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A CLASS OF MEROMORPHIC MULTIVALENT FUNCTIONS DEFINED BY A DIFFERENTIAL OPERATOR

A. Ebadian, S.Shams, R. Asadi

Abstract. In this paper, we define a class of meromorphically multivalent functions in $\mathbb{U}^* = \{z : z \in \mathbb{C} : 0 < |z| < 1\}$ by using a differential operator.Important properties of this class like coefficient estimates, distortion theorem, radius of starlikeness and convexity,closure theorems, convolution properties are obtained. We also study δ-neighboorhoods and partial sums for this class.

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1. Introduction

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k} z^{p+k} \ (a_{p+k} \geq 0; p \in \mathbb{N} = \{1, 2, ...\}) \tag{1}$$

which are analytic and p-valent in the punctured unit disk

$$\mathbb{U}^* = \{z \ : \ z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$$

A function $f \in \Sigma_p$ is meromorphically starlike of order $\rho (0 \leq \rho < p)$ if

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho$$

A function $f \in \Sigma_p$ is meromorphically convex of order $\theta (0 \leq \theta < p)$ if

$$-\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \theta$$

If $f \in \Sigma_p$ is given by (1) and $g \in \Sigma_p$ is given by

$$g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} b_{p+k} z^{p+k} \ (b_{p+k} \geq 0; p \in \mathbb{N}) \tag{2}$$

then the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k} b_{p+k} z^{p+k} = (g * f)(z) \ (z \in \mathbb{U}^*; p \in \mathbb{N}) \tag{3}$$

For functions $f(z) \in \Sigma_p$; Aouf [1] defined the following differential operator:

$$\begin{aligned}
S^0_{\lambda,p} f(z) &= f(z) \\
S^1_{\lambda,p} f(z) &= (1-\lambda) f(z) + \frac{\lambda}{p} z f'(z) + \frac{2\lambda}{z^p} \\
&= \frac{1}{z^p} + \sum_{k=0}^{\infty} (\frac{p+\lambda k}{p}) a_{p+k} z^{p+k} \\
&= S_{\lambda,p} f(z) \ \ (\lambda \geq 0; p \in \mathbb{N}) \\
S^2_{\lambda,p} f(z) &= S_{\lambda,p}(S^1_{\lambda,p} f(z))
\end{aligned}$$

and

$$\begin{aligned}
S^n_{\lambda,p} f(z) &= S_{\lambda,p}(S^{n-1}_{\lambda,p} f(z)) \\
&= (1-\lambda) S^{n-1}_{\lambda,p} f(z) + \frac{\lambda}{p} z (S^{n-1}_{\lambda,p} f(z))' + \frac{2\lambda}{z^p} \ (\lambda \geq 0; n, p \in \mathbb{N})
\end{aligned}$$

It can be easily seen that

$$S^n_{\lambda,p} f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} (\frac{p+\lambda k}{p})^n a_{p+k} z^{p+k} \ (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, p \in \mathbb{N}) \tag{4}$$

Also Orhan et al. [4] defined the differential operator $T^n_{\sigma\mu p}$ in the following way:

$$\begin{aligned}
&T^0_{\sigma\mu p} f(z) = f(z) \\
&T^1_{\sigma\mu p} f(z) = T_{\sigma\mu p} f(z) = \sigma\mu \frac{[z^{p+1} f(z)]''}{z^{p-1}} + (\sigma - \mu) \frac{[z^{p+1} f(z)]'}{z^p} + (1 - \sigma + \mu) f(z)
\end{aligned} \tag{5}$$

and, in general

$$T^n_{\sigma\mu p} f(z) = T_{\sigma\mu p}(T^{n-1}_{\sigma\mu p} f(z)) \tag{6}$$

where $0 \leq \mu \leq \sigma$ and $n \in \mathbb{N}_0$.
If the function$f(z) \in \Sigma_p$ is given by (1) then from (5) and (6) we obtain

$$T^n_{\sigma\mu p} f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \Psi_k(\sigma, \mu, n, p) a_{k+p} z^{k+p} \tag{7}$$

where

$$\Psi_k(\sigma, \mu, n, p) = [1 + (k + 2p)(\sigma - \mu + (k + 2p + 1)\sigma\mu)]^n \tag{8}$$

Making use of the differential operators $S^n_{\lambda,p} f(z)$ and $T^n_{\sigma\mu p} f(z)$ defined as in (4) and (7) respectively, we defined the following differential operator for the functions $f(z) \in \Sigma_p$:

$$D^n_{\lambda,\sigma,\mu,\omega,p} f(z) = (1 - \omega) S^n_{\lambda,p} f(z) + \omega T^n_{\sigma\mu p} f(z) \tag{9}$$

for $n \in \mathbb{N}, \lambda \geq 0, 0 \leq \mu \leq \sigma, 0 \leq \omega \leq 1$.
Let $f(z)$ be given by (1), then by using (4), (7) and (9) it is easy to see that

$$D^n_{\lambda,\sigma,\mu,\omega,p} f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \Phi_k(n, \lambda, \sigma, \mu, \omega, p) a_{p+k} z^{p+k} \tag{10}$$

where

$$\Phi_k(n, \lambda, \sigma, \mu, \omega, p) = (1 - \omega)(\frac{p + \lambda k}{p}) + \omega \Psi_k(\sigma, \mu, n, p) \tag{11}$$

$$\text{and } \Psi_k(\sigma, \mu, n, p) = [1 + (k + 2p)(\sigma - \mu + (k + 2p + 1)\sigma\mu)]^n \tag{12}$$

for $n \in \mathbb{N}, \lambda \geq 0, 0 \leq \mu \leq \sigma, 0 \leq \omega \leq 1$, or in terms of convolution as

$$D^n_{\lambda,\sigma,\mu,\omega,p} f(z) = (f * h)(z)$$

where

$$h(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \Phi_k(n, \lambda, \sigma, \mu, \omega, p) z^{p+k}$$

With the aid of the differential operator $D^n_{\lambda,\sigma,\mu,\omega,p} f(z)$ we define the following subclass of multivalent meromorphic functions.

Definition 1. *A function $f(z) \in \Sigma_p$ is said to be in the class $\mathcal{H}_p(\alpha, \beta, \gamma)$ if it satisfies the following inequality:*

$$\left| \frac{z^{p+2} \big(D^n_{\lambda,\sigma,\mu,\omega,p} f(z)\big)'' + z^{p+1} \big(D^n_{\lambda,\sigma,\mu,\omega,p} f(z)\big)' - p^2}{\gamma z^{p+1} \big(D^n_{\lambda,\sigma,\mu,\omega,p} f(z)\big)' + \alpha(1 + \gamma)p - p} \right| < \beta \tag{13}$$

where $0 \leq \alpha < 1, 0 < \beta \leq 1, 0 \leq \gamma \leq 1$.

Meromorphically multivalent functions have been extensively studied, for example, recently by Najafzadeh and Ebadian [3], Atshan and Kulkarni [2], Orhan et al. [4] and Auof [1].

2. Coefficients Estimates

Theorem 1. *A function $f(z)$ defined by* (1) *is in the class $\mathcal{H}_p(\alpha, \beta, \gamma)$ if and only if*

$$\sum_{k=0}^{\infty}(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)a_{p+k} \leq \beta p(1-\alpha)(1+\gamma) \tag{14}$$

where $\Phi_k(n, \lambda, \sigma, \mu, \omega, p)$ is given by(11).

Proof. Assume that (14) holds. It is enough to show that

$$\begin{aligned} M = &\left|z^{p+2}\left(D^n_{\lambda,\sigma,\mu,\omega,p}f(z)\right)'' + z^{p+1}\left(D^n_{\lambda,\sigma,\mu,\omega,p}f(z)\right)' - p^2\right| \\ &-\beta\left|\gamma z^{p+1}\left(D^n_{\lambda,\sigma,\mu,\omega,p}f(z)\right)' + \alpha(1+\gamma)p - p\right| < 0 \end{aligned}$$

For $|z| = r < 1$ from (14) we obtain

$$\begin{aligned} M = &\left|\sum_{k=0}^{\infty}(p+k)^2\Phi_k(n,\lambda,\sigma,\mu,\omega,p)a_{p+k}z^{2p+k}\right| \\ &- \beta\left|p(1-\alpha)(1+\gamma) - \gamma\sum_{k=0}^{\infty}(p+k)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)a_{p+k}z^{2p+k}\right| \\ &\leq \sum_{k=0}^{\infty}(p+k)^2\Phi_k(n,\lambda,\sigma,\mu,\omega,p)a_{p+k}r^{2p+k} \\ &- \beta p(1-\alpha)(1+\gamma) + \beta\gamma\sum_{k=0}^{\infty}(p+k)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)a_{p+k}r^{2p+k} \\ &< \sum_{k=0}^{\infty}(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)a_{p+k} - \beta p(1-\alpha)(1+\gamma) < 0 \end{aligned}$$

Hence $f \in \mathcal{H}_p(\alpha, \beta, \gamma)$.

Conversely Let $f(z) \in \mathcal{H}_p(\alpha, \beta, \gamma)$, then (13) holds true, so we have

$$\left| \frac{z^{p+2}\big(D^n_{\lambda,\sigma,\mu,\omega,p}f(z)\big)'' + z^{p+1}\big(D^n_{\lambda,\sigma,\mu,\omega,p}f(z)\big)' - p^2}{\gamma z^{p+1}\big(D^n_{\lambda,\sigma,\mu,\omega,p}f(z)\big)' + \alpha(1+\gamma)p - p} \right|$$

$$= \left| \frac{\sum_{k=0}^{\infty}(p+k)^2\Phi_k(n,\lambda,\sigma,\mu,\omega,p)a_{p+k}z^{2p+k}}{p(1-\alpha)(1+\gamma) - \gamma\sum_{k=0}^{\infty}(p+k)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)a_{p+k}z^{2p+k}} \right| < \beta$$

Since $\mathrm{Re}(z) \leq |z|$ for all z, it follows that

$$\mathrm{Re}\left\{ \frac{\sum_{k=0}^{\infty}(p+k)^2\Phi_k(n,\lambda,\sigma,\mu,\omega,p)a_{p+k}z^{2p+k}}{p(1-\alpha)(1+\gamma) - \gamma\sum_{k=0}^{\infty}(p+k)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)a_{p+k}z^{2p+k}} \right\} < \beta$$

Now by letting $z \to 1^-$ through real axis, we obtain

$$\sum_{k=0}^{\infty}(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)a_{p+k} \leq \beta p(1-\alpha)(1+\gamma)$$

Hence the result follows.

Corollary 2. *If $f(z)$ defined by (1) is in the class $\mathcal{H}_p(\alpha, \beta, \gamma)$ then*

$$a_{p+k} \leq \frac{\beta p(1-\alpha)(1+\gamma)}{(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}$$

This result is sharp for the function $f(z)$ given by

$$f(z) = \frac{1}{z^p} + \frac{\beta p(1-\alpha)(1+\gamma)}{(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)} z^{p+k} \tag{15}$$

where $\Phi_k(n,\lambda,\sigma,\mu,\omega,p)$ is given by (11).

3. Distortion Theorem

Theorem 3. *If $f(z)$ defined by (1) is in the class $\mathcal{H}_p(\alpha, \beta, \gamma)$ then for $0 < |z| = r < 1$ we have*

$$\frac{1}{r^p} - \frac{\beta p(1-\alpha)(1+\gamma)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} r^p \leq |f(z)| \leq \frac{1}{r^p} + \frac{\beta p(1-\alpha)(1+\gamma)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} r^p \tag{16}$$

and

$$\frac{p}{r^{p+1}} - \frac{\beta p(1-\alpha)(1+\gamma)}{(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} r^{p-1} \leq |f'(z)| \leq \frac{p}{r^{p+1}} + \frac{\beta p(1-\alpha)(1+\gamma)}{(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} r^{p-1} \tag{17}$$

where

$$\Phi_0(n,\lambda,\sigma,\mu,\omega,p) = \left[\Big(1-\omega\Big) + \omega\Big[1+2p(\sigma-\mu+(2p+1)\sigma\mu\Big]^n\right] \tag{18}$$

The bounds are attained for the function $f(z)$ given by

$$f(z) = \frac{1}{z^p} + \frac{\beta p(1-\alpha)(1+\gamma)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} z^p \tag{19}$$

Proof. In view of Theorem 1 we have

$$p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)\sum_{k=0}^{\infty} a_{p+k} \leq \sum_{k=0}^{\infty}(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)a_{p+k}$$
$$\leq \beta p(1-\alpha)(1+\gamma)$$

which is equivalent to

$$\sum_{k=0}^{\infty} a_{p+k} \leq \frac{\beta p(1-\alpha)(1+\gamma)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} \tag{20}$$

Thus for $0<|z|=r<1$ we get

$$\begin{aligned} |f(z)| &\leq \frac{1}{r^p} + \sum_{k=0}^{\infty} a_{p+k} r^{p+k} \\ &\leq \frac{1}{r^p} + r^p \sum_{k=0}^{\infty} a_{p+k} \\ &\leq \frac{1}{r^p} + \frac{\beta p(1-\alpha)(1+\gamma)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} r^p \end{aligned} \tag{21}$$

and

$$\begin{aligned} |f(z)| &\geq \frac{1}{r^p} - \sum_{k=0}^{\infty} a_{p+k} r^{p+k} \\ &\geq \frac{1}{r^p} - r^p \sum_{k=0}^{\infty} a_{p+k} \\ &\geq \frac{1}{r^p} - \frac{\beta p(1-\alpha)(1+\gamma)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} r^p \end{aligned} \tag{22}$$

which together, yield (16). Furthermore, it follows from Theorem 1 that

$$\sum_{k=0}^{\infty}(p+k)a_{p+k} \leq \frac{\beta p(1-\alpha)(1+\gamma)}{(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} \tag{23}$$

Hence

$$\begin{aligned} |f'(z)| &\leq \frac{p}{r^{p+1}} + \sum_{k=0}^{\infty}(p+k)a_{p+k}r^{p+k-1} \\ &\leq \frac{p}{r^{p+1}} + r^{p-1}\sum_{k=0}^{\infty}(p+k)a_{p+k} \\ &\leq \frac{p}{r^{p+1}} + \frac{\beta p(1-\alpha)(1+\gamma)}{(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)}r^{p-1} \end{aligned} \tag{24}$$

and

$$\begin{aligned} |f'(z)| &\geq \frac{p}{r^{p+1}} - \sum_{k=0}^{\infty}(p+k)a_{p+k}r^{p+k-1} \\ &\geq \frac{p}{r^{p+1}} - r^{p-1}\sum_{k=0}^{\infty}(p+k)a_{p+k} \\ &\geq \frac{p}{r^{p+1}} - \frac{\beta p(1-\alpha)(1+\gamma)}{(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)}r^{p-1} \end{aligned} \tag{25}$$

which, together, yield (17). It can be easily seen that the function $f(z)$ defined by (19) is extermal for Theorem 3.

4. Radius of Starlikeness and Convexity

Theorem 4. *Let the function $f(z)$ defined by (1) be in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$ then $f(z)$ is meromorphically p-valent starlike of order $\rho(0 \leq \rho < p)$ in $0 < |z| < r(n,\lambda,\sigma,\mu,\omega,p,\rho,\alpha,\beta,\gamma)$, where*

$$r(n,\lambda,\sigma,\mu,\omega,p,\rho,\alpha,\beta,\gamma) = \inf_k \left[\frac{(p-\rho)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)(p+k)(p+k+\beta\gamma)}{\beta p(1-\alpha)(1+\gamma)(3p+k-\rho)}\right]^{\frac{1}{2p+k}} \tag{26}$$

$$(k \geq 0; p \in \mathbb{N}; n \in \mathbb{N}_0)$$

The result is sharp.

Proof. It is sufficient to show that

$$\left|\frac{(zf'(z))+pf(z)}{f(z)}\right| \leq p-\rho \quad for \ 0<|z|<r(n,\lambda,\sigma,\mu,\omega,p,\rho,\alpha,\beta,\gamma)$$

Note that

$$\left|\frac{zf'(z)+pf(z)}{f(z)}\right| = \left|\frac{\sum_{k=0}^{\infty}(2p+k)a_{p+k}z^{p+k}}{z^{-p}+\sum_{k=0}^{\infty}a_{p+k}z^{p+k}}\right|$$

$$\leq \frac{\sum_{k=0}^{\infty}(2p+k)a_{p+k}r^{2p+k}}{1-\sum_{k=0}^{\infty}a_{p+k}r^{2p+k}}$$

Thus, $\left|\frac{zf'(z)+pf(z)}{f(z)}\right| \leq p-\rho$ if

$$\sum_{k=0}^{\infty}\frac{(3p+k-\rho)}{(p-\rho)}a_{p+k}r^{2p+k} \leq 1 \tag{27}$$

Theorem 1 ensures that

$$\sum_{k=0}^{\infty}\frac{(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{\beta p(1-\alpha)(1+\gamma)}a_{p+k} \leq 1, \tag{28}$$

in view of (28) it follows that (27) will be true if

$$\frac{(3p+k-\rho)}{(p-\rho)}r^{2p+k} \leq \frac{(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{\beta p(1-\alpha)(1+\gamma)} \tag{29}$$

or if

$$r \leq \left[\frac{(p-\rho)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)(p+k)(p+k+\beta\gamma)}{\beta p(1-\alpha)(1+\gamma)(3p+k-\rho)}\right]^{\frac{1}{2p+k}} \tag{30}$$

Setting $r(n,\lambda,\sigma,\mu,\omega,p,\rho,\alpha,\beta,\gamma)$ in (30) the result follows. The result is sharp for the extermal function $f(z)$ given by (15).

Theorem 5. *Let the function $f(z)$ defined by (1) be in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$ then $f(z)$ is meromorphically p-valent convex of order $\theta (0 \le \theta < p)$ in $0 < |z| < r\ (n,\lambda,\sigma,\mu,\omega,p,\theta,\alpha,\beta,\gamma)$, where*

$$r(n,\lambda,\sigma,\mu,\omega,p,\theta,\alpha,\beta,\gamma) = \inf_k \left[\frac{p(p-\theta)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)(p+k+\beta\gamma)}{\beta p(1-\alpha)(1+\gamma)(3p+k-\theta)}\right]^{\frac{1}{2p+k}} \tag{31}$$

$$(k \ge 0; p \in \mathbb{N}; n \in \mathbb{N}_0)$$

The result is sharp.

Proof. It is sufficient to show that

$$\left|\frac{(zf'(z))' + pf'(z)}{f'(z)}\right| \le p - \theta$$

for $0 < |z| < r\ (n,\lambda,\sigma,\mu,\omega,p,\theta,\alpha,\beta,\gamma)$.
Note that

$$\left|\frac{(zf'(z))' + pf'(z)}{f'(z)}\right| = \left|\frac{\sum_{k=0}^{\infty}(p+k)(2p+k)a_{p+k}z^{p+k-1}}{-pz^{-(p+1)} + \sum_{k=0}^{\infty}(p+k)a_{p+k}z^{p+k-1}}\right|$$

$$\le \frac{\sum_{k=0}^{\infty}(p+k)(2p+k)a_{p+k}r^{2p+k}}{p - \sum_{k=0}^{\infty}(p+k)a_{p+k}r^{2p+k}}$$

Thus $\left|\frac{(zf'(z))' + pf'(z)}{f'(z)}\right| \le p - \theta$ if

$$\sum_{k=0}^{\infty}\frac{(p+k)(3p+k-\theta)}{p(p-\theta)}a_{p+k}r^{2p+k} \le 1 \tag{32}$$

Theorem 1 ensures that

$$\sum_{k=0}^{\infty}\frac{(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{\beta p(1-\alpha)(1+\gamma)}a_{p+k} \le 1, \tag{33}$$

in view of (33) it follows that (32) will be true if

$$\frac{(p+k)(3p+k-\theta)}{p(p-\theta)} r^{2p+k} \leq \frac{(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{\beta p(1-\alpha)(1+\gamma)} \tag{34}$$

or if

$$r \leq \left[\frac{p(p-\theta)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)(p+k+\beta\gamma)}{\beta p(1-\alpha)(1+\gamma)(3p+k-\theta)}\right]^{\frac{1}{2p+k}} \tag{35}$$

Setting $r(n,\lambda,\sigma,\mu,\omega,p,\theta,\alpha,\beta,\gamma)$ in (35) the result follows.The result is sharp for the extermal function $f(z)$ given by (15).

5. Closure Theorems

Theorem 6. *Let*

$$f_{p-1}(z) = \frac{1}{z^p} \tag{36}$$

and

$$f_{p+k}(z) = \frac{1}{z^p} + \frac{\beta p(1-\alpha)(1+\gamma)}{(p+k)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)(p+k+\beta\gamma)} z^{p+k} \quad (k \geq 0; p \in \mathbb{N}; n \in \mathbb{N}_0) \tag{37}$$

Then $f(z)$ is in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$ if and only if it can be expressed of the form

$$f(z) = \sum_{k=-1}^{\infty} c_{p+k} f_{p+k}(z) \tag{38}$$

where

$$c_{p+k} \geq 0 \text{ and } \sum_{k=-1}^{\infty} c_{p+k} = 1$$

Proof. Let $f(z) = \sum_{k=-1}^{\infty} c_{p+k} f_{p+k}(z)$ where $c_{p+k} \geq 0$ *and* $\sum_{k=-1}^{\infty} c_{p+k} = 1$, then

$$\begin{aligned} f(z) &= \sum_{k=-1}^{\infty} c_{p+k} f_{p+k}(z) \\ &= \frac{1}{z^p} + \sum_{k=0}^{\infty} c_{p+k} \frac{\beta p(1-\alpha)(1+\gamma)}{(p+k)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)(p+k+\beta\gamma)} z^{p+k} \end{aligned}$$

Since

$$\sum_{k=0}^{\infty} c_{p+k} \frac{\beta p(1-\alpha)(1+\gamma)}{(p+k)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)(p+k+\beta\gamma)} \times \frac{(p+k)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)(p+k+\beta\gamma)}{\beta p(1-\alpha)(1+\gamma)}$$
$$= \sum_{k=0}^{\infty} c_{p+k} = 1 - c_{p-1} \leq 1$$

which by Theorem 1 shows $f(z) \in \mathcal{H}_p(\alpha,\beta,\gamma)$.
Conversely, suppose $f(z) \in \mathcal{H}_p(\alpha,\beta,\gamma)$, then by Corollary 2 we have

$$a_{p+k} \leq \frac{\beta p(1-\alpha)(1+\gamma)}{(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}$$

Setting

$$c_{p+k} = \frac{(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{\beta p(1-\alpha)(1+\gamma)} a_{p+k}$$

and

$$c_{p-1} = 1 - \sum_{k=0}^{\infty} c_{p+k},$$

then

$$\begin{aligned}
f(z) &= \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k} z^{p+k} \\
&= \frac{1}{z^p} + \sum_{k=0}^{\infty} \frac{\beta p(1-\alpha)(1+\gamma)}{(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)} c_{p+k} z^{p+k} \\
&= \frac{1}{z^p} + \sum_{k=0}^{\infty} (f_{p+k}(z) - \frac{1}{z^p}) c_{p+k} \\
&= \frac{1}{z^p}\left(1 - \sum_{k=0}^{\infty} c_{p+k}\right) + \sum_{k=0}^{\infty} c_{p+k} f_{p+k}(z) \\
&= \frac{1}{z^p} c_{p-1} + \sum_{k=0}^{\infty} c_{p+k} f_{p+k}(z) \\
&= \sum_{k=-1}^{\infty} c_{p+k} f_{p+k}(z)
\end{aligned}$$

This completes the proof of Theorem 6.

Theorem 7. *The class $\mathcal{H}_p(\alpha,\beta,\gamma)$ is closed under convex linear combinations.*

Proof. Let each of the functions

$$f_j(z)=\frac{1}{z^p}+\sum_{k=0}^{\infty}a_{p+k,j}z^{p+k}\ (a_{p+k,j}\geq 0; j=1,2) \tag{39}$$

be in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z)=(1-t)f_1(z)+tf_2(z)\ (0\leq t\leq 1) \tag{40}$$

is also in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$. Since

$$h(z)=\frac{1}{z^p}+\sum_{k=0}^{\infty}[(1-t)(a_{p+k,1}+ta_{p+k,2}]z^{p+k}\ (0\leq t\leq 1) \tag{41}$$

with the help of Theorem 1 we have

$$\begin{aligned}
&\sum_{k=0}^{\infty}(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)[(1-t)a_{p+k,1}+ta_{p+k,2}]\\
&=(1-t)\sum_{k=0}^{\infty}(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)a_{p+k,1}\\
&+t\sum_{k=0}^{\infty}(p+k)[p+k+\beta\gamma]\Phi_k(n,\lambda,\sigma,\mu,\omega,p)a_{p+k,2}\\
&\leq(1-t)\beta p(1-\alpha)(1+\gamma)+t\beta p(1-\alpha)(1+\gamma)=\beta p(1-\alpha)(1+\gamma)
\end{aligned}$$

which shows that $h(z)\in\mathcal{H}_p(\alpha,\beta,\gamma)$. Hence the result follows.

6. Convolution Properties

Theorem 8. *Let the functions $f_j(z)(j=1,2)$ defined by (39) be in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$, then $(f_1*f_2)(z)\in\mathcal{H}_p(\phi,\beta,\gamma)$, where*

$$\phi=1-\frac{p\beta(1+\gamma)(1-\alpha)^2}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} \tag{42}$$

and $\Phi_0(n,\lambda,\sigma,\mu,\omega,p)$ is given by (18). The result is sharp for the functions $f_j(z)(j=1,2)$ given by

$$f_j(z)=\frac{1}{z^p}+\frac{p\beta(1+\gamma)(1-\alpha)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)}z^p \tag{43}$$

Proof. Employing the technique used earlier by Schlid and Silverman, we will find the largest ϕ such that

$$\sum_{k=0}^{\infty} \frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\phi)} a_{p+k,1}a_{p+k,2} \leq 1 \tag{44}$$

for $f_j(z) \in \mathcal{H}_p(\alpha,\beta,\gamma)(j=1,2)$. Since $f_j(z) \in \mathcal{H}_p(\alpha,\beta,\gamma)(j=1,2)$, we readily see that

$$\sum_{k=0}^{\infty} \frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)} a_{p+k,j} \leq 1 \ (j=1,2) \tag{45}$$

By applying the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=0}^{\infty} \frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)} \sqrt{a_{p+k,1}a_{p+k,2}} \leq 1 \tag{46}$$

This implies that we need only to show that

$$\frac{a_{p+k,1}a_{p+k,2}}{(1-\phi)} \leq \frac{\sqrt{a_{p+k,1}a_{p+k,2}}}{(1-\alpha)} \tag{47}$$

or, equivalently, that

$$\sqrt{a_{p+k,1}a_{p+k,2}} \leq \frac{1-\phi}{(1-\alpha)} \tag{48}$$

Hence by the inequality (46) it is sufficient to prove the following inequality

$$\frac{p\beta(1+\gamma)(1-\alpha)}{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)} \leq \frac{1-\phi}{1-\alpha} \tag{49}$$

which it implies that

$$\phi \leq 1 - \frac{p\beta(1+\gamma)(1-\alpha)^2}{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)} \tag{50}$$

Now define the function $\Lambda(k)$ by

$$\Lambda(k) = 1 - \frac{p\beta(1+\gamma)(1-\alpha)^2}{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)} \tag{51}$$

We note that $\Lambda(k)$ is an increasing function of k, therefore we conclude that

$$\phi \leq \Lambda(0) = 1 - \frac{p\beta(1+\gamma)(1-\alpha)^2}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} \tag{52}$$

which completes the proof of Theorem.

Theorem 9. *Let the functions* $f_1(z)$ *and* $f_2(z)$ *defined by* (39) *be in the classes* $\mathcal{H}_p(\alpha_1,\beta,\gamma),\mathcal{H}_p(\alpha_2,\beta,\gamma)$*, respectively. then* $(f_1 * f_2)(z) \in \mathcal{H}_p(\eta,\beta,\gamma)$*, where*

$$\eta = 1 - \frac{p\beta(1+\gamma)(1-\alpha_1)(1-\alpha_2)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} \tag{53}$$

and $\Phi_0(n,\lambda,\sigma,\mu,\omega,p)$ *is given by* (18)*. The result is sharp for the functions* $f_j(z)(j = 1,2)$ *given by*

$$f_1(z) = \frac{1}{z^p} + \frac{p\beta(1+\gamma)(1-\alpha_1)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} z^p \tag{54}$$

and

$$f_2(z) = \frac{1}{z^p} + \frac{p\beta(1+\gamma)(1-\alpha_2)}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} z^p \tag{55}$$

Proof. Using similar arguments to those in the proof of Theorem 8 we get the result.

Theorem 10. *If* $f_1(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,1} z^{p+k} \in \mathcal{H}_p(\alpha,\beta,\gamma)$ *and*
$f_2(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{p+k,2} z^{p+k}$ $(0 \leq a_{p+k,2} \leq 1; k = 0,1,2,...; p \in \mathbb{N})$ *then*
$(f_1 * f_2)(z) \in \mathcal{H}_p(\alpha,\beta,\gamma)$.

Proof. Since

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)} a_{p+k,1} a_{p+k,2} \\
\leq &\sum_{k=0}^{\infty} \frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)} a_{p+k,1} \leq 1
\end{aligned}$$

Then Theorem 1, implies that $(f_1 * f_2)(z) \in \mathcal{H}_p(\alpha,\beta,\gamma)$.

Corollary 11. *If* $f(z) \in \mathcal{H}_p(\alpha,\beta,\gamma)$*, then the integral operator*

$$\mathcal{F}_{c,p}(z) = \frac{c}{z^{p+c}} \int_0^z t^{c+p-1} f(t)dt, \quad c > 0 \tag{56}$$

is also in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$.

Proof. It is easy to check that

$$\mathcal{F}_{c,p}(z) = f(z) * \Big(\frac{1}{z^p} + \sum_{k=0}^{\infty} \frac{c}{c+2p+k} z^{p+k}\Big) \tag{57}$$

Since $0 < \dfrac{c}{c+2p+k} \le 1$, then by Theorem 10 , the proof is trivial.

Theorem 12. *Let the functions $f_j(z)(j = 1,2)$ defined by* (39) *be in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$ and*

$$p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p) - 2p\beta(1+\gamma)(1-\alpha) \ge 0 \tag{58}$$

then the function $h(z)$ defined by

$$h(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} (a_{p+k,1}^2 + a_{p+k,2}^2) z^{p+k} \tag{59}$$

belongs to the class $\mathcal{H}_p(\alpha,\beta,\gamma)$, where $\Phi_0(n,\lambda,\sigma,\mu,\omega,p)$ is given by (18).

Proof. Since $f_1(z) \in \mathcal{H}_p(\alpha,\beta,\gamma)$, we get

$$\sum_{k=0}^{\infty} \frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)} a_{p+k,1} \le 1 \tag{60}$$

and so

$$\sum_{k=0}^{\infty} \left[\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}\right]^2 a_{p+k,1}^2 \le 1 \tag{61}$$

Similarly, since $f_2(z) \in \mathcal{H}_p(\alpha,\beta,\gamma)$, we have

$$\sum_{k=0}^{\infty} \left[\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}\right]^2 a_{p+k,2}^2 \le 1 \tag{62}$$

Hence

$$\sum_{k=0}^{\infty} \frac{1}{2}\left[\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}\right]^2 (a_{p+k,1}^2 + a_{p+k,2}^2) \le 1 \tag{63}$$

In view of Theorem 1, it is sufficient to show that

$$\sum_{k=0}^{\infty} \left[\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}\right] (a_{p+k,1}^2 + a_{p+k,2}^2) \le 1 \tag{64}$$

Thus the inequality (64) will be satisfied if, for $k = 0, 1, 2, ...$

$$\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)} \leq \frac{1}{2}\left[\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}\right]^2 \tag{65}$$

or if

$$(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p) - 2p\beta(1+\gamma)(1-\alpha) \geq 0 \tag{66}$$

for $k = 0, 1, 2, ...$The left hand side of (66) is an increasing function of k, hence it is satisfied for all k if

$$p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p) - 2p\beta(1+\gamma)(1-\alpha) \geq 0 \tag{67}$$

which is true by our assumption. Hence the proof is complete.

Theorem 13. *Let the functions $f_j(z)(j = 1, 2)$ defined by* (39) *be in the class $\mathcal{H}_p(\alpha,\beta,\gamma)$ then the function $h(z)$ defined by* (59) *belongs to the class $\mathcal{H}_p(\tau,\beta,\gamma)$, where*

$$\tau = 1 - \frac{2p\beta(1+\gamma)(1-\alpha)^2}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} \tag{68}$$

and $\Phi_0(n,\lambda,\sigma,\mu,p)$ is given by (18). *The result is sharp for the functions$f_j(z)(j = 1, 2)$ defined by* (43).

Proof. Noting that

$$\sum_{k=0}^{\infty} \frac{\Big[(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)\Big]^2}{\Big[p\beta(1+\gamma)(1-\alpha)\Big]^2} a_{p+k,j}^2 \tag{69}$$

$$\leq \left[\sum_{k=0}^{\infty} \frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)} a_{p+k,j}\right]^2 \leq 1 \tag{70}$$

for $f_j(z) \in \mathcal{H}_p(\alpha,\beta,\gamma)(j = 1, 2)$, we have

$$\sum_{k=0}^{\infty} \frac{\Big[(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)\Big]^2}{2\Big[p\beta(1+\gamma)(1-\alpha)\Big]^2} (a_{p+k,1}^2 + a_{p+k,2}^2) \leq 1 \tag{71}$$

Therefore we have to find the largest τ such that

$$\frac{1}{1-\tau} \leq \frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{2p\beta(1+\gamma)(1-\alpha)^2} \tag{72}$$

That is

$$\tau \leq 1 - \frac{2p\beta(1+\gamma)(1-\alpha)^2}{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)} \tag{73}$$

If we define $L(k)$ by

$$L(k) = 1 - \frac{2p\beta(1+\gamma)(1-\alpha)^2}{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)} \tag{74}$$

we see that $L(k)$ is an increasing function of k, thus we conclude that

$$\tau \leq L(0) = 1 - \frac{2p\beta(1+\gamma)(1-\alpha)^2}{p(p+\beta\gamma)\Phi_0(n,\lambda,\sigma,\mu,\omega,p)} \tag{75}$$

which completes the proof of Theorem.

7. Neighborhoods and Partial Sums

Definition 2. *For $\delta > 0$ and a non-negative sequence $\mathcal{S} = \{s_k\}_{k=0}^{\infty}$ where*

$$s_k = \frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)} \tag{76}$$
$$(k \geq 0, p \in \mathbb{N}, 0 \leq \alpha < 1, 0 < \beta \leq 1, 0 \leq \gamma \leq 1, \lambda \geq 0, 0 \leq \mu \leq \sigma)$$

the δ-neighborhood of a function $f \in \Sigma_p$ is defined by

$$\mathcal{N}_\delta(f) = \left\{ g \in \Sigma_p \ : \ g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} b_{p+k} z^{p+k} \text{ and } \sum_{k=0}^{\infty} s_k |b_{p+k} - a_{p+k}| \leq \delta \right\} \tag{77}$$

Theorem 14. *Let $f \in \mathcal{H}_p(\alpha,\beta,\gamma)$ be given by (1). If f satisfies*

$$\frac{f(z) + \epsilon z^{-p}}{1+\epsilon} \in \mathcal{H}_p(\alpha,\beta,\gamma) \ (\epsilon \in \mathbb{C}, |\epsilon| < \delta, \delta > 0) \tag{78}$$

then

$$\mathcal{N}_\delta(f) \subset \mathcal{H}_p(\alpha,\beta,\gamma) \tag{79}$$

Proof. It is not difficult to see that a function $f \in \mathcal{H}_p(\alpha,\beta,\gamma)$ if and only if

$$\frac{z^{p+2}\big(D^n_{\lambda,\sigma,\mu,\delta,p}f(z)\big)'' + z^{p+1}\big(D^n_{\lambda,\sigma,\mu,\delta,p}f(z)\big)' - p^2}{\beta\gamma z^{p+1}\big(D^n_{\lambda,\sigma,\mu,\delta,p}f(z)\big)' + \beta\alpha(1+\gamma)p - \beta p} \neq \nu \quad (\nu \in \mathbb{C}, |\nu| = 1) \tag{80}$$

which is equivalent to

$$\frac{(f*h)(z)}{z^{-p}} \neq 0 \tag{81}$$

where

$$h(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} c_{p+k} z^{p+k}$$

such that

$$c_{p+k} = \frac{(p+k)(p+k-\nu\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\nu\beta(1+\gamma)(1-\alpha)}$$

which implies

$$|c_{p+k}| \leq \frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}$$

Furthermore, under the hypotheses (78), using (81) we obtain

$$\frac{1}{z^{-p}}\Big(\frac{f(z)+\epsilon z^{-p}}{1+\epsilon} * h(z)\Big) \neq 0 \tag{82}$$

We have

$$\begin{aligned}\left|\frac{1}{1+\epsilon}\frac{(f*h)(z)}{z^{-p}} + \frac{\epsilon}{1+\epsilon}\right| &\geq \frac{1}{|1+\epsilon|}\left|\frac{(f*h)(z)}{z^{-p}}\right| - \frac{|\epsilon|}{|1+\epsilon|} \\ &> \frac{1}{1+\delta}\left|\frac{(f*h)(z)}{z^{-p}}\right| - \frac{\delta}{1+\delta}\end{aligned}$$

For holding (82) we must have

$$\frac{1}{1+\delta}\left|\frac{(f*h)(z)}{z^{-p}}\right| - \frac{\delta}{1+\delta} \geq 0$$

Therefore $\left|\dfrac{(f*h)(z)}{z^{-p}}\right| \geq \delta.$

Now if we let

$$g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} b_{p+k} z^{p+k} \in \mathcal{N}_\delta(f)$$

then we have

$$\begin{aligned}\delta - \left|\frac{(g*h)(z)}{z^{-p}}\right| &\leq \left|\frac{((f-g)*h)(z)}{z^{-p}}\right| \\ &= \left|\sum_{k=0}^{\infty}(a_{p+k}-b_{p+k})c_{p+k}z^{2p+k}\right| \\ &\leq \sum_{k=0}^{\infty}|a_{p+k}-b_{p+k}||c_{p+k}||z|^{2p+k} \\ &< \sum_{k=0}^{\infty}\frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}|a_{p+k}-b_{p+k}| \leq \delta\end{aligned}$$

Thus $\dfrac{(g*h)(z)}{z^{-p}} \neq 0$ which implies $g(z) \in \mathcal{H}_p(\alpha,\beta,\gamma)$ and the proof of the theorem is completed.

Theorem 15. *Let $f \in \Sigma_p$ be given by* (1) *and the partial sums $k_0(z)$ and $k_q(z)$ be defined by*

$$k_0(z) = \frac{1}{z^p}$$

and

$$k_q(z) = \frac{1}{z^p} + \sum_{k=0}^{q-1} a_{p+k}z^{p+k} \quad (q>0)$$

also suppose that

$$\sum_{k=0}^{\infty}\theta_{p+k}a_{p+k} \leq 1 \tag{83}$$

where

$$\theta_{p+k} = \frac{(p+k)(p+k+\beta\gamma)\Phi_k(n,\lambda,\sigma,\mu,\omega,p)}{p\beta(1+\gamma)(1-\alpha)}$$

then, for $q>0$, we have

$$\Re\left\{\frac{f(z)}{k_q(z)}\right\} > 1 - \frac{1}{\theta_q} \tag{84}$$

and

$$\Re\left\{\frac{k_q(z)}{f(z)}\right\} > \frac{\theta_q}{1+\theta_q} \tag{85}$$

Proof. Under the hypotheses we can see from (83) that

$$\theta_{p+k+1} > \theta_{p+k} > 1 \quad (k = 0, 1, 2, ...)$$

Therefore by using (83) again we obtain

$$\sum_{k=0}^{q-1} a_{p+k} + \theta_q \sum_{k=q}^{\infty} a_{p+k} \leq \sum_{k=0}^{\infty} \theta_{p+k} a_{p+k} \leq 1 \tag{86}$$

Let

$$w(z) = \theta_q \left[\frac{f(z)}{k_q(z)} - \left(1 - \frac{1}{\theta_q}\right) \right] = 1 + \frac{\theta_q \sum\limits_{k=q}^{\infty} a_{p+k} z^{2p+k}}{1 + \sum\limits_{k=0}^{q-1} a_{p+k} z^{2p+k}} \tag{87}$$

Applying (86) and (87) we find

$$\left| \frac{w(z)-1}{w(z)+1} \right| = \left| \frac{\theta_q \sum\limits_{k=q}^{\infty} a_{p+k} z^{2p+k}}{2 + 2\sum\limits_{k=0}^{q-1} a_{p+k} z^{2p+k} + \theta_q \sum\limits_{k=q}^{\infty} a_{p+k} z^{2p+k}} \right| \leq \frac{\theta_q \sum\limits_{k=q}^{\infty} a_{p+k}}{2 - 2\sum\limits_{k=0}^{q-1} a_{p+k} - \theta_q \sum\limits_{k=q}^{\infty} a_{p+k}} \leq 1 \tag{88}$$

which shows that $\Re w(z) > 0$. From (87) we immediately obtain (84).
Similarly letting

$$\varphi(z) = (1 + \theta_q) \left[\frac{k_q(z)}{f(z)} - \frac{\theta_q}{1 + \theta_q} \right]$$

we can prove (85). This completes the proof.

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A. Ebadian
Department of Mathematics, Payame Noor University,
Tehran, Iran
email: *ebadian.ali@gmail.com*

S.Shams
Department of Mathematics, Faculty of Science,
University of Urmia,
Urmia,Iran
email: *sa40shams@yahoo.com*

R. Asadi
Department of Mathematics, Faculty of Science,
University of Urmia,
Urmia,Iran
email: *reza810asady@gmail.com*