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SOME FIXED POINT RESULTS FOR GENERALIZED φ-FUZZY CONTRACTION IN FUZZY METRIC SPACES

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Abstract. In this paper, we introduce generalized φ-fuzzy contraction in fuzzy metric spaces. We derive some results about existence and uniqueness of a fixed point for this class of self mappings in fuzzy metric spaces. We also give some examples which support our main results.

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1. Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [27] in 1965. Since then, using this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [8] and Kramosil and Michalek [16] have introduced the concept of fuzzy topological spaces induced by fuzzy metric, which have very important applications in quantum particle physics, particularly in connections with both string and ϵ^{∞} theory, given and studied by El Naschie [6, 7].

The contraction type mappings in fuzzy metric spaces play a crucial role in fixed point theory. In 1988, Grabiec [9] first defined the Banach contraction in a fuzzy metric space and extended fixed point theorems of Banach and Edelstein to fuzzy metric spaces. Following Grabiec's approach, Mishra et al. [21] obtained some common fixed point theorems for asymptotically commuting mappings on fuzzy metric spaces in 1994. In 1998, Vasuki [26] offered a generalization of Grabiec's fuzzy Banach contraction theorem and proved a common fixed point theorem for a sequence of mappings in a fuzzy metric space. Afterwards, Gregori and Sapena [10] introduced the notion of fuzzy contractive mapping and gave some fixed point theorems for complete fuzzy metric spaces in the sense of George and Veeramani, and also for Kramosil and Michalek's fuzzy metric spaces which are complete in

Grabiec's sense. Soon after, Mihet [18] proposed a fuzzy Banach theorem for (weak) B-contraction in M-complete fuzzy metric spaces. Recently, further studies have been done by different authors [1, 2, 19, 20, 23, 24, 25].

First, we recall some well-known definitions and results in the theory of fuzzy metric space (abbreviated, FM space) which are used later in this paper.

Throughout this paper, we denote $\mathbb{R}$ the set of all real numbers, and by $\mathbb{R}^+$ the set of all nonnegative real numbers.

Definition 1. *A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (abbreviated, t-norm) if the following conditions are satisfied:*

(i) $\Delta(a,b) = \Delta(b,a)$,

(ii) $\Delta(a,\Delta(b,c)) = \Delta(\Delta(a,b),c)$,

(iii) $\Delta(a,b) \geq \Delta(c,d)$, *whenever* $a \geq c$ *and* $b \geq d$,

(iv) $\Delta(a,1) = a$,

for every $a,b,c,d \in [0,1]$.

Two typical examples of continuous t-norm are $\Delta_p(a,b) = ab$ and $\Delta_m(a,b) = \min\{a,b\}$. It is evident that, as regards the pointwise ordering, $\Delta \leq \Delta_m$, for each t-norm Δ.

An arbitrary t-norm Δ can be extended (by (iii)) in a unique way to an n-ary operation. For $(a_1,\cdots,a_n) \in [0,1]^n$ $(n \in \mathbb{N})$, the value $\Delta^n(a_1,\cdots,a_n)$ is defined by $\Delta^1(a_1) = a_1$ and $\Delta^n(a_1,\cdots,a_n) = \Delta(\Delta^{n-1}(a_1,\cdots,a_{n-1}),a_n)$. For each $a \in [0,1]$, the sequence $(\Delta^n(a))$ is defined by $\Delta^n(a) = \Delta^n(a,\cdots,a)$.

Definition 2. *A t-norm Δ is said to be of Hadžić type (H-type) if the sequence of functions $(\Delta^n(a))$ is equicontinuous at $a = 1$, that is*

$$\forall \varepsilon \in (0,1) \;\; \exists\, \delta \in (0,1) : \;\; a > 1-\delta \Rightarrow \Delta^n(a) > 1-\varepsilon \quad (n \in \mathbb{N}).$$

The t-norm Δ_m is a trivial example of a t-norm of H-type, but there are t-norms Δ of H-type with $\Delta \neq \Delta_m$, see [11]. It is easy to see that if Δ is of H-type, then Δ satisfies $\sup_{a\in(0,1)} \Delta(a,a) = 1$.

Definition 3. *(George and Veeramani [8]) The 3-tuple (X,M,Δ) is said to be a fuzzy metric space (abbreviated, FM space) if X is a nonempty set, Δ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying the following conditions:*

(FM1) $M(x,y,t) > 0$,

(FM2) $M(x,y,t) = 1$ *iff* $x = y$,

(FM3) $M(x, y, t) = M(y, x, t)$,

(FM4) $\Delta(M(x, y, t), M(y, z, s)) \leq M(x, z, t + s)$,

(FM5) $M(x, y, .) : (0, \infty) \to [0, 1]$ *is continuous.*

for every $x, y, z \in X$ *and* $t, s > 0$.

Example 1. *[8] Let* (X, d) *be a metric space. Define*

$$M(x, y, t) = \frac{kt^n}{kt^n + md(x, y)}, \qquad k, m, n \in \mathbb{R}^+,$$

then (X, M, Δ_m) *is a FM space.*

Proposition 1. *[9] Let* (X, M, Δ) *be a FM space. Then for all* $x, y \in X$, $M(x, y, \cdot)$ *is nondecreasing.*

Definition 4. *Let* (X, M, Δ) *be a FM space. An open ball with center* x *and radius* λ $(0 < \lambda < 1)$ *in* X *is the set* $U_x(\varepsilon, \lambda) = \{y \in X : M(x, y, \epsilon) > 1 - \lambda\}$, *for all* $\varepsilon > 0$. *It is easy to see that* $\mathfrak{U} = \{U_x(\varepsilon, \lambda) : \ x \in X, \ \varepsilon > 0, \ \lambda \in (0, 1)\}$ *determines a Hausdorff topology for* X *[8].*

Definition 5. *A sequence* (x_n) *in a FM space* (X, M, Δ) *is said to be convergent to a point* $x \in X$ *if and only if for every* $\varepsilon > 0$ *and* $\lambda \in (0, 1)$, *there exists* $n_0(\varepsilon, \lambda) \in \mathbb{N}$ *such that* $M(x_n, x, \varepsilon) > 1 - \lambda$ *for all* $n \geq n_0(\varepsilon, \lambda)$ *or* $\lim_{n\to\infty} M(x_n, x, t) = 1$ *for all* $t > 0$, *in this case we say that limit of the sequence* (x_n) *is* x.

Definition 6. *A sequence* (x_n) *in a FM space* (X, M, Δ) *is said to be Cauchy sequence if and only if for every* $\varepsilon > 0$ *and* $\lambda \in (0, 1)$, *there exists* $n_0(\varepsilon, \lambda) \in \mathbb{N}$ *such that* $M(x_{n+p}, x_n, \varepsilon) > 1-\lambda$ *for all* $n \geq n_0(\varepsilon, \lambda)$ *and every* $p \in \mathbb{N}$ *or* $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$, *for all* $t > 0$ *and* $p \in \mathbb{N}$.

Also, a FM space (X, M, Δ) *is said to be complete if and only if every Cauchy sequence in* X, *is convergent.*

Proposition 2. *The limit of a convergent sequence in a FM space* (X, M, Δ) *is unique.*

Proof. It is obvious.

Lemma 1. *[9] If* $\lim\limits_{n\to\infty} x_n = x$ *and* $\lim\limits_{n\to\infty} y_n = y$, *then* $\lim\limits_{n\to\infty} M(x_n, y_n, t) = M(x, y, t)$ *for all* $t > 0$.

Lemma 2. *[15] Let* $n \in \mathbb{N}$, $g_n : (0, \infty) \to (0, \infty)$ *and* $F_n, F : \mathbb{R} \to [0, 1]$. *Assume that* $\sup\{F(t) : t > 0\} = 1$ *and for any* $t > 0$, $\lim\limits_{n\to\infty} g_n(t) = 0$ *and* $F_n(g_n(t)) \geq F(t)$. *If each* F_n *is nondecreasing, then* $\lim\limits_{n\to\infty} F_n(t) = 1$ *for any* $t > 0$.

Lemma 3. *Let (X, M, Δ) be a FM space and $\varphi : (0, \infty) \to (0, \infty)$ be a mapping such that $\lim_{n\to\infty} \varphi^n(t) = 0$. If $x, y \in X$ and $M(x, y, \varphi(t)) \geq M(x, y, t)$ for all $t > 0$. Then $x = y$.*

Proof. By using the above lemma, the result follows.

Proposition 3. *Let $n \in \mathbb{N}$, $M : (0, \infty) \to [0, 1]$, $g_1, g_2, \ldots, g_n : \mathbb{R} \to [0, 1]$ and M, g_i are nondecreasing, left continuous and $\lim_{t\to\infty} M(t) = 1$ $(i = 1, 2, \ldots, n)$. If $\varphi : (0, \infty) \to (0, \infty)$ is a mapping such that $\varphi(t) < t$, $\lim_{n\to\infty} \varphi^n(t) = 0$ and*

$$M(\varphi(t)) \geq \min\{g_1(t), g_2(t), \ldots, g_n(t), M(t)\}, \qquad \forall t > 0.$$

Then $M(\varphi(t)) \geq \min\{g_1(t), g_2(t), \ldots, g_n(t)\}$ for all $t > 0$.

Proof. By way of contradiction, we assume that the conclusion is false. Hence, there exists $t_0 > 0$ such that $\min\{g_1(t_0), g_2(t_0), \ldots, g_n(t_0)\} > M(\varphi(t_0))$. So by the hypothesis we have $M(\varphi(t_0)) \geq M(t_0)$. As M is nondecreasing and $\varphi(t_0) < t_0$, one then has that $M(t) = M(t_0)$ for all $\varphi(t_0) \leq t \leq t_0$. So in fact $\min\{g_1(t_0), g_2(t_0), \ldots, g_n(t_0)\} > M(t_0)$. Let $m = \sup\{t > 0 : M(t) = M(t_0)\}$, by the hypothesis we have $m < \infty$. Choose $t_1 \in (\varphi(m), m)$ and $t_2 > m$ such that $\varphi(t_2) < t_1$, so we have, as M is nondecreasing and $t_1 < m$,

$$M(\varphi(t_2)) \leq M(t_1) \leq M(t_0) < M(t_2).$$

This implies

$$M(\varphi(t_2)) \geq \min\{g_1(t_2), g_2(t_2), \ldots, g_n(t_2)\},$$

(as $M(\varphi(t_2)) \geq \min\{g_1(t_2), g_2(t_2), \ldots, g_n(t_2), M(t_2)\}$). Since

$$\min\{g_1(t_0), g_2(t_0), \ldots, g_n(t_0)\} > M(t_0),$$

we have

$$\begin{aligned} \min\{g_1(t_0), g_2(t_0), \ldots, g_n(t_0)\} > M(t_0) &\geq M(\varphi(t_2)) \\ &\geq \min\{g_1(t_2), g_2(t_2), \ldots, g_n(t_2)\} \\ &\geq \min\{g_1(t_0), g_2(t_0), \ldots, g_n(t_0)\}, \end{aligned}$$

a contradiction, the result follows.

Banach in 1922 proved the celebrated result which is well-known in the literature as the classical Banach's fixed point principle or the classical Banach's contraction principle. The classical Banach's contraction principle has been generalized in many ways over the years. In 1988, Grabiec [9] introduced a fixed point of a contraction mapping on a FM space and proved a fixed point theorem which is generalization of the classical Banach's contraction principle.

Theorem 4. *[9] Let (X, M, Δ) be a complete FM space such that for all $x, y \in X$, $\lim_{t\to\infty} M(x, y, t) = 1$. If T is a contraction mapping of X into itself, that is there exists a constant $0 < c < 1$ such that*

$$M(Tx, Ty, ct) \geq M(x, y, t), \qquad \forall t > 0, \forall x, y \in X.$$

Then there is a unique $x^* \in X$ such that $Tx^* = x^*$.

In 1968, Browder [4] proved the following distinguished generalization of classical Banach's contraction principle.

Theorem 5. *[4] Let (X, d) be a complete metric space. If $T : X \to X$ is a φ-contraction, that is,*

$$d(Tx, Ty) \leq \varphi(d(x, y)),$$

for all $x, y \in X$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing and right continuous function satisfying $0 < \varphi(t) < t$, for all $t > 0$. Then T has a unique fixed point x_0 and $\lim_{n\to\infty} T^n x = x_0$ for any $x \in X$.

Subsequently, his result was extended in 1969 by Boyd and Wong [3] by weakening the hypothesis on φ it suffices that φ is right upper semicontinuous (not necessarily monotonic); i.e.,

$$\limsup_{s\to t^+} \varphi(s) \leq \varphi(t), \quad \forall t \in \mathbb{R}^+.$$

In 1975, Matkowski [17] showed that sometimes in Theorem 5, the condition of the right continuous φ may be omitted.

Theorem 6. *[17] Let (X, d) be a complete metric space and $T : X \to X$ be a φ-contraction, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function satisfying $\lim\limits_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. Then T has a unique fixed point x_0 and $\lim_{n\to\infty} T^n x = x_0$ for any $x \in X$.*

In the following Jachymski [15] obtained nice fuzzy version of Matkowski's theorem.

Theorem 7. *[15] Let (X, M, Δ) be a complete FM space under a t-norm Δ of H-type. If $T : X \to X$ is a φ-fuzzy contraction, that is,*

$$M(Tx, Ty, \varphi(t)) \geq M(x, y, t), \qquad \forall t > 0,\ \forall x, y \in X, \tag{1}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a mapping such that, for any $t > 0$, $0 < \varphi(t) < t$ and $\lim\limits_{n\to\infty} \varphi^n(t) = 0$. Then T has a unique fixed point x_0 and $\lim_{n\to\infty} T^n x = x_0$ for any $x \in X$.

Recently, Ricarte and Romaguera [22] established the following new fuzzy version of Matkowski's theorem by using a type of contraction introduced in the fuzzy intuitionistic context by Huang et al. [14], and that generalizes C-contractions as defined by Hicks in [13].

Theorem 8. *[22] Let (X, M, Δ) be a complete FM space and $T : X \to X$ a self-map such that*

$$M(x, y, t) > 1 - t \implies M(Tx, Ty, \varphi(t)) > 1 - \varphi(t),$$

for all $x, y \in X$ and $t > 0$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function satisfying $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. Then T has a unique fixed point.

In 2014, Castro-Company et al. [5] obtained a generalization of Theorem 8 to preordered fuzzy quasi-metric spaces which is applied to deduce, among other results, a procedure to show in a direct and easy way the existence of solution for the recurrence equations that are typically associated to Quicksort and Divide and Conquer algorithms, respectively.

Definition 7. *Let (X, M, Δ) be a FM space and $T : X \to X$. We say that T is generalized φ-fuzzy contraction if for every $x, y \in X$ and $t > 0$,*

$$M(Tx, Ty, \varphi(t)) \geq \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), M(x, Ty, t), M(Tx, y, t)\}, \tag{2}$$

where $\varphi : (0, \infty) \to (0, \infty)$ is a mapping.

The following example shows that a generalized φ-fuzzy contraction need not be a φ-fuzzy contraction.

Example 2. *Let $X = [0, \infty)$, $T : X \to X$ be defined by $Tx = x + 1$, and let $\varphi : (0, \infty) \to (0, \infty)$ be defined by*

$$\varphi(t) = \begin{cases} \frac{t}{1+t}, & 0 < t \leq 1, \\ t - 1, & 1 < t. \end{cases}$$

For each $x, y \in X$, let $M(x, y, t) = \frac{t}{t+|x-y|}$ for all $t > 0$. Since for all $x, y \in X$, $\max\{|x - y - 1|, |y - x - 1|\} = |x - y| + 1$, then

$$M(Tx, Ty, \varphi(t)) \geq \min\{M(x, Ty, t), M(Tx, y, t)\}.$$

Thus,

$$M(Tx, Ty, \varphi(t)) \geq \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), M(x, Ty, t), M(Tx, y, t)\},$$

which satisfies (2). *If* $t = 2, x = 0$ *and* $y = \frac{3}{2}$, *then* $M(T0, T\frac{3}{2}, \varphi(2)) = \frac{2}{5}$ *and* $M(0, \frac{3}{2}, 2) = \frac{4}{7}$. *Thus,* $M(T0, T\frac{3}{2}, \varphi(2)) < M(0, \frac{3}{2}, 2)$, *which does not satisfy* (1).

As the following example shows, there exists T that does not satisfy (2) with $\varphi(t) = kt$, $0 < k < 1$.

Example 3. *Let* $X = [0, \infty)$, $T : X \to X$ *be defined by* $Tx = 2x$, *and let* $\varphi : (0, \infty) \to (0, \infty)$ *be defined by* $\varphi(t) = kt$, $0 < k < 1$. *Let* $M(x, y, t) = \frac{t}{t+|x-y|}$ *for each* $x, y \in X$ *and for all* $t > 0$. *If* $x = 0, y = 1$ *and* $t = \frac{2}{k} > 0$, *then for simple calculations,* $M(T0, T1, \varphi(\frac{2}{k})) = \frac{1}{2}$ *and,*

$$\min\{M(0,1,\frac{2}{k}), M(0,T0,\frac{2}{k}), M(1,T1,\frac{2}{k}), M(0,T1,\frac{2}{k}), M(T0,1,\frac{2}{k})\}$$
$$= \frac{2}{2+k} > \frac{1}{2}.$$

Therefore, for $x = 0, y = 1$ *and* $t = \frac{2}{k} > 0$, *the mapping* T *does not satisfy* (2). *Thus, we showed that there exists* T *that does not satisfy* (2) *with* $\varphi(t) = kt$, $0 < k < 1$.

Definition 8. *Let* (X, M, Δ) *be a FM space. For every* $x_0 \in X$, *let* $O(x_0, T) = \{T^n x_0 : n \in \mathbb{N} \cup \{0\}\}$. *The set* $O(x_0, T)$ *is the orbit of the mapping* $T : X \to X$ *at* x_0. *Let* $D_{O(x_0,T)} : \mathbb{R} \to [0, 1]$ *be a diameter of* $O(x_0, T)$, *i.e.,* $D_{O(x_0,T)}(t) = \sup_{s<t} \inf_{x,y \in O(x_0,T)} M(x, y, s)$. *If* $\sup_{t\in\mathbb{R}} D_{O(x_0,T)}(t) = 1$, *then the orbit* $O(x_0, T)$ *is a fuzzy bounded subset of* X. *Hence* $O(x_0, T)$ *is a fuzzy bounded set if and only if* $\lim_{t\to\infty} D_{O(x_0,T)}(t) = 1$.

In this paper, we introduce generalized φ-fuzzy contraction in FM spaces. We derive some results about existence and uniqueness of a fixed point for this class of self mappings in FM spaces. We show that if $\varphi : (0, \infty) \to (0, \infty)$ is a injective mapping such that $\varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$. Then every generalized φ-fuzzy contraction with the bounded orbit has unique fixed point. Finally, we give some examples which support our main results.

2. Main results

Now we state and prove our main theorem about existence and uniqueness of a fixed point for generalized φ-fuzzy contraction in FM space under certain conditions.

Theorem 9. *Let* (X, M, Δ) *be a complete FM space such that* $\lim_{t\to\infty} M(x, y, t) = 1$, *for all* $x, y \in X$ *and let* $T : X \to X$ *be a generalized* φ*-fuzzy contraction mapping* *where* $\varphi : (0, \infty) \to (0, \infty)$ *is a injective mapping such that* $\varphi(t) < t$ *and* $\lim_{n\to\infty} \varphi^n(t) = 0$ *for each* $t > 0$. *If there exists* $x_0 \in X$ *with the bounded orbit, then there is a unique* $x^* \in X$ *such that* $Tx^* = x^*$. *Moreover,* $(T^n x_0)$ *converges to* x^*.

Proof. Let $u_n = T^n x_0$, if there exists $n \in \mathbb{N}$, such that $u_{n+1} = u_n$, then there is a $x^* \in X$ such that $Tx^* = x^*$ and $(T^n x_0)$ converges to x^*. So we can assume that $u_{n+1} \neq u_n$ for all $n \in \mathbb{N}$.

Now by the condition (2), we have

$$M(u_n, u_{n+1}, \varphi(t)) \geq \min\{M(u_{n-1}, u_n, t), M(u_{n-1}, u_n, t), M(u_n, u_{n+1}, t), M(u_{n-1}, u_{n+1}, t), M(u_n, u_n, t)\},$$

for all $t > 0$, so

$$M(u_n, u_{n+1}, \varphi(t)) \geq \min\{M(u_{n-1}, u_n, t), M(u_n, u_{n+1}, t), M(u_{n-1}, u_{n+1}, t)\},$$

for all $t > 0$. By Proposition 3 we have

$$M(u_n, u_{n+1}, \varphi(t)) \geq \min\{M(u_{n-1}, u_n, t), M(u_{n-1}, u_{n+1}, t)\}, \quad (\forall t > 0). \tag{3}$$

In the following we show by induction that for each $n \in \mathbb{N}$ and for each $t > 0$, there exists $1 \leq m \leq n + 1$ such that

$$M(u_n, u_{n+1}, \varphi^n(t)) \geq M(u_0, u_m, t). \tag{4}$$

If $n = 1$, then by (3), we have

$$\begin{aligned} M(u_1, u_2, \varphi(t)) &\geq \min\{M(u_0, u_1, t), M(u_0, u_2, t)\} \\ &= M(u_0, u_m, t), \end{aligned}$$

for some $1 \leq m \leq 2$ and for all $t > 0$. Thus (4) holds for $n = 1$. Assume towards a contradiction that (4) is not true and take $n_0 > 1$, be the least natural number such that (4) does not hold. So there exists $t_0 > 0$, such that for all $1 \leq m \leq n_0 + 1$, we have

$$M(u_{n_0}, u_{n_0+1}, t_0) < M(u_0, u_m, \varphi^{-n_0}(t_0)). \tag{5}$$

If $\min\{M(u_{n_0-1}, u_{n_0}, \varphi^{-1}(t_0)), M(u_{n_0-1}, u_{n_0+1}, \varphi^{-1}(t_0))\} = M(u_{n_0-1}, u_{n_0}, \varphi^{-1}(t_0))$, then by the hypothesis we have

$$M(u_{n_0}, u_{n_0+1}, t_0) \geq M(u_{n_0-1}, u_{n_0}, \varphi^{-1}(t_0)) \geq M(u_0, u_m, \varphi^{-n_0}(t_0)),$$

for some $1 \leq m \leq n_0$, a contradiction. Thus

$$\min\{M(u_{n_0-1}, u_{n_0}, \varphi^{-1}(t_0)), M(u_{n_0-1}, u_{n_0+1}, \varphi^{-1}(t_0))\} = M(u_{n_0-1}, u_{n_0+1}, \varphi^{-1}(t_0)).$$

Also form (3), we have

$$M(u_{n_0}, u_{n_0+1}, t_0) \geq M(u_{n_0-1}, u_{n_0+1}, \varphi^{-1}(t_0)). \tag{6}$$

By the condition (2), we get

$$\begin{aligned} M(u_{n_0-1}, u_{n_0+1}, \varphi^{-1}(t)) \geq \min\{&M(u_{n_0-2}, u_{n_0}, \varphi^{-2}(t)), M(u_{n_0-2}, u_{n_0-1}, \varphi^{-2}(t)),\\ &M(u_{n_0}, u_{n_0+1}, \varphi^{-2}(t)), M(u_{n_0-2}, u_{n_0+1}, \varphi^{-2}(t)),\\ &M(u_{n_0}, u_{n_0-1}, \varphi^{-2}(t))\}, \end{aligned} \tag{7}$$

for all $t > 0$. If

$$\begin{aligned} \min\{&M(u_{n_0-2}, u_{n_0}, \varphi^{-2}(t_0)), M(u_{n_0-2}, u_{n_0-1}, \varphi^{-2}(t_0)), M(u_{n_0-2}, u_{n_0+1}, \varphi^{-2}(t_0)),\\ &M(u_{n_0}, u_{n_0+1}, \varphi^{-2}(t_0)), M(u_{n_0}, u_{n_0-1}, \varphi^{-2}(t_0))\}\\ =&M(u_{n_0}, u_{n_0-1}, \varphi^{-2}(t_0)), \end{aligned}$$

then from (6) and the above, we have

$$\begin{aligned} M(u_{n_0}, u_{n_0+1}, t_0) \geq& M(u_{n_0-1}, u_{n_0+1}, \varphi^{-1}(t_0))\\ \geq& M(u_{n_0}, u_{n_0-1}, \varphi^{-2}(t_0)) = M(u_{n_0-1}, u_{n_0}, \varphi^{-2}(t_0))\\ \geq& M(u_{n_0}, u_m, \varphi^{-(n_0+1)}(t_0))\\ \geq& M(u_{n_0}, u_m, \varphi^{-n_0}(t_0)), \end{aligned}$$

for some $1 \leq m \leq n_0$, a contradiction. If

$$\begin{aligned} \min\{&M(u_{n_0-2}, u_{n_0}, \varphi^{-2}(t_0)), M(u_{n_0-2}, u_{n_0-1}, \varphi^{-2}(t_0)), M(u_{n_0}, u_{n_0+1}, \varphi^{-2}(t_0)),\\ &M(u_{n_0-2}, u_{n_0+1}, \varphi^{-2}(t_0))\} = M(u_{n_0}, u_{n_0+1}, \varphi^{-2}(t_0)), \end{aligned}$$

then form (6), (7) and the above, we have

$$M(u_{n_0}, u_{n_0+1}, t_0) \geq M(u_{n_0}, u_{n_0+1}, \varphi^{-2}(t_0)),$$

since $\varphi^{-2}(t_0) > t_0$, then $M(u_{n_0}, u_{n_0+1}, t_0) = M(u_{n_0}, u_{n_0+1}, \varphi^{-2}(t_0))$. By (3),

$$M(u_{n_0}, u_{n_0+1}, \varphi^{-2}(t_0)) \geq \min\{M(u_{n_0-1}, u_{n_0}, \varphi^{-3}(t_0)), M(u_{n_0-1}, u_{n_0+1}, \varphi^{-3}(t_0))\}.$$

If $\min\{M(u_{n_0-1}, u_{n_0}, \varphi^{-3}(t_0)), M(u_{n_0-1}, u_{n_0+1}, \varphi^{-3}(t_0))\} = M(u_{n_0-1}, u_{n_0}, \varphi^{-3}(t_0))$, then by the hypothesis we have

$$\begin{aligned} M(u_{n_0}, u_{n_0+1}, t_0) \geq& M(u_{n_0-1}, u_{n_0}, \varphi^{-3}(t_0)) \geq M(u_0, u_m, \varphi^{-(n_0+2)}(t_0))\\ \geq& M(u_0, u_m, \varphi^{-n_0}(t_0)), \end{aligned}$$

for some $1 \leq m \leq n_0$, a contradiction. Thus

$$\min\{M(u_{n_0-1}, u_{n_0}, \varphi^{-3}(t_0)), M(u_{n_0-1}, u_{n_0+1}, \varphi^{-3}(t_0))\} = M(u_{n_0-1}, u_{n_0+1}, \varphi^{-3}(t_0)).$$

Also form (3), we have

$$M(u_{n_0}, u_{n_0+1}, t_0) = M(u_{n_0}, u_{n_0+1}, \varphi^{-2}(t_0)) \geq M(u_{n_0-1}, u_{n_0+1}, \varphi^{-3}(t_0)). \tag{8}$$

By the condition (2), we get

$$\begin{aligned} M(u_{n_0-1}, u_{n_0+1}, \varphi^{-3}(t_0)) \geq \min\{&M(u_{n_0-2}, u_{n_0}, \varphi^{-4}(t_0)), M(u_{n_0-2}, u_{n_0-1}, \varphi^{-4}(t_0)),\\ &M(u_{n_0}, u_{n_0+1}, \varphi^{-4}(t_0)), M(u_{n_0-2}, u_{n_0+1}, \varphi^{-4}(t_0)),\\ &M(u_{n_0}, u_{n_0-1}, \varphi^{-4}(t_0))\}. \end{aligned}$$

If

$$\begin{aligned} &\min\{M(u_{n_0-2}, u_{n_0}, \varphi^{-4}(t_0)), M(u_{n_0-2}, u_{n_0-1}, \varphi^{-4}(t_0)), M(u_{n_0}, u_{n_0+1}, \varphi^{-4}(t_0)),\\ &\quad M(u_{n_0-2}, u_{n_0+1}, \varphi^{-4}(t_0)), M(u_{n_0}, u_{n_0-1}, \varphi^{-4}(t_0))\}\\ &=M(u_{n_0}, u_{n_0-1}, \varphi^{-4}(t_0)), \end{aligned}$$

then from (8) and the above, we obtain

$$\begin{aligned} M(u_{n_0}, u_{n_0+1}, t_0) \geq &M(u_{n_0-1}, u_{n_0+1}, \varphi^{-3}(t_0)) \geq M(u_{n_0-1}, u_{n_0}, \varphi^{-4}(t_0))\\ \geq &M(u_0, u_m, \varphi^{-(n_0+3)}(t_0)) \geq M(u_0, u_m, \varphi^{-n_0}(t_0)), \end{aligned}$$

for some $1 \leq m \leq n_0$, a contradiction. If

$$\begin{aligned} &\min\{M(u_{n_0-2}, u_{n_0}, \varphi^{-4}(t_0)), M(u_{n_0-2}, u_{n_0-1}, \varphi^{-4}(t_0)), M(u_{n_0}, u_{n_0+1}, \varphi^{-4}(t_0)),\\ &\quad M(u_{n_0-2}, u_{n_0+1}, \varphi^{-4}(t_0)), M(u_{n_0}, u_{n_0-1}, \varphi^{-4}(t_0))\} = M(u_{n_0}, u_{n_0+1}, \varphi^{-4}(t_0)), \end{aligned}$$

then from (8) and the above, we have

$$M(u_{n_0}, u_{n_0+1}, t_0) = M(u_{n_0}, u_{n_0+1}, \varphi^{-4}(t_0)).$$

Again by (3), we have

$$M(u_{n_0}, u_{n_0+1}, \varphi^{-4}(t_0)) \geq \min\{M(u_{n_0-1}, u_{n_0}, \varphi^{-5}(t_0)), M(u_{n_0-1}, u_{n_0+1}, \varphi^{-5}(t_0))\}.$$

Therefore by continuing this process, we see that if

$$\begin{aligned} &\min\{M(u_{n_0-2}, u_{n_0}, \varphi^{-k}(t_0)), M(u_{n_0-2}, u_{n_0-1}, \varphi^{-k}(t_0)), M(u_{n_0}, u_{n_0+1}, \varphi^{-k}(t_0)),\\ &\quad M(u_{n_0-2}, u_{n_0+1}, \varphi^{-k}(t_0)), M(u_{n_0}, u_{n_0-1}, \varphi^{-k}(t_0))\}\\ &=M(u_{n_0}, u_{n_0-1}, \varphi^{-k}(t_0)), \end{aligned}$$

for some $k \geq 2$, then

$$M(u_{n_0}, u_{n_0+1}, t_0) \geq M(u_0, u_m, \varphi^{-(n_0+k-1)}(t_0)) \geq M(u_0, u_m, \varphi^{-n_0}(t_0)),$$

for some $1 \leq m \leq n_0$, a contradiction. If

$$\begin{aligned}\min\{&M(u_{n_0-2}, u_{n_0}, \varphi^{-k}(t_0)), M(u_{n_0-2}, u_{n_0-1}, \varphi^{-k}(t_0)), M(u_{n_0}, u_{n_0+1}, \varphi^{-k}(t_0)),\\ &M(u_{n_0-2}, u_{n_0+1}, \varphi^{-k}(t_0)), M(u_{n_0}, u_{n_0-1}, \varphi^{-k}(t_0))\}\\ =&M(u_{n_0}, u_{n_0+1}, \varphi^{-k}(t_0)),\end{aligned}$$

for all $k \geq 2$, then $M(u_{n_0}, u_{n_0+1}, t_0) = M(u_{n_0}, u_{n_0+1}, \varphi^{-k}(t_0))$. Now letting $k \to \infty$, then $M(u_{n_0}, u_{n_0+1}, t_0) = 1$, which is contradiction with (5). Otherwise, if there exists $k \geq 2$ such that

$$\begin{aligned}\min\{&M(u_{n_0-2}, u_{n_0}, \varphi^{-k}(t_0)), M(u_{n_0-2}, u_{n_0-1}, \varphi^{-k}(t_0)), M(u_{n_0}, u_{n_0+1}, \varphi^{-k}(t_0)),\\ &M(u_{n_0-2}, u_{n_0+1}, \varphi^{-k}(t_0)), M(u_{n_0}, u_{n_0-1}, \varphi^{-k}(t_0))\} \geq\\ \min\{&M(u_{n_0-2}, u_{n_0}, \varphi^{-k}(t_0)), M(u_{n_0-2}, u_{n_0-1}, \varphi^{-k}(t_0)), M(u_{n_0-2}, u_{n_0+1}, \varphi^{-k}(t_0))\},\end{aligned}$$

since $t < \varphi^{-1}(t) < \varphi^{-2}(t) < \cdots$, then we have

$$\begin{aligned}\min\{&M(u_{n_0-2}, u_{n_0}, \varphi^{-k}(t_0)), M(u_{n_0-2}, u_{n_0-1}, \varphi^{-k}(t_0)), M(u_{n_0}, u_{n_0+1}, \varphi^{-k}(t_0)),\\ &M(u_{n_0-2}, u_{n_0+1}, \varphi^{-k}(t_0)), M(u_{n_0}, u_{n_0-1}, \varphi^{-k}(t_0))\} \geq\\ \min\{&M(u_{n_0-2}, u_{n_0}, \varphi^{-2}(t_0)), M(u_{n_0-2}, u_{n_0-1}, \varphi^{-2}(t_0)), M(u_{n_0-2}, u_{n_0+1}, \varphi^{-2}(t_0))\}.\end{aligned}$$

Therefore

$$\begin{aligned}M(u_{n_0-1}, u_{n_0+1}, \varphi^{-1}(t_0)) \geq \min\{&M(u_{n_0-2}, u_{n_0}, \varphi^{-2}(t_0)), M(u_{n_0-2}, u_{n_0-1}, \varphi^{-2}(t_0)),\\ &M(u_{n_0-2}, u_{n_0+1}, \varphi^{-2}(t_0))\}.\end{aligned}$$

From (6) and the above, we get

$$\begin{aligned}M(u_{n_0}, u_{n_0+1}, t_0) \geq&M(u_{n_0-1}, u_{n_0+1}, \varphi^{-1}(t_0))\\ \geq&\min\{M(u_{n_0-2}, u_{n_0}, \varphi^{-2}(t_0)), M(u_{n_0-2}, u_{n_0-1}, \varphi^{-2}(t_0)),\\ &M(u_{n_0-2}, u_{n_0+1}, \varphi^{-2}(t_0))\}\\ =&M(u_{n_0-2}, u_m, \varphi^{-2}(t_0)),\end{aligned} \tag{9}$$

for some $1 \leq m \leq n_0 + 1$. Therefore by continuing this process, we see that for each $1 \leq k \leq n_0$, there exists $1 \leq m \leq n_0 + 1$ such that

$$M(u_{n_0}, u_{n_0+1}, t_0) \geq M(u_{n_0-k}, u_m, \varphi^{-k}(t_0)). \tag{10}$$

If $k = n_0$ in (10), then this is a contradiction by (5). So (4) holds for all $n \in \mathbb{N}$. Then from (4) we get

$$M(u_n, u_{n+1}, t) \geq M(u_n, u_{n+1}, \varphi^n(t)) \geq M(u_0, u_m, t) \geq D_{O(x_0,T)}(t).$$

Let $\varepsilon > 0$ and $\lambda \in (0, 1)$ be given, since $D_{O(x_0,T)}(t) \to 1$ as $t \to \infty$, then there exists $t_1 > 0$ such that

$$D_{O(x_0,T)}(t_1) > 1 - \lambda.$$

Since $\varphi^n(t_1) \to 0$ as $n \to \infty$, then there is $N \in \mathbb{N}$ such that $\varphi^n(t_1) < \varepsilon$ whenever $n \geq N$. So

$$\begin{aligned} M(u_n, u_{n+1}, \varepsilon) &\geq M(u_n, u_{n+1}, \varphi^n(t_1)) \\ &\geq D_{O(x_0,T)}(t_1) \\ &> 1 - \lambda. \end{aligned}$$

Thus we proved that for each $\varepsilon > 0$ and for each $\lambda \in (0, 1)$, there exists a positive integer N such that

$$M(u_n, u_{n+1}, \varepsilon) > 1 - \lambda, \qquad \forall n \geq N.$$

This means that $\lim_{n\to\infty} M(u_n, u_{n+1}, t) = 1$ for all $t > 0$. On the other hand

$$M(u_n, u_{n+p}, t) \geq \Delta\left(M(u_n, u_{n+1}, \frac{t}{p}), M(u_{n+1}, u_{n+2}, \frac{t}{p}), \ldots, M(u_{n+p-1}, u_{n+p}, \frac{t}{p})\right),$$

for all $p \geq 1$, now taking the limits as $n \to \infty$, by the hypothesis we get

$$\lim_{n\to\infty} M(u_n, u_{n+p}, t) = 1.$$

Hence (u_n) is a Cauchy sequence and by the hypothesis there exists an element $x^* \in X$ such that $\lim_{n\to\infty} u_n = x^*$. Again by (2) we have

$$\begin{aligned} M(Tu_n, Tx^*, \varphi(t)) \geq \min\{&M(u_n, x^*, t), M(u_n, Tu_n, t), M(x^*, Tx^*, t), \\ &M(u_n, Tx^*, t), M(x^*, Tu_n, t)\}. \end{aligned}$$

Since $\lim_{n\to\infty} Tu_n = \lim_{n\to\infty} u_{n+1} = x^*$, then by Lemma 1 we get

$$M(x^*, Tx^*, \varphi(t)) \geq M(x^*, Tx^*, t),$$

for all $t > 0$ now by Lemma 3, $Tx^* = x^*$. Let $y^* \in X$ such that $Ty^* = y^*$ then from (2) we have

$$\begin{aligned} M(Tx^*, Ty^*, \varphi(t)) \geq &\min\{M(x^*, y^*, t), M(Tx^*, x^*, t), M(Ty^*, y^*, t), \\ &M(Tx^*, y^*, t), M(x^*, Ty^*, t)\} \\ =&M(x^*, y^*, t), \end{aligned}$$

then $M(x^*, y^*, \varphi(t)) \geq M(x^*, y^*, t)$, now by Lemma 3, $x^* = y^*$, so the desired result is obtained.

Example 4. *Consider $X = [-1, 1]$ and define $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and for all $t > 0$, where d is Euclidean metric. Then (X, M, Δ_m) is a complete FM space. Define self mapping T on X as follows:*

$$Tx = \begin{cases} 0\ ; & -1 \leq x < 0, \\ \frac{x}{16(1+x)}\ ; & 0 \leq x < \frac{4}{5} \text{ or } \frac{7}{8} < x \leq 1, \\ \frac{x}{16}\ ; & \frac{4}{5} \leq x \leq \frac{7}{8}. \end{cases}$$

To verify T is generalized φ-fuzzy contraction with $\varphi(t) = \frac{1}{8}t$, we need to consider several possible cases.

Case 1. Let $x, y \in [-1, 0)$. Then

$$d(Tx, Ty) = |Tx - Ty| = 0 \leq \frac{1}{8}|x - y| = \frac{1}{8}d(x, y).$$

Case 2. Let $x \in [-1, 0)$ and $y \in [0, \frac{4}{5}) \cup (\frac{7}{8}, 1]$. Then

$$d(Tx, Ty) = |Tx - Ty| = \frac{y}{16(1 + y)} \leq \frac{1}{8}|y - 0| = \frac{1}{8}d(y, Tx).$$

Case 3. Let $x \in [-1, 0)$ and $y \in [\frac{4}{5}, \frac{7}{8}]$. Then

$$d(Tx, Ty) = |Tx - Ty| = \frac{y}{16} \leq \frac{1}{8}|y - 0| = \frac{1}{8}d(y, Tx).$$

Case 4. Let $x, y \in [0, \frac{4}{5}) \cup (\frac{7}{8}, 1]$. Then

$$d(Tx, Ty) = |Tx - Ty| = |\frac{x}{16(1 + x)} - \frac{y}{16(1 + y)}| \leq \frac{1}{8}|x - y| = \frac{1}{8}d(x, y).$$

Case 5. Let $x \in [0, \frac{4}{5}) \cup (\frac{7}{8}, 1]$ and $y \in [\frac{4}{5}, \frac{7}{8}]$. Then

$$d(Tx, Ty) = |Tx - Ty| = |\frac{x}{16(1 + x)} - \frac{y}{16}| \leq \frac{1}{16}(\frac{x}{1 + x} + y) \leq \frac{1}{16}(\frac{1}{2} + \frac{7}{8}) \leq \frac{11}{128},$$

and
$$\frac{123}{160} = \frac{4}{5} - \frac{1}{16}\frac{1}{2} \le y - \frac{x}{16(1+x)} \le |y - \frac{x}{16(1+x)}| = d(y, Tx).$$
Thus
$$d(Tx, Ty) \le \frac{11}{128} \le \frac{123}{1280} = \frac{1}{8} \times \frac{123}{160} \le \frac{1}{8} d(y, Tx).$$
Case 6. Let $x, y \in [\frac{4}{5}, \frac{7}{8}]$. *Then*
$$d(Tx, Ty) = |Tx - Ty| = |\frac{x}{16} - \frac{y}{16}| \le \frac{1}{8}|x - y| = \frac{1}{8} d(x, y).$$
Hence, we obtain
$$d(Tx, Ty) \le \frac{1}{8} \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \quad (x, y \in [-1, 1]),$$
or in other words
$$M(Tx, Ty, \frac{1}{8}t) \ge \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), M(x, Ty, t), M(y, Tx, t)\},$$
for every $x, y \in X$ *and* $t > 0$. *Also,* $0 \in X$ *has the bounded orbit, so* T *has a unique fixed point 0 in* X*, by Theorem 9.*

Example 5. *Let* $X = [-1, 1]$, $T : X \to X$ *and* $\varphi : (0, \infty) \to (0, \infty)$ *be mappings defined as follows:*
$$T(x) = \begin{cases} 0, & -1 \le x < 0, \\ \frac{x}{1+x}, & 0 \le x < \frac{4}{5} \text{ or } \frac{7}{8} < x \le 1, \\ \frac{-1}{16}x, & \frac{4}{5} \le x \le \frac{7}{8}, \end{cases} \qquad \varphi(t) = \begin{cases} t - \frac{t^2}{8}, & 0 < t \le 1, \\ \frac{7}{8}t, & 1 < t. \end{cases}$$
Let $M(x, y, t) = \frac{t}{t + |x - y|}$ *for all* $t > 0$ *and all* $x, y \in X$. *Then* (X, M, Δ_m) *is a complete FM space. It is easy to see that all of the assumptions of Theorem 9 are satisfied, and so* T *has a unique fixed point (*$x = 0$ *is a unique fixed point of* T*). On the other hand, we can show that* T *does not satisfy* (1).

Lemma 10. *[12] Let* X *be a nonempty set and* $T : X \to X$ *a mapping. Then there exists a subset* $E \subseteq X$ *such that* $T(E) = T(X)$ *and* $T : E \to X$ *is one-to-one.*

Theorem 11. *Let* (X, M, Δ) *be a complete FM space such that* $\lim_{t\to\infty} M(x, y, t) = 1$*, for all* $x, y \in X$ *and let self mappings* T *and* S *satisfy the following condition:*
$$M(Tx, Ty, \varphi(t)) \ge \min\{M(Sx, Sy, t), M(Sx, Tx, t), M(Sy, Ty, t), M(Sx, Ty, t), M(Tx, Sy, t)\},$$

for all $x, y \in X$, where $\varphi : (0,\infty) \to (0,\infty)$ is a mapping the same as in Theorem 9. If $TX \subseteq SX$ and SX is a complete subset of X and there is a sequence $(x_n) \subseteq X$ such that $Tx_{n-1} = Sx_n$ and $\sup_{t\in\mathbb{R}} D_{\{T^n x_{n-1}:\, n\in\mathbb{N}\}}(t) = 1$, then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible (i.e., they commut at their coincidence points), then T and S have a unique common fixed point.

Proof. By Lemma 10, there exists $E \subseteq X$ such that $SE = SX$ and $S : E \to X$ is one-to-one. Now, define a mapping $U : SE \to SE$ by $U(Sx) = Tx$. Since S is one to one on E, U is well defined. Also we have

$$\begin{aligned} M(U(Sx), U(Sy), \varphi(t)) =& M(Tx, Ty, \varphi(t)) \\ &\geq \min\{M(Sx, Sy, t), M(Sx, Tx, t), M(Sy, Ty, t), \\ &\qquad M(Sx, Ty, t), M(Tx, Sy, t)\}, \end{aligned}$$

for all $Sx, Sy \in SE$. Since $Tx_{n-1} = Sx_n$, so we get $U^n(Sx_0) = T^n x_{n-1}$. By our assumptions $\sup_{t\in\mathbb{R}} D_{O(Sx_0,U)}(t) = \sup_{t\in\mathbb{R}} D_{\{T^n x_{n-1}:\, n\in\mathbb{N}\}}(t) = 1$. Since $SE = SX$ is complete, by using Theorem 9, there exists $x^* \in X$ such that $U(Sx^*) = Sx^*$. Then $Tx^* = Sx^*$, and so T and S have a coincidence point, which is also unique.

If T and S are weakly compatible, since $Tx^* = Sx^*$, then we have

$$T(Tx^*) = TSx^* = STx^* = S(Sx^*).$$

Thus, $Tx^* = Sx^*$ is also a confidence point of T and S. By uniqueness of coincidence point of T and S, we get $Tx^* = Sx^* = x^*$.

Theorem 12. *Let (X, M, Δ) be a complete FM space such that $\lim_{t\to\infty} M(x, y, t) = 1$, for all $x, y \in X$. Suppose that $T : X \to X$ is a mapping satisfy the following condition:*

$$\begin{aligned} M(Tx, Ty, \alpha(t)t) \geq \min\{&M(x, y, t), M(x, Tx, t), M(y, Ty, t), \\ &M(x, Ty, t), M(Tx, y, t)\}, \end{aligned} \tag{11}$$

for all $t > 0$ and $x, y \in X$, where $\alpha : (0,\infty) \to (0, 1)$ is strictly decreasing function. If there exists $x_0 \in X$ with the bounded orbit, then there is a unique $x^* \in X$ such that $Tx^* = x^*$. Moreover, $(T^n x_0)$ converges to x^*.

Proof. Set $\varphi(t) = \alpha(t)t$, it is sufficient to prove that φ satisfying the hypothesis of Theorem 9. In fact, since $\alpha(t) < 1$, then $\varphi(t) < t$, for all $t > 0$. On the other hand, for all $n \in \mathbb{Z}^+$, we see that $0 < \varphi^{n+1}(r) = \varphi(\varphi^n(r)) < \varphi^n(r)$, thus the sequence

$\{\varphi^n(r)\}$ is convergent for each $r > 0$. Let $\lim_{n\to\infty} \varphi^n(r) = a \geq 0$, then $\lim_{t\to a^+} \varphi(t) = a$. Suppose that $a > 0$, then by the monotony of α, we have

$$a = \lim_{t\to a^+} \varphi(t) = \lim_{t\to a^+} \alpha(t)t \leq \lim_{t\to a^+} \alpha\left(\frac{a}{2}\right) t = \alpha\left(\frac{a}{2}\right) a < a.$$

This is a contradiction. Thus $\lim_{n\to\infty} \varphi^n(r) = 0$. Then by Theorem 9, the result follows.

Example 6. *Consider $X = [0, 3]$ and define $M(x, y, t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and for all $t > 0$. Then (X, M, Δ_m) is a complete FM space. Let $\varphi(t) = \frac{t}{2}$, define continuous self mappings S and T on X as*

$$Tx = \frac{1}{6}x + 1, \qquad Sx = \frac{1}{3}(x + \frac{12}{5}), \qquad (x \in X).$$

Thus we have

$$M(Tx, Ty, \varphi(t)) = \frac{\frac{t}{2}}{\frac{t}{2} + \frac{1}{6}|x-y|} = \frac{t}{t + \frac{1}{3}|x-y|} = M(Sx, Sy, t).$$

It is easy to see that $TX \subseteq SX$, T and S are weakly compatible. Hence, we conclude that all the conditions of Theorem 11 hold, so T and S have a unique common fixed point $\frac{6}{5}$ in X.

Example 7. *Let $X = [0, \infty)$ and*

$$M(x, y, t) = \begin{cases} \frac{t}{t+|x-y|}, & t \leq |x-y|, \\ 1, & t > |x-y|. \end{cases}$$

Then (X, M, Δ_m) is a complete FM space. Define $Tx = \frac{x}{1+x}$ and $\alpha(t) = \frac{1}{1+t}$. By definition of T we have

$$|Tx - Ty| = \frac{|x-y|}{1 + |x-y| + 2\min\{x, y\} + xy} \leq \frac{|x-y|}{1 + |x-y|}.$$

Clearly, if $\alpha(t)t > |Tx - Ty|$, then (11) holds. Suppose now that $\alpha(t)t \leq |Tx - Ty|$. Then we have

$$\frac{t}{1+t} \leq |Tx - Ty| \leq \frac{|x-y|}{1 + |x-y|},$$

so $t \leq |x-y|$ and by definition of M we get

$$\begin{aligned} M(Tx,Ty,\alpha(t)t) &= \frac{\alpha(t)t}{\alpha(t)t+|Tx-Ty|} \\ &= \frac{t}{t+(1+t)|Tx-Ty|} \\ &\geq \frac{t}{t+(1+t)\frac{|x-y|}{1+|x-y|}} \\ &\geq \frac{t}{t+(1+|x-y|)\frac{|x-y|}{1+|x-y|}} \\ &= \frac{t}{t+|x-y|} \\ &= M(x,y,t). \end{aligned}$$

Thus we proved that T satisfies (11). *Therefore, we showed that the mapping T satisfies all hypotheses of Theorem 12 and has a unique fixed point* 0.

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