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INCLUSION RELATIONSHIP AND ARGUMENT PROPERTIES FOR CERTAIN CLASSES OF MEROMORPHIC FUNCTIONS DEFINED BY USING DIFFER-INTEGRAL OPERATOR

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Abstract. In this paper, we introduced a new classes of $p-$valent meromorphic functions defined by using differ-integral operator. We investigate various inclusion relationship for these classes and some argument properties are considered. Also, some applications involving integral operators are studied.

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1. Introduction

Let $\Sigma_{p,n}$ denote the class of multivalent meromorphic functions of the form:

$$f(z) = z^{-p} + \sum_{k=n}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, 3, ...\};\ n > -p), \tag{1}$$

which are analytic in the punctured unit disk $\mathbb{U}^* = \{z : z \in \mathbb{C},\ \text{and } 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$. We note that $\Sigma_{p,-p+1} = \Sigma_p$. For a function $f \in \Sigma_{p,n}$, we can define

$$\begin{aligned}
\Sigma S^*_{p,n}(\alpha) &= \{f \in \Sigma_{p,n} : Re\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad (0 \le \alpha < p,\, z \in \mathbb{U})\}, \\
\Sigma C_{p,n}(\alpha) &= \{f \in \Sigma_{p,n} : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < -\alpha \quad (0 \le \alpha < p,\, z \in \mathbb{U})\},
\end{aligned}$$

$$\Sigma K_{p,n}(\beta,\alpha)$$
$$= \{f \in \Sigma_{p,n} : \exists\, g \in \Sigma S^*_{p,n}(\alpha) \text{ such that } Re\left(\frac{zf'(z)}{g(z)}\right) < -\beta \quad (0 \le \alpha, \beta < p,\ z \in \mathbb{U})\},$$

and

$$\Sigma K^*_{p,n}(\beta,\alpha)$$
$$= \{f \in \Sigma_{p,n} : \exists\, g \in \Sigma C_{p,n}(\alpha) \text{ such that } Re\left(\frac{(zf'(z))'}{g'(z)}\right) < -\beta \quad (0 \le \alpha, \beta < p,\ z \in \mathbb{U})\}.$$

These classes are meromorphic starlike, Convex, close-to-convex and quasi-convex functions, respectively (see [3], [9], [15]-[17]).

Definition 1. *For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $\omega(z)$ which is analytic in $\mathbb{U}$, satisfying the following conditions:*

$$\omega(0) = 0 \qquad \textit{and} \qquad |\omega(z)| < 1; \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(\omega(z)); \qquad (z \in \mathbb{U}).$$

Indeed it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \quad \Longrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, we have the following equivalence (see [4], [13], [14]):

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \quad \Longleftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let M be the class of functions $\varphi(z)$ which are analytic, univalent in $\mathbb{U}$ and for which $\varphi(\mathbb{U})$ is convex with $\varphi(0) = 1$ and $R[\varphi(z)] > 0$ for $z \in \mathbb{U}$. By making use of the principle of subordination between analytic functions, we define the subclasses $\Sigma S^*_{p,n}(\alpha;\phi)$, $\Sigma C_{p,n}(\alpha;\phi)$, $\Sigma K_{p,n}(\alpha,\beta,\phi,\psi)$ and $\Sigma K^*_{p,n}(\alpha,\beta;\phi,\psi)$ of the class $\Sigma_{p,n}$ for $0 \le \alpha, \beta < p$ and $\phi, \psi \in M$, which are defined by

$$\Sigma S^*_{p,n}(\alpha;\phi) = \{f \in \Sigma_{p,n} : \frac{1}{p-\alpha}\left(\frac{-zf'(z)}{f(z)} - \alpha\right) \prec \phi(z) \text{ in } \mathbb{U}\}, \tag{2}$$

$$\Sigma C_{p,n}(\alpha;\phi) = \{f \in \Sigma_{p,n} : \frac{1}{p-\alpha}\left(-\left[1 + \frac{zf''(z)}{f'(z)}\right] - \alpha\right) \prec \phi(z) \text{ in } \mathbb{U}\}, \tag{3}$$

$$\Sigma K_{p,n}(\alpha,\beta;\phi,\psi)=\{f\in\Sigma_{p,n}:\exists\, g\in\Sigma S^*_{p,n}(\alpha;\phi) \text{ such that } \frac{1}{p-\beta}\left(\frac{-zf'(z)}{g(z)}-\beta\right)\prec\psi(z) \text{ in } \mathbb{U}\}, \tag{4}$$

and

$$\Sigma K^*_{p,n}(\alpha,\beta;\phi,\psi)=\{f\in\Sigma_{p,n}:\exists\, g\in\Sigma C_{p,n}(\alpha;\phi) \text{ such that } \frac{1}{p-\beta}\left(\frac{-(zf'(z))'}{g'(z)}-\beta\right)\prec\psi(z) \text{ in } \mathbb{U}\}. \tag{5}$$

We note that

$$f(z)\in\Sigma C_{p,n}(\alpha;\phi) \iff -\frac{zf'(z)}{p}\in\Sigma S^*_{p,n}(\alpha;\phi),$$
$$f(z)\in\Sigma K^*_{p,n}(\alpha,\beta;\phi,\psi) \iff -\frac{zf'(z)}{p}\in\Sigma K_{p,n}(\alpha,\beta;\phi,\psi).$$

We observe that for special choices for the parameters p,α,β and the functions ϕ and ψ involved in these definitions. For example, the classes

$$\Sigma S^*_{1,0}\left(\alpha;\frac{1+z}{1-z}\right)=\Sigma S^*(\alpha),\quad \Sigma C_{1,0}\left(\alpha;\frac{1+z}{1-z}\right)=\Sigma C(\alpha),$$

$$\Sigma K_{1,0}\left(\alpha,\beta;\frac{1+z}{1-z},\frac{1+z}{1-z}\right)=\Sigma K(\alpha,\beta),\quad \Sigma K^*_{1,0}\left(\alpha,\beta;\frac{1+z}{1-z},\frac{1+z}{1-z}\right)=\Sigma K^*(\alpha,\beta).$$

For $\mu>0$, a, $c\in\mathbb{C}$ be such that $Re(c-a)\geq 0, Re(a)\geq \mu p$, $p\in\mathbb{N}$ and $f(z)\in\Sigma_{p,n}$ is given by (1), the integral operator

$$J^{a,c}_{p,\mu}:\Sigma_{p,n}\to\Sigma_{p,n}$$

defined as following ([8])

- For $Re(c-a)>0$ by

$$J^{a,c}_{p,\mu}f(z)=\frac{\Gamma(c-\mu p)}{\Gamma(a-\mu p)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}f(zt^{\mu})dt; \tag{6}$$

- For $a=c$ by

$$J^{a,a}_{p,\mu}f(z)=f(z). \tag{7}$$

Using (6) and (7), it is easily that the operator $J^{a,c}_{p,\mu}f(z)$ can expresse as following

$$J^{a,c}_{p,\mu}f(z) = z^{-p} + \frac{\Gamma(c-\mu p)}{\Gamma(a-\mu p)}\sum_{k=n}^{\infty}\frac{\Gamma(a+\mu k)}{\Gamma(c+\mu k)}a_k z^k, \tag{8}$$

where $\mu > 0$, a, $c \in \mathbb{C}, Re(c-a) \geq 0, Re(a) \geq \mu p\, (p \in \mathbb{N})$.
It is readily verified from (8) that

$$z(J^{a,c}_{p,\mu}f(z))' = \frac{a-\mu p}{\mu}J^{a+1,c}_{p,\mu}f(z) - \frac{a}{\mu}J^{a,c}_{p,\mu}f(z). \tag{9}$$

$$z(J^{a,c+1}_{p,\mu}f(z))' = \frac{c-\mu p}{\mu}J^{a,c}_{p,\mu}f(z) - \frac{c}{\mu}J^{a,c+1}_{p,\mu}f(z), \tag{10}$$

We also note that the operator $J^{a,c}_{p,\mu}f(z)$ generalizes several previously studied familiar operators, and we will show some of the interesting particular cases as follows

(i) $J^{a,c}_{1,\mu}f(z) = I_\mu(a,c)f(z)(a, c \in \mathbb{C}, \mu > 0, Re(a) > \mu, Re(c-a) \geq 0)$(see [7]),

(ii) $J^{a+p,c+p}_{p,1}f(z) = \ell_p(a,c)f(z)(a \in \mathbb{R}, c \in \mathbb{R}\backslash\mathbb{Z}_0^-, \mathbb{Z}_0^- = \{0,1,2,...\}, p \in \mathbb{N})$(see [12]),

(iii) $J^{n+2p,p+1}_{p,1}f(z) = D^{n+p-1}f(z)$ (n is an integer, $n > -p, p \in \mathbb{N}$) (see [1],[2],[18]),

(iV) $J^{a,a+1}_{p,1}f(z) = J^a_p f(z)$ (Re$(a) > p$, $p \in \mathbb{N}$) (see [10]).

Using the operator $J^{a,c}_{p,\mu}$, we introduce the following subclasses of the multivalent meromorphic functions as follows

$$\Sigma S^{*a,c}_{p,\mu}(\alpha;\phi) = \{f : f \in \Sigma_{p,n} \text{ and } J^{a,c}_{p,\mu}f \in \Sigma S^*_{p,n}(\alpha;\phi)\}, \tag{11}$$

where $\mu > 0; a, c \in \mathbb{C}; Re(c-a) \geq 0; Re(a) > p\mu; p \in \mathbb{N}; n > -p; \phi \in M$.

$$\Sigma C^{a,c}_{p,\mu}(\alpha;\phi) = \{f : f \in \Sigma_{p,n} \text{ and } J^{a,c}_{p,\mu}f \in \Sigma C_{p,n}(\alpha;\phi)\}, \tag{12}$$

where $\mu > 0; a, c \in \mathbb{C}; Re(c-a) \geq 0; Re(a) > p\mu; p \in \mathbb{N}; n > -p; \phi \in M$.

$$\Sigma K^{a,c}_{p,\mu}(\alpha,\beta;\phi,\psi) = \{f : f \in \Sigma_{p,n} \text{ and } J^{a,c}_{p,\mu}f \in \Sigma K_{p,n}(\alpha,\beta;\phi,\psi)\}, \tag{13}$$

where $\mu > 0; a, c \in \mathbb{C}; Re(c-a) \geq 0; Re(a) > p\mu; p \in \mathbb{N}; n > -p; \phi \in M$.

$$\Sigma K^{*a,c}_{p,\mu}(\alpha,\beta;\phi,\psi) = \{f : f \in \Sigma_{p,n} \text{ and } J^{a,c}_{p,\mu}f \in \Sigma K^*_{p,n}(\beta,\alpha;\phi,\psi)\}, \tag{14}$$

where $\mu > 0; a, c \in \mathbb{C}; Re(c-a) \geq 0; Re(a) > p\mu; p \in \mathbb{N}; n > -p; \phi \in M$.

We note that

$$f(z) \in \Sigma C_{p,\mu}^{a,c}(\alpha;\phi) \iff -\frac{zf'(z)}{p} \in \Sigma S_{p,\mu}^{*a,c}(\alpha;\phi), \tag{15}$$

$$f(z) \in \Sigma K_{p,\mu}^{*a,c}(\alpha,\beta;\phi,\psi) \iff -\frac{zf'(z)}{p} \in \Sigma K_{p,\mu}^{a,c}(\alpha,\beta;\phi,\psi). \tag{16}$$

In particular, set

$$\Sigma S_{p,\mu}^{*a,c}\left(\alpha, \frac{1+Az}{1+Bz}\right) = \Sigma S_{p\mu}^{*a,c}(\alpha;A,B) \qquad (-1 < B < A \leq 1), \tag{17}$$

and

$$\Sigma C_{p,\mu}^{a,c}\left(\alpha, \frac{1+Az}{1+Bz}\right) = \Sigma C_{p\mu}^{a,c}(\alpha;A,B) \qquad (-1 < B < A \leq 1), \tag{18}$$

In this paper, we investigate several inclusion relationship properties of the classes mentioned above. Some applications involving integral operator are also considered. We drive interesting arguments properties of p-valent meromorphic functions defined by the integral operator $J_{p,\mu}^{a,c}$.

2. The main results

In this section, we give several inclusion relationships for $p-$valent meromorphic functions classes, which are associated with the operator $J_{p,\mu}^{a,c}$.

Lemma 1. *[5]Let ϕ be convex univalent in $\mathbb{U}$ with $\phi(0) = 1$ and $\Re\{\beta\phi(z) + v\} > 0$ $(\beta, v \in \mathbb{C})$. If $q(z)$ is analytic in $\mathbb{U}$ with $q(0) = 1$, then*

$$q(z) + \frac{zq'(z)}{\beta q(z) + v} \prec \phi(z),$$

which implies that

$$q(z) \prec \phi(z).$$

Lemma 2. *[14] Let ϕ be convex univalent in $\mathbb{U}$ and ω be analytic in $\mathbb{U}$ with $\Re\{\omega(z)\} \geq 0$. If $q(z)$ is analytic in $\mathbb{U}$ with $q(0) = \phi(0)$, then*

$$q(z) + \omega(z)zq'(z) \prec \phi(z),$$

implies that

$$q(z) \prec \phi(z).$$

Unless otherwise mentioned we shall assume throughout this section that $\mu > 0$, $a, c \in \mathbb{C}$, $Re(c-a) \geq 0$, $Re(a) \geq \mu p$, $p \in \mathbb{N}$, $0 \leq \alpha, \beta < p$ and the powers are understood as principle values.

Theorem 3. *Let* $\phi \in M$, $Re(\frac{c}{\mu}), Re(\frac{a}{\mu}) > \alpha$ *with*

$$\max_{z\in U}(Re\{\phi(z)\}) < \min\left\{\frac{Re(\frac{a}{\mu})-\alpha}{p-\alpha}, \frac{Re(\frac{c}{\mu})-\alpha}{p-\alpha}\right\},$$

then

$$\Sigma S^{*a+1,c}_{p,\mu}(\alpha;\phi) \subset \Sigma S^{*a,c}_{p,\mu}(\alpha;\phi) \subset \Sigma S^{*a,c+1}_{p,\mu}(\alpha;\phi).$$

Proof. (i) we prove the first inclusion relationship

$$\Sigma S^{*a+1,c}_{p,\mu}(\alpha;\phi) \subset \Sigma S^{*a,c}_{p,\mu}(\alpha;\phi).$$

Let $f(z) \in \Sigma S^{*a+1,c}_{p,\mu}(\alpha;\phi)$ and set

$$\frac{z(J^{a,c}_{p,\mu}f(z))'}{J^{a,c}_{p,\mu}f(z)} = -\alpha - (p-\alpha)q(z), \tag{19}$$

where $q(z)$ is analytic in $\mathbb{U}$ with $q(0) = 1$. Then by applying Eq.(9) in (19), we obtain

$$\frac{a-p\mu}{\mu}\frac{J^{a+1,c}_{p,\mu}f(z)}{J^{a,c}_{p,\mu}f(z)} = -\alpha + \frac{a}{\mu} - (p-\alpha)q(z). \tag{20}$$

Differentiating (20) logarithmically with respect to z, we obtain

$$\frac{z(J^{a+1,c}_{p,\mu}f(z))'}{J^{a+1,c}_{p,\mu}f(z)} = -\alpha - (p-\alpha)q(z) - \frac{(p-\alpha)zq'(z)}{-\alpha + \frac{a}{\mu} - (p-\alpha)q(z)}$$

$$\frac{1}{p-\alpha}\left[-\frac{z(J^{a+1,c}_{p,\mu}f(z))'}{J^{a+1,c}_{p,\mu}f(z)} - \alpha\right] = q(z) + \frac{zq'(z)}{-\alpha + \frac{a}{\mu} - (p-\alpha)q(z)}. \tag{21}$$

Since

$$\max_{z\in U}(\Re\{\phi(z)\}) < \frac{\frac{Re(a)}{\mu} - \alpha}{p-\alpha} \qquad (Re(a) > 0;\ 0 \leq \alpha < p).$$

Applying Lemma 1 to Eq.(21), it follows that $q(z) \prec \phi(z)$, that is $f(z) \in \Sigma S^{*a,c}_{p,\mu}(\alpha;\phi)$.

(ii) For the second inclusion relationship asserted by Theorem 3, using arguments similar to those detailed above with (10), we have

$$\Sigma S^{*a,c}_{p,\mu}(\alpha;\phi) \subset \Sigma S^{*a,c+1}_{p,\mu}(\alpha;\phi).$$

This completes the proof of Theorem 3.

Theorem 4. *Let* $\phi \in M$, $Re(\frac{c}{\mu}), Re(\frac{a}{\mu}) > \alpha$ *with*

$$\max_{z\in U}(Re\{\phi(z)\}) < \min\left\{\frac{Re(\frac{a}{\mu})-\alpha}{p-\alpha}, \frac{Re(\frac{c}{\mu})-\alpha}{p-\alpha}\right\},$$

then

$$\Sigma C_{p,\mu}^{a+1,c}(\alpha;\phi) \subset \Sigma C_{p,\mu}^{a,c}(\alpha;\phi) \subset \Sigma C_{p,\mu}^{a,c+1}(\alpha;\phi).$$

Proof. Applying (15) and Theorem 3, we obtain that

$$\begin{aligned} f(z) \in \Sigma C_{p,\mu}^{a+1,c}(\alpha;\phi) \;&\Leftrightarrow\; J_{p,\mu}^{a+1,c} f(z) \in \Sigma C_{p,n}(\alpha;\phi) \\ &\Leftrightarrow\; \frac{-z}{p}\left(J_{p,\mu}^{a+1,c} f(z)\right)' \in \Sigma S_{p,n}^{*}(\alpha;\phi) \\ &\Leftrightarrow\; J_{p,\mu}^{a+1,c}\left(\frac{-zf'(z)}{p}\right) \in \Sigma S_{p,n}^{*}(\alpha;\phi) \\ &\Rightarrow\; \frac{-zf'(z)}{p} \in \Sigma S_{p,\mu}^{*a+1,c}(\alpha;\phi) \\ &\Rightarrow\; \frac{-zf'(z)}{p} \in \Sigma S_{p,\mu}^{*a,c}(\alpha;\phi) \end{aligned}$$

$$\begin{aligned} &\Leftrightarrow\; J_{p,\mu}^{a,c}\left(\frac{-zf'(z)}{p}\right) \in \Sigma S_{p,n}^{*}(\alpha;\phi) \\ &\Leftrightarrow\; \frac{-z}{p}\left(J_{p,\mu}^{a,c} f(z)\right)' \in \Sigma S_{p,n}^{*}(\alpha;\phi) \\ &\Leftrightarrow\; J_{p,\mu}^{a,c} f(z) \in \Sigma C_{p,n}(\alpha;\phi) \\ &\Leftrightarrow\; f(z) \in \Sigma C_{p,\mu}^{a,c}(\alpha;\phi). \end{aligned}$$

The second part of the theorem can be proved by using similar arguments. The proof of Theorem 4 is completed.

Corollary 5. *Suppose that*

$$\frac{1+A}{1+B} < \min\left(\frac{Re(\frac{a}{\mu})-\alpha}{p-\alpha}, \frac{Re(\frac{c}{\mu})-\alpha}{p-\alpha}\right),$$

with $Re(\frac{c}{\mu}), Re(\frac{a}{\mu}) > \alpha; -1 < B < A \leq 1$. *Then, for the function classes defined by (17) and (18),*

$$\Sigma S_{p,\mu}^{*a+1,c}(\alpha;A,B) \subset \Sigma S_{p,\mu}^{*a,c}(\alpha;A,B) \subset \Sigma S_{p,\mu}^{*a,c+1}(\alpha;A,B),$$

and

$$\Sigma C_{p,\mu}^{a+1,c}(\alpha;A,B) \subset \Sigma C_{p,\mu}^{a,c}(\alpha;A,B) \subset \Sigma C_{p,\mu}^{a,c+1}(\alpha;A,B).$$

Theorem 6. *Let* $\phi, \psi \in M$, $Re\{\frac{a}{\mu}\}, Re\{\frac{c}{\mu}\} > \alpha$ *with*

$$\max_{z\in U}\left(Re\{\phi(z)\}\right) < \min\left(\frac{Re(\frac{a}{\mu})-\alpha}{p-\alpha}, \frac{Re(\frac{c}{\mu})-\alpha}{p-\alpha}\right),$$

then

$$\Sigma K_{p,\mu}^{a+1,c}(\alpha,\beta;\phi,\psi) \subset \Sigma K_{p,\mu}^{a,c}(\alpha,\beta;\phi,\psi) \subset \Sigma K_{p,\mu}^{a,c+1}(\alpha,\beta;\phi,\psi).$$

Proof. (i)We begin by showing the first inclusion relationship:

$$\Sigma K_{p,\mu}^{a+1,c}(\alpha,\beta;\phi,\psi) \subset \Sigma K_{p,\mu}^{a,c}(\alpha,\beta;\phi,\psi).$$

Let $f(z) \in \Sigma K_{p,\mu}^{a+1,c}(\alpha,\beta;\phi,\psi)$, then from the definition of the class $\Sigma K_{p,\mu}^{a+1,c}(\alpha,\beta;\phi,\psi)$ there exists a function $k(z) \in \Sigma S_{p,n}^{*}(\alpha,\phi)$ such that

$$\frac{1}{p-\beta}\left[\frac{-(zJ_{p,\mu}^{a+1,c}f(z))'}{k(z)} - \beta\right] \prec \psi(z).$$

Choose the function $g(z)$ such that $J_{b,\mu}^{a+1,c}g(z) = k(z)$. Then $g(z) \in \Sigma S_{p,\mu}^{*a+1,c}(\alpha;\phi)$ and

$$\frac{1}{p-\beta}\left[\frac{-z(J_{p,\mu}^{a+1,c}f(z))'}{J_{p,\mu}^{a+1,c}g(z)} - \beta\right] \prec \psi(z). \tag{22}$$

Let

$$p(z) = \frac{1}{p-\beta}\left[\frac{-z(J_{p,\mu}^{a,c}f(z))'}{J_{p,\mu}^{a,c}g(z)} - \beta\right], \tag{23}$$

where the function $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. Using (9), we find that

$$\begin{aligned}
&\frac{1}{p-\beta}\left[\frac{-z(J_{p,\mu}^{a+1,c}f(z))'}{J_{p,\mu}^{a+1,c}g(z)} - \beta\right] = \frac{1}{p-\beta}\left[\frac{J_{p,\mu}^{a+1,c}(-zf'(z))}{J_{p,\mu}^{a+1,c}g(z)} - \beta\right]\\
&= \frac{1}{p-\beta}\left[\frac{z[J_{p,\mu}^{a,c}(-zf'(z))]' + a/\mu J_{p,\mu}^{a,c}(-zf'(z))}{z[J_{p,\mu}^{a,c}g(z)]' + a/\mu J_{p,\mu}^{a,c}g(z)} - \beta\right]\\
&= \frac{1}{p-\beta}\left[\frac{z[J_{p,\mu}^{a,c}(-zf'(z))]'/J_{p,\mu}^{a,c}g(z)}{z[J_{p,\mu}^{a,c}g(z)]'/J_{p,\mu}^{a,c}g(z) + a/\mu}\right] + \frac{1}{p-\beta}\left[\frac{a/\mu J_{p,\mu}^{a,c}(-zf'(z))/J_{p,\mu}^{a,c}g(z)}{z[J_{p,\mu}^{a,c}g(z)]'/J_{p,\mu}^{a,c}g(z) + a/\mu} - \beta\right].
\end{aligned} \tag{24}$$

Since

$$g(z) \in \Sigma S_{p,\mu}^{*a+1,c}(\alpha;\phi) \subset \Sigma S_{p,\mu}^{*a,c}(\alpha;\phi)$$

By Theorem 3, then we set

$$q(z) = \frac{1}{p-\alpha}\left[\frac{-z[J_{p,\mu}^{a,c}(g(z))]'}{J_{p,\mu}^{a,c}(g(z))} - \alpha\right]$$

where $q(z) \prec \phi(z)$ in $\mathbb{U}$ with assumption that $\phi(z) \in M$. Then by (23) and (24), we observe that

$$J_{p,\mu}^{a,c}(-zf'(z)) = (p-\beta)p(z)J_{p,\mu}^{a,c}g(z) + \beta J_{p,\mu}^{a,c}g(z), \tag{25}$$

and

$$\frac{1}{p-\beta}\left[\frac{-z[J_{p,\mu}^{a+1,c}(f(z))]'}{J_{p,\mu}^{a+1,c}g(z)} - \beta\right] = \frac{1}{p-\beta}\left[\frac{z[J_{p,\mu}^{a,c}(-zf'(z))]'/J_{p,\mu}^{a,c}g(z)}{-(p-\alpha)q(z)-\alpha+a/\mu}\right] + \frac{1}{p-\beta}\left[\frac{a/\mu[(p-\beta)p(z)+\beta]}{-(p-\alpha)q(z)-\alpha+a/\mu} - \beta\right] \tag{26}$$

Differentiating both side of (25) with respect to z and dividing by $J_{p,\mu}^{a,c}g(z)$, we obtain

$$\frac{z[J_{p,\mu}^{a,c}(-zf'(z))]'}{J_{p,\mu}^{a,c}g(z)} = (p-\beta)zp'(z) - [(p-\alpha)q(z)+\alpha][(p-\beta)p(z)+\beta]. \tag{27}$$

Now, using Eqs.(22), (26) and (27), we get

$$\frac{1}{p-\beta}\left[\frac{-z[J_{p,\mu}^{a+1,c}f(z)]'}{J_{p,\mu}^{a+1,c}g(z)} - \beta\right] = \frac{zp'(z)}{a/\mu - \alpha - (p-\alpha)q(z)} + p(z) \prec \psi(z) \quad (z \in \mathbb{U}). \tag{28}$$

Since $Re\{\frac{a}{\mu}\} > \alpha$, and $q(z) \prec \phi(z)$ in $\mathbb{U}$ with

$$\max_{z\in\mathbb{U}}\{Re(\phi(z))\} < \frac{Re\{a/\mu\}-\alpha}{p-\alpha}.$$

We have

$$Re\{a/\mu - \alpha - (p-\alpha)q(z)\} > 0 \qquad (z \in \mathbb{U}).$$

Hence by taking

$$\zeta(z) = \frac{1}{a/\mu - (p-\alpha)q(z),}$$

in (28) and then by applying Lemma 2, we can show that $p(z) \prec \psi(z)$ in $\mathbb{U}$, so that $f(z) \in \Sigma K_{p,\mu}^{a,c}(\alpha,\beta;\phi,\psi)$.

(ii)For the second inclusion relationship asserted by Theorem 6, using arguments similar to those detailed above with (10), we obtain

$$\Sigma K_{p,\mu}^{a,c}(\alpha,\beta;\phi,\psi) \subset \Sigma K_{p,\mu}^{a,c+1}(\alpha,\beta;\phi,\psi).$$

Theorem 7. *Let* $\phi, \psi \in M$*,* $Re\{\frac{a}{\mu}\}, Re\{\frac{c}{\mu}\} > \alpha$ *with*

$$\max_{z\in U}(Re\{\phi(z)\}) < \min\left(\frac{Re\{a/\mu\}-\alpha}{p-\alpha}, \frac{Re\{c/\mu\}-\alpha}{p-\alpha}\right),$$

then

$$\Sigma K_{p,\mu}^{*a+1,c}(\alpha,\beta;\phi,\psi) \subset \Sigma K_{p,\mu}^{*a,c}(\alpha,\beta;\phi,\psi) \subset \Sigma K_{p,\mu}^{*a,c+1}(\alpha,\beta;\phi,\psi).$$

Proof. Applying (16) and Theorem 6 obtain that

$$\begin{aligned} f(z) \in \Sigma K_{p,\mu}^{*a+1,c}(\alpha,\beta;\phi,\psi) &\Leftrightarrow J_{p,\mu}^{a+1,c} f(z) \in \Sigma K_{p,n}^{*}(\alpha,\beta;\phi,\psi) \\ &\Leftrightarrow \frac{-z}{p}\left(J_{p,\mu}^{a+1,c} f(z)\right)' \in \Sigma K_{p,n}(\alpha,\beta;\phi,\psi) \\ &\Leftrightarrow J_{p,\mu}^{a+1,c}\left(\frac{-zf'(z)}{p}\right) \in \Sigma K_{p,n}(\alpha,\beta;\phi,\psi) \\ &\Rightarrow \frac{-zf'(z)}{p} \in \Sigma K_{p,\mu}^{a+1,c}(\alpha,\beta;\phi,\psi) \\ &\Rightarrow \frac{-zf'(z)}{p} \in \Sigma K_{p,\mu}^{a,c}(\alpha,\beta;\phi,\psi) \end{aligned}$$

$$\begin{aligned} &\Leftrightarrow J_{p,\mu}^{a,c}\left(\frac{-zf'(z)}{p}\right) \in \Sigma K_{p,n}(\alpha,\beta;\phi,\psi) \\ &\Leftrightarrow \frac{-z}{p}\left(J_{p,\mu}^{a,c} f(z)\right)' \in \Sigma K_{p,n}(\alpha;\phi) \\ &\Leftrightarrow J_{p,\mu}^{a,c} f(z) \in \Sigma K_{p,n}^{*}(\alpha,\beta;\phi,\psi) \\ &\Leftrightarrow f(z) \in \Sigma K_{p,\mu}^{*a,c}(\alpha,\beta;\phi,\psi). \end{aligned}$$

The second part of the theorem can be proved by using similar arguments. The proof of Theorem 7 is completed.

3. Inclusion properties involving the integral operators $F_{\nu,p}$

In this section, we consider the integral operator $F_{\nu,p}$ (see [10]) defined by

$$F_{\nu,p} f(z) = \frac{\nu}{z^{\nu+p}} \int_0^z t^{\nu+p-1} f(t)dt \quad (f(z) \in \Sigma_{p,n}, \nu > 0; p \in \mathbb{N}). \tag{29}$$

From the above equation, it is easily verified that

$$z\left(J_{p,\mu}^{a,c} F_{\nu,p} f(z)\right)' = \nu J_{p,\mu}^{a,c} f(z) - (\nu+p) J_{p,\mu}^{a,c} F_{\nu,p} f(z). \tag{30}$$

By using (30), we can prove the following theorems.

Theorem 8. *Let* $\phi \in M$ *with* $\max_{z\in U}\left(Re\{\phi(z)\}\right) < 1 + \frac{\nu}{p-\alpha}$, $(\nu > 0; 0 \le \alpha < p)$. *If* $f(z) \in \Sigma S^{*a,c}_{p,\mu}(\alpha;\phi)$, *then* $F_{\nu,p}f(z) \in \Sigma S^{*a,c}_{p,\mu}(\alpha;\phi)$.

Proof. Let $f(z) \in \Sigma S^{*a,c}_{p,\mu}(\alpha;\phi)$ and set

$$p(z) = \frac{1}{p-\alpha}\left[-\frac{z(J^{a,c}_{p,\mu}F_{\nu,p}f(z))'}{J^{a,c}_{p,\mu}F_{\nu,p}f(z)} - \alpha\right], \tag{31}$$

where $p(z)$ is analytic in U with $p(0) = 1$. By using (30) and (31), we have

$$-\nu\frac{J^{a,c}_{p,\mu}f(z)}{J^{a,c}_{p,\mu}F_{\nu,p}f(z)} = (p-\alpha)p(z) - (\nu + p - \alpha). \tag{32}$$

Differntiating (32) logarithmically with respec to z, we obtain

$$\frac{z[J^{a,c}_{p,\mu}f(z)]'}{J^{a,c}_{p,\mu}f(z)} - \frac{z[J^{a,c}_{p,\mu}F_{\nu,p}f(z)]'}{J^{a,c}_{p,\mu}F_{\nu,p}f(z)} = \frac{(p-\alpha)zp'(z)}{(p-\alpha)p(z) - (\nu+p-\alpha)}$$

$$\frac{1}{p-\alpha}\left[\frac{-z[J^{a,c}_{p,\mu}f(z)]'}{J^{a,c}_{p,\mu}f(z)} - \alpha\right] = p(z) + \frac{zp'(z)}{(\nu+p-\alpha) - (p-\alpha)p(z)} \;\; (z \in U).$$

Hence by Lemma 1, we conclude that $p(z) \prec \phi(z)$ in U for

$$\max_{z\in U}\left(Re\{\phi(z)\}\right) < \frac{\nu}{p-\alpha} + 1 \quad (\nu > 0, 0 \le \alpha < p),$$

which implies that

$$F_{\nu,p}f(z) \in \Sigma S^{*a,c}_{p,\mu}(\alpha;\phi).$$

Theorem 9. *Let* $\phi \in M$ *with*

$$\max_{z\in U}(Re\{\phi(z)\}) < 1 + \frac{\nu}{p-\alpha} \quad (\nu > 0; 0 \le \alpha < p).$$

If $f(z) \in \Sigma C^{a,c}_{p,\mu}(\alpha;\phi)$, *then*

$$F_{\nu,p}f(z) \in \Sigma C^{a,c}_{p,\mu}(\alpha;\phi).$$

Proof. By applying Theorem 8, it follows that

$$\begin{aligned} f(z) \in \Sigma C^{a,c}_{p,\mu}(\alpha;\phi) \;&\Leftrightarrow\; \frac{-z}{p}f'(z) \in \Sigma S^{*a,c}_{p,\mu}(\alpha;\phi) \\ &\Rightarrow\; F_{\nu,p}\left(\frac{-z}{p}f'(z)\right) \in \Sigma S^{*a,c}_{p,\mu}(\alpha;\phi) \\ &\Leftrightarrow\; \frac{-z}{p}\left(F_{\nu,p}f(z)\right)' \in \Sigma S^{*a,c}_{p,\mu}(\alpha;\phi) \end{aligned}$$

$$F_{\nu,p}f(z) \in \Sigma C^{a,c}_{p,\mu}(\alpha;\phi),$$

which proves Theorem 9.

From Theorem 8 and 9, we obtain the following corollary.

Corollary 10. *Suppose that*

$$\frac{1+A}{1+B} < 1 + \frac{\nu}{p-\alpha},$$

with $\nu > 0; 0 \leq \alpha < p; -1 < B < A \leq 1$. *Then, for the function classes defined by (17) and (18),*

$$f(z) \in \Sigma S^{*a+1,c}_{p,\mu}(\alpha; A, B) \Rightarrow F_{\nu,p}f(z) \in \Sigma S^{*a+1,c}_{p,\mu}(\alpha; A, B),$$

and

$$f(z) \in \Sigma C^{a+1,c}_{p,\mu}(\alpha; A, B) \Rightarrow F_{\nu,p}f(z) \in \Sigma C^{a+1,c}_{p,\mu}(\alpha; A, B).$$

Theorem 11. *Let* $\phi, \psi \in M$ *with*

$$\max_{z\in U}(Re\{\phi(z)\}) < 1 + \frac{\nu}{p-\alpha} \quad (\nu > 0; 0 \leq \alpha, \beta < p).$$

If $f(z) \in \Sigma K^{a,c}_{p,\mu}(\alpha, \beta; \phi, \psi)$, *then*

$$F_{\nu,p}f(z) \in \Sigma K^{a,c}_{p,\mu}(\alpha, \beta; \phi, \psi).$$

Proof. Let $f(z) \in \Sigma K^{a,c}_{p,\mu}(\alpha, \beta; \phi, \psi)$. Then in view of the definition of the function class there exists a function $g(z) \in \Sigma S^{*a,c}_{p,\mu}(\alpha; \phi)$ such that

$$\frac{1}{p-\beta}\left[-\frac{z(J^{a,c}_{p,\mu}f(z))'}{J^{a,c}_{p,\mu}g(z)} - \beta\right] \prec \psi(z) \quad (z \in U). \tag{33}$$

Setting

$$p(z) = \frac{1}{p-\beta}\left[-\frac{z(J^{a,c}_{p,\mu}F_{\nu,p}f(z))'}{J^{a,c}_{p,\mu}F_{\nu,p}g(z)} - \beta\right],$$

where the function $p(z)$ is analytic in U with $p(0) = 1$. Applying Eq.(30), we obtain

$$\begin{aligned}
&\frac{1}{p-\beta}\left[\frac{-z(J^{a,c}_{p,\mu}f(z))'}{J^{a,c}_{p,\mu}g(z)} - \beta\right] = \frac{1}{p-\beta}\left[\frac{J^{a,c}_{p,\mu}(-zf'(z))}{J^{a,c}_{p,\mu}g(z)} - \beta\right] \\
&= \frac{1}{p-\beta}\left[\frac{z[J^{a,c}_{p,\mu}F_{\nu,p}(-zf'(z))]' + (\nu+p)J^{a,c}_{p,\mu}F_{\nu,p}(-zf'(z))}{z[J^{a,c}_{p,\mu}F_{\nu,p}g(z)]' + (\nu+p)J^{a,c}_{p,\mu}F_{\nu,p}g(z)} - \beta\right] \\
&= \frac{1}{p-\beta}\left[\frac{z[J^{a,c}_{p,\mu}F_{\nu,p}(-zf'(z))]'/J^{a,c}_{p,\mu}F_{\nu,p}g(z)}{z[J^{a,c}_{p,\mu}F_{\nu,p}g(z)]'/J^{a,c}_{p,\mu}F_{\nu,p}g(z) + (\nu+p)}\right] + \\
&\qquad \frac{1}{p-\beta}\left[\frac{(\nu+p)J^{a,c}_{p,\mu}F_{\nu,p}(-zf'(z))/J^{a,c}_{p,\mu}g(z)}{z[J^{a,c}_{p,\mu}F_{\nu,p}g(z)]'/J^{a,c}_{p,\mu}F_{\nu,p}g(z) + (\nu+p)} - \beta\right].
\end{aligned} \tag{34}$$

Since

$$g(z) \in \Sigma S^{*a,c}_{p,\mu}(\alpha;\phi),$$

By Theorem 8, we find $F_{\nu,p}g(z) \in \Sigma S^{*a,c}_{p,\mu}(\alpha;\phi)$ and set

$$q(z) = \frac{1}{p-\alpha}\left[\frac{-z[J^{a,c}_{p,\mu}F_{\nu,p}(g(z))]'}{J^{a,c}_{p,\mu}F_{\nu,p}(g(z))} - \alpha\right],$$

where $q(z) \prec \phi(z)$ in U with assumption that $\phi(z) \in M$. Then by using the same techniques as in the proof of Theorem 6 from Eqs.(33) and (34), we obtain

$$\frac{1}{p-\beta}\left[\frac{-z[J^{a,c}_{p,\mu}(f(z))]'}{J^{a,c}_{p,\mu}g(z)} - \beta\right] = p(z) + \frac{zp'(z)}{(\nu+p-\alpha)-(p-\alpha)q(z)} \prec \psi(z). \tag{35}$$

Hence, by setting

$$\omega(z) = \frac{1}{(\nu+p-\alpha)-(p-\alpha)q(z)},$$

in Eq.(35) and apply Lemma 2, we find that $p(z) \prec \psi(z)$ in U, which yields

$$F_{\nu,p}f(z) \in \Sigma K^{a,c}_{p,\mu}(\alpha,\beta;\phi;\psi).$$

Theorem 12. *Let $\phi, \psi \in M$ with*

$$\max_{z\in U}(Re\{\phi(z)\}) < 1 + \frac{\nu}{p-\alpha} \quad (\nu > 0; 0 \le \alpha, \beta < p).$$

If $f(z) \in \Sigma K^{*a,c}_{p,\mu}(\alpha,\beta;\phi,\psi)$, then

$$F_{\nu,p}f(z) \in \Sigma K^{*a,c}_{p,\mu}(\alpha,\beta;\phi,\psi).$$

Proof. By applying Theorem 11, we obtain that

$$\begin{aligned} f(z) \in \Sigma K^{*a,c}_{p,\mu}(\alpha,\beta;\phi,\psi) \quad &\Leftrightarrow \quad \frac{-z}{p}f'(z) \in \Sigma K^{a,c}_{p,\mu}(\alpha,\beta;\phi,\psi) \\ &\Rightarrow \quad F_{\nu,p}\left(\frac{-z}{p}f'(z)\right) \in \Sigma K^{a,c}_{p,\mu}(\alpha,\beta;\phi,\psi) \\ &\Leftrightarrow \quad \frac{-z}{p}\left(F_{\nu,p}f(z)\right)' \in \Sigma K^{a,c}_{p,\mu}(\alpha,\beta;\phi,\psi) \end{aligned}$$

$$F_{\nu,p}f(z) \in \Sigma K^{*a,c}_{p,\mu}(\alpha,\beta;\phi,\psi),$$

which proves Theorem 9.

4. Argument properties for the operator $J_{p,\mu}^{a,c}$

Following the technique used by El-Ashwah [6], we will study some argument results involving the operator $J_{p,\mu}^{a,c}$. Unless otherwise mentioned, we shall assume throughtout this section that $p \in \mathbb{N}$, $\theta > 0$, γ, α, $\delta > 0$, a, $c \in \mathbb{R}^+$, $(c-a) > 0$, $a > \mu p$.
To derive our main theorems, we need the following lemma.

Lemma 13. *[11] Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \neq 0$. Further suppose that*

$$|arg(p(z) + \eta z p'(z))| < \frac{\pi}{2}(\theta + \frac{2}{\pi}\arctan(\eta\theta)) \quad (\eta,\, \theta > 0),$$

then

$$|arg\, p(z)| < \frac{\pi}{2}\theta.$$

Theorem 14. *Let $g(z) \in \Sigma_p$. Suppose $f(z) \in \Sigma_p$ satisfies the following condition*

$$\left|arg\left(\left\{\frac{J_{p,\mu}^{a,c}f(z)}{J_{p,\mu}^{a,c}g(z)}\right\}^{\gamma}\left[1+\delta\left\{\frac{J_{p,\mu}^{a+1,c}f(z)}{J_{p,\mu}^{a,c}f(z)} - \frac{J_{p,\mu}^{a+1,c}g(z)}{J_{p,\mu}^{a,c}g(z)}\right\}\right]\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\delta\mu}{\gamma(a-\mu p)}\alpha\right]\right)$$

then

$$\left|arg\left\{\frac{J_{p,\mu}^{a,c}f(z)}{J_{p,\mu}^{a,c}g(z)}\right\}^{\gamma}\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}).$$

Proof. Define a function

$$p(z) = \left\{\frac{J_{p,\mu}^{a,c}f(z)}{J_{p,\mu}^{a,c}g(z)}\right\}^{\gamma}, \quad \gamma \neq 0, \tag{36}$$

then $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, is analytic in $\mathbb{U}$ with $p(0) = 1$ and $p'(0) \neq 0$.
Differentiating (36) logarithmically with respect to z and multiplying by z, we have

$$\frac{1}{\gamma}\frac{zp'(z)}{p(z)} = \frac{z(J_{p,\mu}^{a,c}f(z))'}{J_{p,\mu}^{a,c}f(z)} - \frac{z(J_{p,\mu}^{a,c}g(z))'}{J_{p,\mu}^{a,c}g(z)}. \tag{37}$$

Using (9) in (37), we obtain

$$p(z) + \frac{\delta\mu}{\gamma(a-\mu p)}zp'(z) = \left\{\frac{J_{p,\mu}^{a,c}f(z)}{J_{p,\mu}^{a,c}g(z)}\right\}^{\gamma}\left[1+\delta\left\{\frac{J_{p,\mu}^{a+1,c}f(z)}{J_{p,\mu}^{a,c}f(z)} - \frac{J_{p,\mu}^{a+1,c}g(z)}{J_{p,\mu}^{a,c}g(z)}\right\}\right].$$

By using Lemma 13, the proof of Theorem 14 is completed.

Putting $\gamma = 1$ and $g(z) = z^{-p}$ in Theorem 14, we obtain the following corollary:

Corollary 15. *If $f(z) \in \Sigma_p$ satisfies the following condition*

$$\left|arg\left((1-\delta)\left(z^p J_{p,\mu}^{a,c} f(z)\right) + \delta z^p J_{p,\mu}^{a+1,c} f(z)\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\delta\mu}{(a-\mu p)}\alpha\right]\right)$$

then

$$\left|arg\left\{z^p J_{p,\mu}^{a,c} f(z)\right\}\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}).$$

Next, putting $p = 1$ in Corollary 15, we obtain the following corollary:

Corollary 16. *If $f(z) \in \Sigma_1$ satisfies the following condition*

$$\left|arg\left((1-\delta)\left(z J_{1,\mu}^{a,c} f(z)\right) + \delta z J_{1,\mu}^{a+1,c} f(z)\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\delta\mu}{(a-\mu)}\alpha\right]\right)$$

then

$$\left|arg\left\{z J_{1,\mu}^{a,c} f(z)\right\}\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}).$$

Putting $a = c$, $\delta = 1$ and $g(z) = z^{-p}$ in Theorem 14, we obtain the following corollary:

Corollary 17. *If $f(z) \in \Sigma_p$ satisfies the following condition*

$$\left|arg\left((z^p f(z))^\gamma\left(\frac{\mu}{a-\mu p}\frac{zf'(z)}{f(z)} + \frac{a}{a-\mu p}\right)\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\mu}{\gamma(a-\mu p)}\alpha\right]\right)$$

then

$$|arg\{z^p f(z)\}^\gamma| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}).$$

Finally, putting $f(z) = z^{-p}, \gamma = 1$ in Theorem 14, we obtain the following corollary:

Corollary 18. *If $g(z) \in \Sigma_p$ and $\frac{z^{-p}}{J_{p,\mu}^{a,c} g(z)} \neq 0$, satisfies the following condition*

$$\left|arg\left((1+\delta)\left(\frac{1}{z^p J_{p,\mu}^{a,c} g(z)}\right) - \delta\left(\frac{1}{z^p J_{p,\mu}^{a,c} g(z)}\right)\left(\frac{J_{p,\mu}^{a+1,c} g(z)}{J_{p,\mu}^{a,c} g(z)}\right)\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\delta\mu}{(a-\mu p)}\alpha\right]\right)$$

then

$$\left|arg\left\{\frac{z^{-p}}{J_{p,\mu}^{a,c} g(z)}\right\}\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}).$$

Theorem 19. *Suppose $g(z) \in \Sigma_p$ and $f(z) \in \Sigma_p$ satisfies the following condition*

$$\left|arg\left(\left\{\frac{J_{p,\mu}^{a,c+1} f(z)}{J_{p,\mu}^{a,c+1} g(z)}\right\}^\gamma\left[1 + \delta\left\{\frac{J_{p,\mu}^{a,c} f(z)}{J_{p,\mu}^{a,c+1} f(z)} - \frac{J_{p,\mu}^{a,c} g(z)}{J_{p,\mu}^{a,c+1} g(z)}\right\}\right]\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\delta\mu}{\gamma(c-\mu p)}\alpha\right]\right)$$

then

$$\left|arg\left(\frac{J_{p,\mu}^{a,c+1} f(z)}{J_{p,\mu}^{a,c+1} g(z)}\right)\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}).$$

Proof. The proof is similar to the proof of Theorem 14, so we omit it.

Putting $\gamma = 1$ and $g(z) = z^{-p}$ in Theorem 19, we obtain the following corollary:

Corollary 20. *If $f(z) \in \Sigma_p$ satisfies the following condition*

$$\left|arg\left((1-\delta)\left(z^p J_{p,\mu}^{a,c+1} f(z)\right) + \delta z^p J_{p,\mu}^{a,c} f(z)\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\delta\mu}{(c-\mu p)}\alpha\right]\right)$$

then

$$\left|arg\left\{z^p J_{p,\mu}^{a,c+1} f(z)\right\}\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}).$$

Next, putting $p = 1$ in Corollary 20, we obtain the following corollary:

Corollary 21. *If $f(z) \in \Sigma_1$ satisfies the following condition*

$$\left|arg\left((1-\delta)\left(zJ_{1,\mu}^{a,c+1} f(z)\right) + \delta z J_{1,\mu}^{a,c} f(z)\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\delta\mu}{(c-\mu)}\alpha\right]\right)$$

then

$$\left|arg\left\{zJ_{1,\mu}^{a,c+1} f(z)\right\}\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}).$$

Putting $a = c$, $\delta = \mu = 1$ and $g(z) = z^{-p}$ in Theorem 19, we obtain the following corollary:

Corollary 22. *If $f(z) \in \Sigma_{p,n}$ satisfies the following condition*

$$\left|arg\left(\left(z^p J_p^a f(z)\right)^\gamma\left(\frac{f(z)}{J_p^a f(z)}\right)\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{1}{\gamma(a-p)}\alpha\right]\right)$$

then

$$\left|arg\left\{z^p f(z)\right\}^\gamma\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}).$$

Finally, putting $f(z) = z^{-p}$, $\gamma = 1$ in Theorem 19, we obtain the following corollary:

Corollary 23. *If $g(z) \in \Sigma_p$ and $\frac{z^{-p}}{J_{p,\mu}^{a,c+1} g(z)} \neq 0$, satisfies the following condition*

$$\left|arg\left((1+\delta)\left(\frac{1}{z^p J_{p,\mu}^{a,c+1} g(z)}\right) - \delta\left(\frac{1}{z^p J_{p,\mu}^{a,c+1} g(z)}\right)\left(\frac{J_{p,\mu}^{a,c} g(z)}{J_{p,\mu}^{a,c+1} g(z)}\right)\right)\right|$$
$$< \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\delta\mu}{(c-\mu p)}\alpha\right]\right)$$

then

$$\left|arg\left\{\frac{z^{-p}}{J_{p,\mu}^{a,c+1} g(z)}\right\}\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}).$$

Theorem 24. *Let* $0 < \delta \leq 1$. *Suppose* $f(z) \in \Sigma_p$, $(p \in \mathbb{N})$ *satisfies the following condition*

$$\left|(z^p J_{p,\mu}^{a,c} f(z))^\gamma\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\mu\delta}{a-\mu p}\alpha\right]\right), \ (z \in \mathbb{U}),$$

then

$$\left|arg\left(\frac{a-\mu p}{\mu\delta} z^{-\frac{a-\mu p}{\mu\delta}} \int_0^z t^{\frac{a-\mu p}{\mu\delta}-1}(t^p J_{p,\mu}^{a,c} f(t))^\gamma dt\right)\right| < \frac{\pi}{2}\alpha.$$

Proof. Consider the function

$$p(z) = \frac{a-\mu p}{\mu\delta} z^{-\frac{a-\mu p}{\mu\delta}} \int_0^z t^{\frac{a-\mu p}{\mu\delta}-1}(t^p J_{p,\mu}^{a,c} f(t))^\gamma dt \ (z \in \mathbb{U}), \tag{38}$$

then $p(z) = 1 + c_1 z + \ldots$, is analytic in $\mathbb{U}$ with $p(0) = 1$ and $p'(0) \neq 0$. differentiating (38) with respect to z, we have

$$p(z) + \frac{\mu\delta}{a-\mu p} z p'(z) = (z^p J_{p,\mu}^{a,c} f(z))^\gamma.$$

By using Lemma 13, the proof of Theorem 24 is completed.

Putting $p = \delta = \gamma = 1$, $a = c$ and $\mu = 1$ in Theorem 24, we obtain the following corollary:

Corollary 25.

$$|arg(zf(z))| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\alpha}{a-1}\right]\right),$$

then

$$\left|arg\left(\frac{a-1}{z^{a-1}}\int_0^z t^{a-1} f(t) dt\right)\right| < \frac{\pi}{2}.$$

Theorem 26. *Let* $0 < \delta \leq 1$. *Suppose* $f(z) \in \Sigma_p$, $(p \in \mathbb{N})$ *satisfies the following condition*

$$\left|(z^p J_{p,\mu}^{a,c+1} f(z))^\gamma\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\mu\delta}{c-\mu p}\alpha\right]\right), \ (z \in \mathbb{U}),$$

then

$$\left|arg\left(\frac{c-\mu p}{\mu\delta} z^{-\frac{c-\mu p}{\mu\delta}} \int_0^z t^{\frac{c-\mu p}{\mu\delta}-1}(t^p J_{p,\mu}^{a,c+1} f(t))^\gamma dt\right)\right| < \frac{\pi}{2}\alpha.$$

Proof. The proof is similar to the proof of Theorem 26, so we omit it.

Putting $p = \delta = \gamma = 1$, $a = c$ and $\mu = 1$ in Theorem 26, we obtain the following corollary:

Corollary 27.

$$|arg(zJ_p^a f(z))| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left[\frac{\alpha}{a-1}\right]\right),$$

then

$$\left|arg\left(\frac{a-1}{z^{a-1}}\int_0^z t^{a-1}J_p^a f(t)dt\right)\right| < \frac{\pi}{2}.$$

Theorem 28. *Suppose $f(z) \in \Sigma_p$ satisfies the following condition*

$$\left|arg\left((1-\delta)(z^p J_{p,\mu}^{a,c}F_{\nu,p}(z))^\gamma + \delta[z^p J_{p,\mu}^{a,c}F_{\nu,p}(z)]^\gamma\left[\frac{J_{p,\mu}^{a,c}f(z)}{J_{p,\mu}^{a,c}F_{\nu,p}(z)}\right]\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\arctan\left(\frac{\delta}{\nu\gamma}\alpha\right)\right)$$

then

$$\left|arg\left(z^p J_{p,\mu}^{a,c}F_{\nu,p}f(z)\right)^\gamma\right| < \frac{\pi}{2}\alpha, \quad (z \in \mathbb{U}),$$

where the function $F_{\nu,p}(z)$ is defined by (29).

Proof. Let

$$p(z) = \left(z^p J_{p,\mu}^{a,c}F_{\nu,p}f(z)\right)^\gamma, \quad \gamma \neq 0, \tag{39}$$

Differentiating (39) logarithmically with respect to z and multiplying by z, we have

$$\frac{1}{\gamma}\frac{zp'(z)}{p(z)} = \frac{z(J_{p,\mu}^{a,c}F_{\nu,p}f(z))'}{J_{p,\mu}^{a,c}F_{\nu,p}f(z)} + p. \tag{40}$$

Using (30) in (40), we obtain

$$p(z) + \frac{\delta}{\gamma\nu}zp'(z) = (1-\delta)(z^p J_{p,\mu}^{a,c}F_{\nu,p}f(z))^\gamma + \delta\left(z^p J_{p,\mu}^{a,c}F_{\nu,p}f(z)\right)^\gamma\left[\frac{J_{p,\mu}^{a,c}f(z)}{J_{p,\mu}^{a,c}F_{\nu,p}f(z)}\right].$$

By using Lemma 13, the proof of Theorem 28 is complete.

Remark 1. *By specifying the parameters p, a, c and μ, we obtain various results for different operators remined in the introduction.*

References

[1] M. K. Aouf, *New criteria for multivalent meromorphic starlike functions of order alpha*, Proc. Japan. Acad., 69 (1993), 66-70.

[2] M. K. Aouf and Srivastava, *A new criterion for meromorphically p-valent convex functions of order alpha*, Math. Sci. Res. Hot-Line, 1 (1997), no. 8, 7-12.

[3] M. Darus and I. Faisal, *Differential subordination for meromorphic multivalent quasi-convex functions*, Facta Universitatis (NIS), S. M. I., 26 (2011), 1-7.

[4] P. L. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften, Band 259 springer-Verlag, NewYork, Berlin, Heidelberg and Tokyo, 1983.

[5] P. Eenigenburg, S. S. Miller, P. T. Mocanu and M. O. Reade, *On a Briot-Bouquet differential subordination*, General Inequalities, 3 (1983), (Birkhauser Verlag), 339-348.

[6] R. M. El-Ashwah, *Argument properties for p-valent meromorphic functions defined by differintegral operator*, Southeast Asian Bulletin of Mathematics, 35 (2011), 1-7.

[7] R. M. El-Ashwah, *Certain class of meromorphic univalent functions defined by an Erdelyi-Kober type integral operator*, Open Sci. J. Math. Appl., 3 (2015), no.1, 7-13.

[8] R. M. El-Ashwah and A. H. Hassan, *Some inequalities of certain subclass of meromorphic functions defined by using new integral operator*, Information in Sciences and Comput., 3 (2014), 1-10.

[9] R. W. Ibrahim and M. Darus, *Differential subordination for meromorphic multivalent quasi-convex functions*, Acta Math. Univ. Com., 1 (2010), 39-45.

[10] V. Kumar and S. L. Shukla, *Certain integrals for classes of $p-$valent meromorphic functions*, Bull. Austral. Math. Soc., 25 (1982), 85-97.

[11] A. Y. Lashin, *A pplications of Nunokawa's theorem*, J. Inequal. Pure. Appl. Math., 5 (2004), no.4, 1-5.

[12] J. L. Liu and H. M. Srivastava, *A linear operator and associated families of meromorphically multivalent functions*, J. Math. Anal. Appl., 259 (2001), 566-581.

[13] S. S. Miller and P. T. Mocanu, *Differntial subordination: Theory and applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol.225, Marcel Dekker Incorporated, NewYork and Basel, 2000.

[14] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J., 28 (1981), 157-171.

[15] L. Shi, T. P. Yi and Z. G. Wang, *On certain subclass of meromorphic close-to-convex functions*, Appl. Math. Comut., 219 (2013), 10143-10149.

[16] H. Tang, G. T. Deng and S. H. Li, *On a certain new subclass of meromorphic close-to-convex functions*, Inequal. and Appl. J., (2013), 1-6.

[17] A. K. Wanas, *Differential subordination theorems for new classes of meromorphic multivalent quasi-convex functions and some applications*, Int. J. Adv. Appl. Math. and Mech., 2 (2015), no. 3, 126-133.

[18] B. A. Uralegaddi and C. Somanatha, Certain classes of meromorphic multivalent functions, Tamkang J. Math., 23 (1992), 223-231.

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