If we continue in the similar way in Case(11-1), we have

$$\overline{X} = X + (-s + c_1)T + \left(\frac{-f' + (-s + c_1)k_1}{k_2} \right) B. \tag{113}$$

In this case, the same result is obtained for Case(B). Thus we give the following theorem for both Case(A) and Case (B):

Theorem 12. *Let C be a given unit speed curve with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the timelike rectifying plane, $sp\{T, B\}$ with spacelike(timelike) vector T and timelike(spacelike) vector B, of the curve C is the timelike normal plane of another space curve $\overline{C}$, then $\overline{C}$ has the following form*

$$\overline{X} = X + (-s + c_1)T + \left(\frac{1}{k_2} \left(-\frac{d\overline{s}}{ds} + (-s + c_1)k_1 \right) \right) B \tag{114}$$

for some constant c_1.

$RP = \overline{RP}$

In this case we investigate the answer of the following question: "Can the timelike rectifying plane of a given space curve be the timelike rectifying plane of another space curve?".

Now, we investigate the answer of the question. We assume that the spacelike rectifying plane of the given curve C is the timelike rectifying plane of another space curve $\overline{C}$. Since the rectifying plane of $\overline{C}$ is a timelike plane, we have two cases:

Case (12-1)**.** $\overline{T}$ is spacelike(timelike) vector and $\overline{B}$ is timelike(spacelike) vector

Case (12-2)**.** $\overline{T}$ and $\overline{B}$ are linearly independent null(lightlike) vectors.

Case (12-1). Since $\overline{T}$ is a spacelike(timelike) vector and $\overline{B}$ is a timelike(spacelike) vector, $\overline{N}$ is a spacelike vector. Therefore, $\overline{C}$ is a spacelike(timelike) curve satisfying the Frenet formulae (1). Since $N^{\perp} = sp\{T, B\} = sp\{\overline{T}, \overline{B}\} = \overline{N}^{\perp}$, N is parallel to $\overline{N}$. Thus, we have the following relation

$$\overline{X} = X + aT + bB, \quad a \neq 0, b \neq 0 \tag{115}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. By taking the derivative of (115) with respect to s and applying the Frenet formulae (96), we get

$$\overline{T}f' = (1 + a')T + (ak_1 - bk_2)N + b'B. \tag{116}$$

Multiplying the equation (116) by N, we obtain

$$ak_1 - bk_2 = 0. \tag{117}$$

Substituting (117) in (116), we have

$$\overline{T}f' = (1 + a')T + b'B. \tag{118}$$

If we denote

$$\lambda = \frac{1 + a'}{f'} \text{ and } \mu = \frac{b'}{f'}, \tag{119}$$

we get

$$\overline{T} = \lambda T + \mu B. \tag{120}$$

Differentiating equation (120) with respect to s, we have

$$f'\overline{k_1 N} = \lambda' T + (\lambda k_1 - \mu k_2)N + \mu' B. \tag{121}$$

Multiplying the equation (121) by T and B respectively, we obtain

$$\mu' = 0 \text{ and } \lambda' = 0. \tag{122}$$

Substituting (119) in (122), we have

$$a = -s + c_1 \int f' \, ds + c_2 \tag{123}$$

and

$$b = d_1 \int f' \, ds + d_2 \tag{124}$$

for some constant c_1, c_2, d_1, and d_2.

Case (12-2). Since $\overline{T}$ and $\overline{B}$ are linearly independent null(lightlike) vectors, $\overline{C}$ is a Cartan null curve satisfying the Frenet formulae (3). Since $N^{\perp} = sp\{T, B\} = sp\{\overline{T}, \overline{B}\} = \overline{N}^{\perp}$, N is parallel to $\overline{N}$. Thus we have the following relation

$$\overline{X} = X + aT + bB, \quad a \neq 0, b \neq 0, \tag{125}$$

where $\overline{X}$ and X are the position vectors of the curves $\overline{C}$ and C respectively, and a and b are the non-zero functions of the parameter s. If we continue in the similar way in Case(12-1), we obtain

$$a = -s + c_1 \int f' \, ds + c_2 \tag{126}$$

and

$$b = d_1 \int f' \, ds + d_2 \tag{127}$$

for some constant c_1, c_2, d_1, and d_2. In this case, the same result is obtained for Case(B). Thus we give the following theorem for both Case(A) and Case(B):

Theorem 13. *Without loss of generality, we assume that the curves C and $\overline{C}$ have the same parameter s ($\overline{s} = s$). Let C be a given unit speed curve in $\mathbb{E}_1^3$ with non-zero curvatures k_1, k_2 and Frenet vectors T, N, B. If the timelike rectifying plane, $sp\{T, B\}$ with spacelike(timelike) vector T and timelike(spacelike) vector B, of the curve C is the timelike rectifying plane of another space curve $\overline{C}$, then $\overline{C}$ has the following form*

$$\overline{C} = C + (c_1 s + c_2)\, T + (d_1 s + d_2)\, B. \tag{128}$$

where $c_1,\ c_2, d_1, d_2 \in \mathbb{R}$.

From (117) we get

$$\frac{k_2}{k_1} = \frac{c_1 s + c_2}{d_1 s + d_2}. \tag{129}$$

Thus we give the following conclusions:

If $c_1 d_2 - c_2 d_1 = 0$, then $\frac{k_1}{k_2}$ =constant. In this case, the curve of C is circular helix or a generalized helix in $\mathbb{E}_1^3$.

If we take $d_2 = 0$ in (129) then we get $\dfrac{k_1}{k_2} = \dfrac{c_1}{d_1} + \dfrac{c_2}{d_1}\dfrac{1}{s}$. In this case, the curve of C is a rectifying curve up to parameterization of the curve C.

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