

## JENSEN TYPE INEQUALITY FOR SEMINORMED FUZZY INTEGRALS

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**ABSTRACT.** In this paper, we generalize Jensen type inequality for seminormed fuzzy integrals where  $\mu$  is an arbitrary fuzzy measure and  $T$  is a  $t$ -seminorm on  $[0, 1]$ . In the continue, we bring a corollary and some examples that illustrate the results.

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### 1. INTRODUCTION

The theory of fuzzy measures and fuzzy integrals was introduced by Sugeno [17]. The properties and applications of the Sugeno integral have been studied by many authors, including Ralescu and Adams [13], Román-Flores et al. [4, 14, 15] and Wang and Klir [18]. Many authors generalized the Sugeno integral by using some other operators to replace the special operators  $\wedge$  and/or  $\vee$  (see, e.g., [7, 10, 11]). Suárez and Gill presented two families of fuzzy integrals, the so-called seminormed fuzzy integrals and semiconormed fuzzy integrals [16].

The study of inequalities for Sugeno integral was initiated by Román-flores et al. [14, 15] and then followed by others ( see [1, 2, 3, 8, 9]). In [4], a fuzzy Chebyshev inequality for a special case was obtained which has been generalized by Ouyang and Mesiar [10]. More precisely they proved the following Chebyshev type inequality for seminormed fuzzy integrals and related inequality for semiconormed fuzzy integrals.

**Theorem 1.** *Let  $(X, \mathcal{F}, \mu)$  be a fuzzy measure space and  $f, g : X \rightarrow [0, 1]$  two comonotone measurable functions. Let  $\star : [0, 1]^2 \rightarrow [0, 1]$  be continuous and non-decreasing in both arguments. If the seminorm  $T$  satisfies*

$$T(a \star b, c) \geq (T(a, c) \star b) \vee (a \star T(b, c)), \quad (1.1)$$

*then*

$$\int_{T,A} f \star g d\mu \geq \left( \int_{T,A} f d\mu \right) \star \left( \int_{T,A} g d\mu \right) \quad (1.2)$$

*holds for any  $A \in \mathcal{F}$ .*

**Theorem 2.** *Let  $(X, \mathcal{F}, \mu)$  be a fuzzy measure space and  $f, g : X \rightarrow [0, 1]$  two comonotone measurable functions. Let  $\star : [0, 1]^2 \rightarrow [0, 1]$  be continuous and non-decreasing in both arguments. If the seminorm  $S$  satisfies*

$$S(a \star b, c) \leq (S(a, c) \star b) \wedge (a \star S(b, c)), \quad (1.3)$$

then

$$\int_{S,A} f \star g d\mu \leq \left( \int_{S,A} f d\mu \right) \star \left( \int_{S,A} g d\mu \right) \quad (1.4)$$

holds for any  $A \in \mathcal{F}$ .

This paper is organized as follows: in Section 2, some preliminaries and summarization of previous known results are given. Section 3 include general Jensen type inequality for seminormed fuzzy integrals. Finally, Section 4 contains a short conclusion.

## 2. PRELIMINARIES

In this section, we recall some basic definition and previous results that will be used in this paper.

Let  $X$  be a non-empty set,  $\mathcal{F}$  be a  $\sigma$ -algebra of subsets of  $X$ . Let  $\mathbb{N}$  denote the set of all positive integers. Throughout this paper, all considered subsets are supposed to belong to  $\mathcal{F}$ .

**Definition 1** (Sugeno (25)). *A set function  $\mu : \mathcal{F} \rightarrow [0, 1]$  is called a fuzzy measure if the following properties are satisfied:*

(FM1)  $\mu(\emptyset) = 0$  and  $\mu(X) = 1$ ;

(FM2)  $A \subset B$  implies  $\mu(A) \leq \mu(B)$ ;

(FM3)  $A_n \rightarrow A$  implies  $\mu(A_n) \rightarrow \mu(A)$ .

When  $\mu$  is a fuzzy measure, the triple  $(X, \mathcal{F}, \mu)$  is called a fuzzy measure space.

Let  $(X, \mathcal{F}, \mu)$  be a fuzzy measure space and  $\mathcal{F}_+(X) = \{f | f : X \rightarrow [0, 1] \text{ is measurable with respect to } \mathcal{F}\}$ . In what follows, all considered functions belong to  $\mathcal{F}_+(X)$ . For any  $\alpha \in [0, 1]$ , we will denote the set  $\{x \in X | f(x) \geq \alpha\}$  by  $F_\alpha$  and  $\{x \in X | f(x) > \alpha\}$  by  $F_{\bar{\alpha}}$ . Clearly, both  $F_\alpha$  and  $F_{\bar{\alpha}}$  are non-increasing with respect to  $\alpha$ , i.e.,  $\alpha \leq \beta$  implies  $F_\alpha \supseteq F_\beta$  and  $F_{\bar{\alpha}} \supseteq F_{\bar{\beta}}$ .

**Definition 2** (Sugeno [17]). *Let  $(X, \mathcal{F}, \mu)$  be a fuzzy measure space and  $A \in \mathcal{F}$ , the Sugeno integral of  $f$  over  $A$ , with respect to the fuzzy measure  $\mu$ , is defined by*

$$\int_A f d\mu = \bigvee_{\alpha \in [0,1]} (\alpha \wedge \mu(A \cap F_\alpha)).$$

When  $A = X$ , then

$$\int_X f d\mu = \int f d\mu = \bigvee_{\alpha \in [0,1]} (\alpha \wedge \mu(F_\alpha)).$$

Notice that Ralescu and Adams (see [13]) extended the range of fuzzy measures and the Sugeno integrals from  $[0,1]$  to  $[0,\infty]$ . But we only deal with the original fuzzy measure and the Sugeno integrals which was introduced by Sugeno in 1974.

Note that in the above definition,  $\wedge$  is just the prototypical  $t$ -norm minimum and  $\vee$  the prototypical  $t$ -conorm maximum.

A  $t$ -norm [6] is a function  $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$  satisfying the following conditions:

- (A)  $T(x, 1) = T(1, x) = x \quad \forall x \in [0, 1]$ ;
- (B)  $\forall x_1, x_2, y_1, y_2 \in [0, 1]$ , if  $x_1 \leq x_2, y_1 \leq y_2$ , then  $T(x_1, y_1) \leq T(x_2, y_2)$ ;
- (C)  $T(x, y) = T(y, x)$ ;
- (D)  $T(T(x, y), z) = T(x, T(y, z))$ .

A function  $S : [0, 1] \times [0, 1] \rightarrow [0, 1]$  is called a  $t$ -conorm, if there is a  $t$ -norm  $T$  such that  $S(x, y) = 1 - T(1 - x, 1 - y)$ .

Evidently, a  $t$ -conorm  $S$  satisfies:

- (A')  $S(x, 0) = S(0, x) = x, \quad \forall x \in [0, 1]$  as well as conditions (B), (C) and (D).

A binary operator  $T$  ( $S$ ) on  $[0, 1]$  is called a  $t$ -seminorm ( $t$ -semiconorm) [16] if it satisfies the above conditions (A) and (B) ((A') and (B)). By using the concepts of  $t$ -seminorm and  $t$ -semiconorm, Suárez and Gil proposed two families of fuzzy integrals:

**Definition 3.** Let  $T$  be a  $t$ -seminorm, then the seminormed fuzzy integral of  $f$  over  $A$  with respect to  $T$  and the fuzzy measure  $\mu$  is defined by

$$\int_{T, A} f d\mu = \bigvee_{\alpha \in [0,1]} T(\alpha, \mu(A \cap F_\alpha)). \quad (2.1)$$

When  $A = X$ , then

$$\begin{aligned} \int_{T, X} f d\mu &= \int_T f d\mu \\ &= \bigvee_{\alpha \in [0,1]} T(\alpha, \mu(F_\alpha)). \end{aligned} \quad (2.1).$$

**Definition 4.** Let  $S$  be a  $t$ -semiconorm, then the semiconormed fuzzy integral of  $f$  over  $A$  with respect to  $S$  and the fuzzy measure  $\mu$  is defined by

$$\int_{S, A} f d\mu = \bigwedge_{\alpha \in [0,1]} S(\alpha, \mu(A \cap F_\alpha)).$$

It is easy to see that the Sugeno integral is a special seminormed fuzzy integral. Moreover, Kandel and Byatt in (see [6]) showed another expression of the Sugeno

integral as follows:

$$\int_A f d\mu = \bigwedge_{\alpha \in [0,1]} (\alpha \vee \mu(A \cap F_{\bar{\alpha}})).$$

So the seminormed fuzzy integrals also generalized the concept of the Sugeno integral. Notice that the seminormed fuzzy integral is just the family of the weakest universal fuzzy integrals. Note that if  $\int_{T,A} f d\mu = a$ , then  $T(\alpha, \mu(A \cap F_{\alpha})) \leq a$  for all  $\alpha \in [0, 1]$  and for  $\varepsilon > 0$  there exists  $\alpha_{\varepsilon}$  such that  $T(\alpha_{\varepsilon}, \mu(A \cap F_{\alpha_{\varepsilon}})) \geq a - \varepsilon$ . Also, if  $\int_{S,A} f d\mu = a$ , then  $S(\alpha, \mu(A \cap F_{\bar{\alpha}})) \geq a$  for all  $\alpha \in [0, 1]$  and, for  $\varepsilon > 0$  there exists  $\alpha_{\varepsilon}$  such that  $S(\alpha_{\varepsilon}, \mu(A \cap F_{\bar{\alpha}_{\varepsilon}})) \geq a + \varepsilon$ .

**Remark 1.** From a numerical point of view, if the distribution function  $F$  associated to  $f$ ,  $F(\alpha) = \mu(\{f \geq \alpha\})$  is continuous, then the fuzzy integral can be calculated solving the equation  $F(\alpha) = \alpha$ .

### 3. JENSEN'S INEQUALITY FOR SEMINORMED FUZZY INTEGRALS

In this section, we will prove a Jensen type inequality for seminormed fuzzy integrals.

**Theorem 3.** Let  $(X, \mathcal{F}, \mu)$  be a fuzzy measure space and  $f : X \rightarrow [0, 1]$  and let  $T$  be a  $t$ -seminorm such that  $\int_{T,A} f d\mu = p$  for any  $A \in \mathcal{F}$ . If  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a strictly increasing function such that  $\phi(x) \leq x$  for any  $x \in [0, p]$ , then:

$$\phi\left(\int_{T,A} f d\mu\right) \leq \int_{T,A} \phi(f) d\mu. \quad (3.1)$$

*Proof.* Let  $\int_{T,A} f d\mu = p$ , then for any  $\varepsilon > 0$ , there exists  $p_{\varepsilon}$  such that  $\mu(A \cap F_{p_{\varepsilon}}) = p_1$ , where  $T(p_{\varepsilon}, p_1) \geq p - \varepsilon$ . We define:  $H_a = \{x \mid \phi(f(x)) \geq a\}$ . Hence

$$\begin{aligned} \int_{T,A} \phi(f(x)) d\mu &= \bigvee_{a \in [0,1]} (T(a, \mu(A \cap H_a))) \\ &\geq T(p_{\varepsilon}, p_1) \\ &\geq p - \varepsilon. \end{aligned}$$

from the arbitrariness of  $\varepsilon$ , we have:

$$\int_{T,A} \phi(f(x)) d\mu \geq p. \quad (3.2)$$

Due to hypothesis and from (3.2) we have:

$$\int_{T,A} \phi(f(x)) d\mu \geq p \geq \phi(p).$$

Thus

$$\int_{T,A} \phi(f(x)) d\mu \geq \phi\left(\int_{T,A} f d\mu\right).$$

**Example 1.** Let  $\mu$  be the Lebesgue measure on  $\mathbb{R}$  and  $T$  be defined as  $T(x, y) = \min\{x, y\}$  and consider the function  $f(x) = (1-x)\chi_{[0,1]}(x)$ . If we define:

$$\phi(x) = \begin{cases} x/2 & \text{if } 0 \leq x < 1/2 \\ \frac{2x+1}{4} & \text{if } x \geq 1/2 \end{cases}$$

Then

$$\begin{aligned} \int_T f d\mu &= \int f d\mu \\ &= \bigvee_{\alpha \in [0,1]} (\alpha \wedge \mu(F_\alpha)) \\ &= \bigvee_{\alpha \in [0,1]} (\alpha, 1 - \alpha) \\ &= 1/2. \end{aligned}$$

Consequently

$$\int_T f d\mu = 1/2 \tag{3.3}$$

Thus

$$\begin{aligned} \phi\left(\int_{T,A} f d\mu\right) &= \phi(1/2) \\ &= 1/2. \end{aligned}$$

On the other hand, a straight forward calculus shows that:

$$\phi(f(x)) = \begin{cases} \frac{3-2x}{4} & \text{if } 0 \leq x \leq 1/2 \\ \frac{1-x}{2} & \text{if } 1/2 \leq x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Thus, by (3.3) and because  $T(x, y) = \min\{x, y\}$  we have:

$$\begin{aligned} \int_T \phi(f) d\mu &= \int \phi(f) d\mu \\ &= \bigvee_{\alpha \in [0,1]} (\alpha, \Phi(f)_\alpha) \\ &= 1/2 \\ &= \phi\left(\int_T f d\mu\right). \end{aligned}$$

Which implies that:  $\int_T \phi(f) d\mu \geq \phi\left(\int_T f d\mu\right)$ .

**Corollary 4.** Let  $(X, \mathcal{F}, \mu)$  be a fuzzy measure space and let  $T$  be a  $t$ -seminorm. If  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a strictly increasing function such that  $\phi(x) \leq x$  for any  $x \in [0, \mu(X)]$ , then  $\phi\left(\int_{T,A} f d\mu\right) \leq \int_{T,A} \phi(f) d\mu$ , holds for all  $f : X \rightarrow [0, 1]$ .

The following examples show some applications of the seminormed fuzzy Jensen's inequality (3.1) when using Corollary 4.

**Example 2.** Let  $(X, \mathcal{F}, \mu)$  be a fuzzy probability space (that is to say,  $\mu(X) = 1$ ) and  $T$  is a seminorm and consider  $f : X \rightarrow [0, 1]$ . Then, taking  $\phi(x) = x^\lambda$  with  $\lambda \geq 1$ , We have  $\phi(x) = x^\lambda \leq x$  for all  $x \in [0, 1]$ . Thus, due to Corollary 4 we obtain:

$$\left(\int_{T,A} f d\mu\right)^\lambda \leq \int_{T,A} f^\lambda d\mu$$

for all  $\lambda \geq 1$  and  $f : X \rightarrow [0, 1]$

**Example 3.** Let  $(X, \mathcal{F}, \mu)$  be a fuzzy measure space and  $T$  is a seminorm and  $f \in \mathcal{F}^\mu(x)$ , taking  $\phi(x) = \lambda x$  with  $0 \leq \lambda \leq 1$ , we have  $\phi(x) = \lambda x \leq x$  for all  $x \in [0, \infty)$  and consequently:

$$\lambda \left(\int_{T,A} f d\mu\right) \leq \int_{T,A} \lambda f d\mu$$

for all  $0 \leq \lambda \leq 1$  and  $f \in \mathcal{F}^\mu(x)$ .

To finalize this section, we will show that condition " $\phi$  strictly increasing" and " $\phi(x) \leq x$ , for every  $x \in [0, \int_{T,A} f d\mu]$ " on the function  $\phi$  in Theorem 3 cannot be

avoided, as we will illustrate in the following examples.

Since Sugeno integral is a special case of seminormed fuzzy integral, so we cannot omit  $\phi$  strictly increasing condition of  $\phi$  and we also cannot omit  $\phi(x) \leq x$  for all  $0 \leq x \leq \int_{T,A} f d\mu$  condition of  $\phi$  in Theorem 3.

#### 4. CONCLUSION

We have shown the Jensen inequality for seminormed fuzzy integrals. Observe that seminormed fuzzy integrals are general from of the sugeno integral, thus it is of great interest to determine when the equality

$$\phi\left(\int_{T,A} f d\mu\right) = \int_{T,A} \phi(f) d\mu$$

holds for comonotone function  $f, g$  and for any measurable set  $A$ . Note that we have only proved the Jensen inequality for seminormed fuzzy integrals. Does the Jensen inequality hold for general semiconormed integral? This question deserves further investigation.

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