GAUSSIAN APPROXIMATION FOR THE BUSY PERIOD DISTRIBUTION IN M/G/1 RETRIAL QUEUE

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Abstract. In this work, we investigate some asymptotic results of performance characteristics of retrial systems related to the behavior of the orbit, based on weak Hölder convergence results. We consider two approaches to study the asymptotic distribution of the busy period of an M/G/1 retrial queue. The first approach rely on the modeling of Artalejo and Falin (1996) and an invariance principle for independent random variables. In the second one, we use the evolution of the system in terms of idle periods and busy periods of the server for dependent sequences and we conclude with an Hölderian invariance principle.

1. Introduction

Retrial queueing systems are characterized by the property that a customer who finds the server busy joins the orbit and repeats his request after some random time. Retrial queues have been widely used in several application domains such as telecommunication or computer networks and systems. A review of the main results and description of situations where retrial queues arise can be found in the survey papers (e.g. Yang and Templeton [16], Falin [9]) and in the didactical books (e.g. Falin and Templeton [10] and Artalejo and Gomez-Corall [4]). We also refer the reader to the papers by Artalejo, for an overview of the progress in this area during the periods of 1990-1999 [6] and 2000-2009 [5].

In these retrial queueing systems, one is generally interested in the performance analysis of some characteristics such as the waiting time of a customer, the busy period, the number of served customers, etc. In this work, we are particularly interested in the busy period analytical analysis of the M/G/1 retrial queue. Such analysis is very helpful for an efficient organisation of the system.

The structure of the busy period in the M/G/1 retrial queue and its analysis with Laplace-Stieltjes transform were addressed using different approaches (see Falin and Templeton [11], Falin [10] and Artalejo [2]). Among the approaches used to derive an estimation of the density function of the busy period, one can mention the principle of maximum entropy and the one based on a truncated retrial queue [4].

Let us recall that the busy period during which n customers are served, noted L_n, can be defined as

$$L_n = \sum_{i=1}^{n} (R_i + S_i), \ n \geq 1, \quad R_1 = 0,$$

where R_i and S_i denote the idle and busy periods of the server respectively.

Our goal in this paper is to determine analytically the asymptotic distribution of L_n. As the duration of R_{i+1} depends on both the primary arrivals and on the number of customers in the orbit at the end of the service-time S_i of the last served customer, we cannot apply the Donsker's functional central limit theorem. To circumvent this difficulty, we propose two different approaches.

The first one, called orbit features-based approach, rely on the Artalejo-Falin decomposition [1] of the busy period of the M/G/1 retrial queue as a sequence $\{(L_k^{(i)}, L_k^{(b)})\}$ of alternating idle and busy periods of the orbit. Using this decomposition and the hölderian invariance principle established in [14], we derive the asymptotic distribution of the busy period of the system. More precisely, from the sequence $\{(L_k^{(i)}, L_k^{(b)})\}$, and under some conditions we show that the results established in [14], can be applied to prove the convergence of L_n to a Gaussian law.

In the second approach, called server-state-based approach, the busy period L is given as the sum of n dependent random variables, given that n is the number of customers served during this period. After estimating the number of customers in the orbit during L, we use an Hölderian invariance principle [13] in the case where the kind of dependence of R_i is α-mixing (strong mixing) to prove that the asymptotic law of L_n is Gaussian.

The paper is organized as follows. After some preliminaries on the M/G/1 retrial queueing system, the orbit features-based approach is described in Section 3. In Section 4, we present the server-state-based approach before concluding.

2. Preliminaries

In this section, we briefly introduce the $M/G/1$ retrial queueing system. To this end, we borrow the notations used by Artalejo and Falin in [1]. First, we consider that primary customers arrive according to a Poisson flow of rate $\lambda > 0$. A customer who

finds the server busy must leave the service area and joins the orbit to retry again until finding the server free. The delays between retrials of each customer in the orbit are independent and identically distributed (i.i.d), exponentially distributed with rate $\mu > 0$. A customer finding the server free is served immediately. We also consider that the sequence of service times is a sequence of i.i.d. random variables with distribution B on R_+. Let $B(t)$, $t \geqslant 0$ be the probability distribution of the service time with Laplace-Stieltdjes transform $\beta(s)$, $Re(s) > 0$ and the k-th order moments $\beta_k = (-1)^k \beta^{(k)}(0)$, $k \in \mathbb{N}$. We also assume that $B(0+) = 0$; in other words the service time cannot be zero. Interarrival times, service times and retrial times are mutually independent random variables.

Let η_i be the time at which the ith service completion occurs, ξ_i the time at which the ith service starts, and π_i the arrival time of the ith primary customer. We also take $\rho = \lambda \beta_1$. The system evolves in the following way. At time η_{i-1}, the $(i-1)^{th}$ customer completes its service and the server becomes free. The next customer enters service after some random time R_i, during which the server is free. Let $N(t)$ be the number of customers in the orbit at time t. If the number of customers in the orbit at time η_{i-1} is $N_{i-1} = n \in \mathbb{N}$, then R_i is exponentially distributed with rate $\lambda + n\mu$. The ith service time corresponds to a primary customer with probability $\frac{\lambda}{\lambda+n\mu}$ and it corresponds to a repeated attempt with probability $\frac{n\mu}{\lambda+n\mu}$. The ith service time S_i start at time $\xi_i = \eta_{i-1} + R_i$. Note that repeated attempts that occur during this service time does not modify the state of the system. At time $\eta_i = \xi_i + S_i$, the server becomes idle again. Thus, the evolution of our retrial queue is described in terms of an alternating sequences (R_i, S_i) of idle and busy periods of the server.

3. Orbit features-based approach

The evolution of our retrial queue exhibits an alternating sequence of idle and busy periods of the server. In contrast to the standard M/G/1 queue, it is possible to have an idle server while the orbit might contain customers, and consequently the system, is not empty. Hence, the ordinary busy period of the M/G/1 retrial queue, L, is defined as the period starting at time t_0 with the arrival of a customer who finds the system empty and ends at the first service completion time t_1 at which the system becomes empty again.

The main characteristic of the sequence (R_i, S_i) is the dependence of R_i on N_{i-1}. In order to overcome this difficulty, Artalejo and Falins (1996) introduced the orbit busy period $(L^{(b)})$ and the orbit idle period $(L^{(i)})$, which are defined as follows. The orbit idle period $L^{(i)}$ is the period that starts at time when a customer alone in the orbit produces a repeated call and finds the server idle (thus the orbit

becomes empty), and ends when a primary customer finds the server busy and is constrained to join the orbit. The orbit busy period $L^{(b)}$ is the period that starts at time when a primary customer arrives and finds the server busy and the orbit idle, and ends at the next time at which a repeated attempt finds the server idle and the orbit becomes empty.

In the following, we present a description of this period based on the sequence $(L_k^{(i)}, L_k^{(b)})$ of alternating idle and busy periods of the orbit (see Figure 1). Let T_k be the random duration of the competition between a service time and arrival input which takes place just before the beginning of $L_k^{(b)}$. Thus, T_k ends with the arrival of a primary customer.

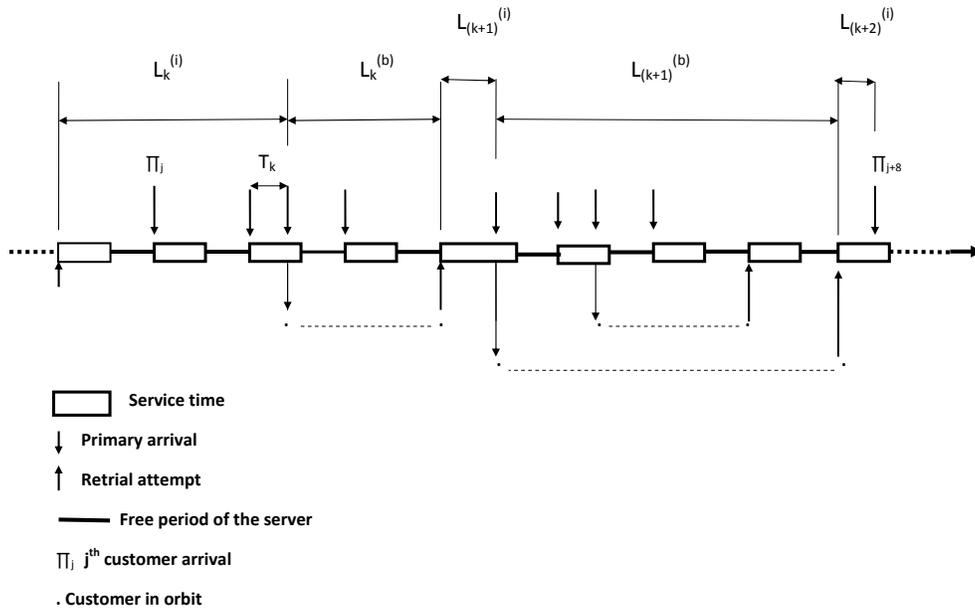Figure 1 shows the evolution of the system.

Figure 1: Evolution of the System

The duration of $L_k^{(i)}$ is determined by the minimum between the service time and the arrival of a new customer. Using conditional probabilities and the properties of

the exponential law, if the duration of $L_k^{(i)}$ ends at completion of the service time, we obtain the following conditional density function of the duration of this competition:

$$f_1(t) = \frac{1}{\beta(\lambda)} B'(t) e^{-\lambda t}.$$

On the contrary, if the duration of $L_k^{(i)}$ ends at the arrival of a customer, then the conditional density function of the duration of this competition is

$$f_2(t) = \frac{1}{1 - \beta(\lambda)} \lambda e^{-\lambda t} (1 - B(t)).$$

In Artalejo and Falin [1], a busy period L is expressed in terms of several orbit busy periods $L_k^{(b)}$ and periods of competition between the service time and the Poisson input process T_k (inside the busy period L, $T_k \equiv L_k^{(i)}$). Ω is the duration of the competition that leads to the end of L, so Ω ends in completion of the service ($\Omega = S_p$, if we served p customers during the busy period).

The distribution of $L_k^{(b)}$ depends on the distribution of T_k, consequently, $L_k^{(b)}$ is not a sequence of i.i.d. random variables. However, if we consider the sequence $C_k^b = T_k + L_k^{(b)}$, then we can write L in terms of i.i.d. random variables. We also denote $M^{(b)}$ the number of orbit busy periods which take place during the busy period L. Then, we can write:

$$L_m = (L/_{M^{(b)}=m-1}) = \left(\sum_{k=1}^{k=m-1} C_k^b \right) + \Omega, \ m \geq 2. \tag{1}$$

Figure 2 illustrates the decomposition of a busy period L in terms of the random variable
$C_k^b = T_k + L_k^{(b)}$ with $M^{(b)} = m - 1 = 3$; τ_1, τ_2, τ_3 are realizations of the variables T_1, T_2, T_3 respectively and ω is a realization of Ω.
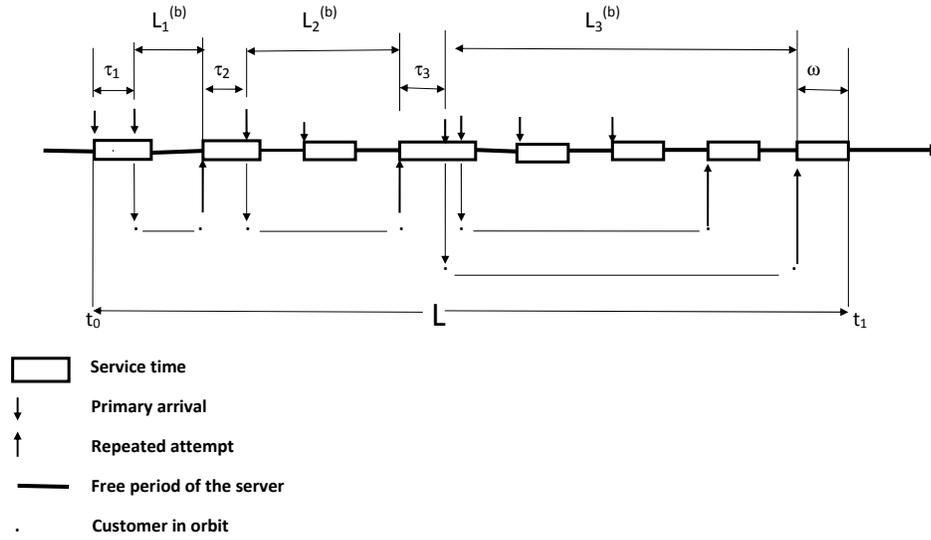
Figure 2: A Busy Period L of the System

In this example, we see that

$$(L/_{M^{(b)}=3}) = T_1 + L_1^{(b)} + T_2 + L_2^{(b)} + T_3 + L_3^{(b)} + \Omega$$

$$(L/_{M^{(b)}=3}) = (\sum_{k=1}^{k=3} C_k^{(b)}) + \Omega.$$

Remark 1.
• *The density function of Ω is $f_1(t)$.*
• *All the random variables T_k, $k \geq 1$ have the same distribution as of T. The density of T is $f_2(t)$.*
• *The random variables $L_k^{(b)}$, $k \geq 1$ are of the same distribution as of $L^{(b)}$. Consequently, the random variables C_k^b, $k \geq 1$ are identically distributed. Henceforth, we denote the corresponding generic random variable by C^b.*

154

Remark 2. *It is shown in Artalejo and Falin [1], that under the condition $\rho < 1$,*

$$E(L^b) = \frac{\beta(\lambda)p_{00}^{-1} - 1}{\lambda(1-\lambda)} \quad and \quad E(T) = \frac{1}{\lambda} + \frac{\beta'(\lambda)}{1-\beta(\lambda)}$$

Consequently,

$$E(C^b) = \frac{\beta(\lambda)p_{00}^{-1} + \lambda\beta'(\lambda) - \beta(\lambda)}{\lambda(1-\beta(\lambda))}, \tag{2}$$

where $p_{00} = (1-\rho)\exp(\frac{-\lambda}{\mu}\int_0^1 \frac{1-\beta(\lambda-\lambda u)}{\beta(\lambda-\lambda u)-u}du)$.

Lemma 1. *If $\rho < 1$, then*

$$E(\Omega) = -\frac{\beta'(\lambda)}{\beta(\lambda)}, \qquad E(\Omega^2) = \frac{\beta''(\lambda)}{\beta(\lambda)}. \tag{3}$$

$$E((C^b)^2) = \frac{\lambda^2\beta''(\lambda) - 2(p_{00}^{-1}-1)\lambda\beta'(\lambda) + \beta(\lambda)(-E(L^2) + 2(p_{00}^{-1}-1)^2)}{\lambda^2(1-\beta(\lambda))}, \tag{4}$$

where

$$E(L^2) = \frac{1}{p_{00}}\left[\frac{1}{1-\rho^2}\left(\frac{2\rho\beta_1}{\mu} + \beta_2\right) - \int_0^1 \frac{2}{\lambda\mu(\beta(\lambda-\lambda t)-t)}\times \right.$$
$$\left.\left(1 - \frac{\lambda(1-t)\beta'(\lambda-\lambda t)}{\beta(\lambda-\lambda t)-t} - \frac{1}{1-\rho}\exp\{\frac{\lambda}{\mu}\int_t^1 \frac{1-\beta(\lambda-\lambda u)}{\beta(\lambda-\lambda u)-u}du\}\right)dt\right].$$

Proof. 1. The Laplace-Stieltjes transform of Ω is given by

$$\Psi(s) = \frac{\beta(s+\lambda)}{\beta(\lambda)}.$$

If we derive this function with respect to s, we obtain

$$\Psi'(s) = \frac{\beta'(s+\lambda)}{\beta(\lambda)}.$$

Since $\Psi'(0) = -E(\Omega)$, then

$$E(\Omega) = -\frac{\beta'(\lambda)}{\beta(\lambda)}.$$

To obtain $E(\Omega^2)$, it is sufficient to derive this function a second time with respect to s and take $s = 0$, so $\Psi''(0) = E(\Omega^2)$.

2. Let $g(s)$ and $\Phi(s)$ be the Laplace transforms of C^b and L respectively. We have the following relation

$$g(s) = \frac{1}{1 - \beta(\lambda)}(1 - \frac{\beta(s + \lambda)}{\Phi(s)}).$$

The first derivative of g with respect to s is given by

$$g'(s) = \frac{-1}{1 - \beta(\lambda)}(\frac{\beta'(s + \lambda)\Phi(s) - \Phi'(s)\beta(s + \lambda)}{(\Phi(s))^2}).$$

If we take the second derivative, we obtain

$$g''(s) = \frac{-1}{1 - \beta(\lambda)} \frac{1}{\Phi^4(s)} \left[\beta''(s + \lambda)\Phi(s) + \beta(s + \lambda)(-\Phi''(s) + 2(\Phi'(s))^2\phi(s)) \right.$$
$$\left. -2\phi'(s)(\phi(s))^2\beta'(s + \lambda) \right].$$

Similarly, we show that

$$E((C^b)^2) = \frac{1}{\lambda^2(1 - \beta(\lambda))}[\lambda^2\beta''(\lambda) - 2(p_{00}^{-1} - 1)\lambda\beta'(\lambda) + \beta(\lambda)(-E(L^2) + 2(p_{00}^{-1} - 1))^2].$$

We shall use the notations for $m \geq 1$,

$$a_m = E(\Omega) + (m - 1)E(C^b) = -\frac{\beta'(\lambda)}{\beta(\lambda)} + (m-)\frac{\beta(\lambda)p_{00}^{-1} + \lambda\beta'(\lambda) - \beta(\lambda)}{\lambda(1 - \beta(\lambda))},$$

$$s_m^2 = Var(\Omega) + (m - 1)Var(C^b),$$

where $Var(\Omega)$ and $Var(C^b)$ are obtained from the formulas (2), (3) and (4).

Our first result is the following theorem.

Theorem 2. *Assume that the random variables Ω and C^b are non degenerate, and there exists $\gamma > 2$ such that $\forall k \in \mathbb{N}^*$,*

$$\mathbb{E}|\Omega|^\gamma < \infty \quad and \quad \mathbb{E}|C^{(b)}|^\gamma < \infty. \tag{5}$$

We define for all $n \in \mathbb{N}^*$, $B_n = \frac{L_n - a_n}{s_n}$.
If $\rho < 1$, then the sequence $(B_n)_{n \geq 1}$ is asymptotically Gaussian.

Before proving theorem 2, let us recall the theorem of Hamadouche and Taleb [14]. We define the Hölder space $H_\alpha[0,1]$, $0 < \alpha \leq 1$, as the space of functions f defined on $[0,1]$, vanishing at zero such that

$$\|f\|_\alpha = \sup_{0 < |t-s| \leq 1} \frac{|f(t) - f(s)|}{|t-s|^\alpha} < +\infty.$$

Theorem 3. *(Hamadouche and Taleb [14])*
Let $(Y_n)_{n \geq 1}$ be a sequence of centered and independent random variables, non identically distributed. We assume that there exists $\gamma > 2$, and two real positives constants d and D such that $\forall j \in \mathbb{N}^*$,

$$d \leq E(Y_j^2) \quad and \quad E|Y_j|^\gamma \leq D < \infty.$$

Then for $n \in \mathbb{N}^*$, the sequence of the smoothed partial sums process

$$\zeta_n(t) = \frac{1}{e_n}\left(\sum_{i=1}^{[nt]} Y_i + (nt - [nt])Y_{[nt]+1}\right), \quad t \in [0,1],$$

converges weakly to a standard Brownian motion $(W_t)_{t \in [0,1]}$ in H_α, for all $0 < \alpha < \frac{1}{2} - \frac{1}{\gamma}$, where $e_n^2 = \sum_{j=1}^n Var(Y_j)$.

Proof. of theorem 2 Using the notations $X_k = C_{k-1}^b$, $k \geq 2$ and $X_1 = \Omega$, we have
$$L_m = \sum_{k=1}^{k=m} X_k.$$
   We consider the sequence $(Y_k = X_k - E(X_k)$, $k \geq 1$, and we shall verify that the conditions of theorem 3 are satisfied. Indeed, the sequence $\zeta_n \in H_\alpha[0,1]$.

1. We have $E(Y_1^2) = Var(X_1) = Var(\Omega)$ and
   for $k \geq 2$, $EY_k^2 = Var(X_k) = Var(C^b)$.

   The variables Ω and C^b are non degenerate and then, we have

   $$\min(Var(\Omega),\ Var(C^b)) > 0.$$

   Then, in order to satisfy condition $d \leq E(Y_j^2)$, it is sufficient to take

   $$d = \min(Var(\Omega),\ Var(C^b)).$$

2. Condition (5) insures that

   $$D = \max(E|\Omega - E(\Omega)|^\gamma,\ E|C^b - E(C^b)|^\gamma) < \infty.$$

Using the fact that $Y_1 = \Omega - E(\Omega)$ and $Y_k = C^b - E(C^b)$ for $k \geq 2$, we obtain

$$E|Y_j|^\gamma \leq D < \infty, \forall j \geq 1.$$

3. Let's now calculate e_m^2

$$e_m^2 = \sum_{k=1}^{k=m} Var(Y_k).$$

Consequently

$$e_m^2 = \sum_{k=1}^{k=m} E(Y_k^2) = \sum_{k=1}^{k=m} Var(X_k) = Var(\Omega) + \sum_{k=2}^{k=m} Var(C^b).$$

It follows that

$$e_m^2 = Var(\Omega) + (m-1)Var(C^b)$$

which means that

$$e_m^2 = s_m^2.$$

Using theorem 3, we deduce that the sequence

$$\frac{1}{s_m} \sum_{k=1}^{[mt]} Y_k + \frac{1}{s_m}(mt - [mt])Y_{[mt]+1}, \quad t \in [0,1], \ m \in \mathbb{N}^*$$

converges to the standard Brownian motion in H_α for all $0 < \alpha < \frac{1}{2} - \frac{1}{\gamma}$.

In the following, we need the continuous mapping theorem given by lemma 4.

Lemma 4. *(Billingsley [7])*
Let F be a continuous function from a metric space E to an other metric space E'. If a sequence (Z_n) of random variables converges weakly to Z in E, then $F(Z_n)$ converges weakly to $F(Z)$ in the space E'.

Let us now consider the continuous function $F : H_\alpha \longrightarrow \mathbb{R}$, given by $F(g) = g(1)$. Using lemma 4, we get $F(\xi_m)$ which converges in distribution in $\mathbb{R}$ to $F(W_t) = W_1 =^d N(0,1)$, where $=^d$ means the equality in distribution, and

$$F(\xi_m) = \frac{\sum_{k=1}^{k=m}(X_k - E(X_k))}{s_m} = \frac{(\sum_{k=1}^{k=m} X_k) - \sum_{k=1}^{k=m} E(X_k)}{s_m}$$

$$= \frac{(\sum\limits_{k=1}^{k=m} X_k) - E(\Omega) - (m-1)E(C^b)}{s_m} = \frac{(\sum\limits_{k=1}^{k=m} X_k) - a_m}{s_m},$$

This concludes the proof of the theorem 2.

4. SERVER-STATE-BASED APPROACH

As mentioned previously, the system evolves according to the sequence $\{(R_i, S_i) : i \geq 0\}$ of busy and idle periods of the server. Thus, a busy period of the system consists in an alternation of service times and idle server periods in which the server is free whereas there are some customers in the orbit.

Let L be a busy period of the M/G/1 retrial queue. Let us denote by I the number of customers served during this period, then the busy period can be expressed as follows

$$L = \sum_{i=1}^{i=I}(R_i + S_i), \quad I \geq 1,$$

where $R_1 = 0$.

The random variable R_i depends on the history of the system until η_{i-1} only through the number of customers in the orbit at this time, N_{i-1}, and has the conditional distribution: $G(x) = P(R_i < x/N_{i-1} = k) = 1 - e^{(-\lambda + k\mu)x}$ with mean $E(R_i/N_{i-1=k}) = \frac{1}{\lambda+k\mu}$.

We recall the following lemma.

Lemma 5. *(Falin and Templeton [11]) The random variables $(S_i + R_i)_{i \geq 2}$ are identically distributed if and only if random variables N_i are identically distributed.*

In the following, we assume that $\rho < 1$ and the number of customers in the orbit during the period L is estimated to $n_e = [\overline{n}_o] + 1$, where $\overline{n}_o = \frac{\lambda^2 \beta_2}{2(1-\rho)} + \frac{\lambda\rho}{\mu(1-\rho)}$ (see [16]). Therefore, the random variables $(U_i = S_i + R_i)$ are identically distributed.

Let us now focus our interest on the asymptotic distribution of the random variable $A_n = \sum\limits_{i=1}^{i=n} U_i$, where R_i are exponential r.v's. with parameter $\frac{1}{\lambda+n_e\mu}$, $i \geq 1$.

Note

$$b_n = \frac{n}{\lambda + n_e\mu} + n\beta_1, \quad n \geq 1$$

and

$$\sigma^2 = \beta_2 - \beta_1^2 + \frac{1}{(\lambda + n_e\mu)^2} + 2\sum_{i=2}^{i=\infty} Cov(R_1, R_i).$$

Let $Z_i = U_i - \mathbb{E}(U_i)$. The sequence (Z_i) of centered random variables are non independent and identically distributed. In the case where (Z_i) is strong mixing, we consider $(\alpha_n)_{n\geq 1}$ the sequence of strong mixing coefficients of $(Z_n)_{n\geq 1}$ and we obtain Theorem 6 below. Before, we need to recall the following definitions.

Definition 1. *The strong mixing coefficient between two σ-fields* **A**, **B** *is defined by*

$$\alpha(\mathbf{A}, \ \mathbf{B}) = sup_{(A,B)\in\mathbf{A}\times\mathbf{B}}|\mathbb{P}(A\cap B) - P(A)P(B)|.$$

Definition 2. *Let $(Y_n)_{n\geq 1}$ be a sequence of random variables defined on the same probability space. The strong mixing coefficient α_n is defined by*

$$\alpha_n = sup_{k\geq 1}\alpha(\mathbf{M_1}^k, \mathbf{M_{k+n}^\infty})$$

where M_n^k is the σ-field spanned by the variables $(X_i, k \leq i \leq n)$.

Definition 3. *The sequence $(Y_n)_{n\geq 1}$ is called α-mixing (or strong mixing) if α_n goes to 0 as n goes to ∞.*

Theorem 6. *Assume that $(R_n + S_n - E(R_n + S_n))_{n\geq 1}$ is an $\alpha-$mixing sequence, strictly stationary random variables and that there are some reals $\gamma > 2$ and $\varepsilon > 0$, such that*

$$\mathbb{E}|S_i|^{\gamma+\varepsilon} < \infty \ ; \ \mathbb{E}|R_i|^{\gamma+\varepsilon} < \infty \quad and \quad \Sigma_{n=1}^\infty (n+1)^{\frac{\gamma}{2}-1}[\alpha_n]^{\frac{\varepsilon}{\gamma+\varepsilon}} < \infty.$$

We define for all $n \in \mathbb{N}^*$, $D_n = \frac{A_n-b_n}{\sigma\sqrt{n}}$.
If $\rho < 1$, then $(D_n)_{n\geq 1}$ is asymptotically Gaussian.

Proof. of theorem 6 For the proof, we need the following theorem:

Lemma 7. *(Hamadouche [13])*
   Let $(Y_n)_{n\geq 1}$ be a strictly stationary $\alpha-$mixing sequence and centered random variables. We assume that there are $\gamma > 2$ and $\varepsilon > 0$ such that $E|Y_1|^{\gamma+\varepsilon} < \infty$ and $\Sigma_{n=1}^\infty (n+1)^{\frac{\gamma}{2}-1}[\alpha_n]^{\frac{\varepsilon}{\gamma+\varepsilon}} < \infty$. Then for $n \in \mathbb{N}^*$, the sequence of smoothed partial sums process

$$\xi_n(t) = \frac{1}{\sigma\sqrt{n}}\left(\sum_{i=1}^{[nt]} Y_i + (nt - [nt])Y_{[nt]+1}\right), \quad t \in [0,1], \ n \geq 1$$

converges to the standard Brownian motion W in H_α^0 for all $0 < \alpha < \frac{1}{2} - \frac{1}{\gamma}$, where $\sigma^2 = EY_1^2 + 2\sum_{j=2}^\infty Cov(Y_1, Y_j) < \infty$.

Under conditions of theorem 6, the sequence of centered random variables $(Z_n = U_n - \mathbb{E}(U_n))_{n \geq 1}$ is a strong mixing and stationary.

The assumption that there exist $\gamma > 2$, $\varepsilon > 0$, such that

$$\mathbb{E}|S_i|^{\gamma+\varepsilon} < \infty \quad and \quad \mathbb{E}|R_i|^{\gamma+\varepsilon} < \infty,$$

gives $E|Z_n|^{\gamma+\varepsilon} < \infty$.

We also have the coefficients $(\alpha_n)_{n \geq 1}$ of the sequence $(Z_n)_{n \geq 1}$ which satisfy the condition $\Sigma_{n=1}^{\infty}(n+1)^{\frac{\gamma}{2}-1}[\alpha_n]^{\frac{\varepsilon}{\gamma+\varepsilon}} < \infty$ of lemma 7.

We conclude, using this lemma, that the sequence

$$\xi_n(t) = \frac{1}{\overline{\sigma}\sqrt{n}}\sum_{i=1}^{[nt]} Z_i + (nt - [nt])Z_{[nt]+1}, \ t \in [0,1], \quad n \geq 1$$

converges weakly to W in H_α, $\alpha < \frac{1}{2} - \frac{1}{\gamma}$, with

$$\overline{\sigma}^2 = E((Z_1)^2) + 2\sum_{i=2}^{i=\infty} Cov(Z_1, Z_i).$$

On the other hand, we have

$$\overline{\sigma}^2 = Var(U_1) + 2\sum_{i=2}^{i=\infty} Cov(U_1, U_i) = \beta_2 - \beta_1^2 + \frac{1}{(\lambda + n_e\mu)^2} + 2\sum_{i=2}^{i=\infty} Cov(R_1, R_i) = \sigma^2.$$

Consider the function defined on $(H_\alpha, \|.\|_\alpha)$; $F : H_\alpha \longrightarrow \mathbb{R}$ given by $F(g) = g(1)$. Using the continuous mapping theorem, we get $F(\xi_n)$ which converges in distribution to $F(W_t) =^d N(0,1)$. After some computations, we obtain $F(\xi_n) = \frac{A_n - b_n}{\sigma\sqrt{n}}$, which converges to $N(0,1)$. This achieves the proof of theorem 6.

5. CONCLUSION

In this paper, we have derived the limiting probability distribution of the busy period of M/G/1 retrial queueing system. To circumvent the difficulty to apply the Donsker's central limit theorem, we proposed two different approaches. The first one, called orbit features-based approach, rely on the Artalejo-Falin decomposition [1] of the busy period of the M/G/1 retrial queue as a sequence of alternating idle and busy periods of the orbit. Using this decomposition and the Hölderian invariance principle established in [14], we derive the asymptotic distribution of the busy period of the system. In the second approach, called server-state-based approach, the busy

period is given as the sum of dependent random variables. After estimating the number of customers in the orbit during the busy period, we used the Hölderian invariance principle [13] (under some conditions) to prove that the asymptotic law of the busy period is Gaussian. These contributions provide an original application of the results established in [13, 14].

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