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$\mathcal{Z}$-FILTER REGULAR SEQUENCE AND GENERALIZED LOCAL COHOMOLOGY

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Abstract. Let R be a commutative Noetherian ring and $\mathcal{Z}$ be an stable under specialization subset of Spec$R$. Two notions of filter regular sequence and generalized local cohomology module with respect to a subset of Spec$R$ be an stable under specialization introduced, and their properties are studied. Some vanishing and non-vanishing theorems are given for this generalized version of generalized local cohomology module.

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1. Introduction

Throughout this paper, R is a commutative Noetherian ring. The theory of local cohomology module is one of the important and interesting subjects for some commutative and homological algebraists. The notion of generalized local cohomology modules

$$\mathrm{H}^i_{\mathfrak{a}}(M,N) := \varinjlim_n Ext^i_R(M/\mathfrak{a}^n M, N),$$

was introduced by Herzog in his Habilitationsschrift [9]. When M is a finitely generated R-module, then

$$\mathrm{H}^i_{\mathfrak{a}}(M,N) \cong \mathrm{H}^i_{\mathfrak{a}}(\mathbf{R}Hom_R(M,N))$$

for all integers i [13, Theorem 3.4]. Takahashi, Yoshino and Yoshizawa [12] have introduced the notion of local cohomology with respect to pairs of ideals. Yoshino and Yoshizawa [14, Theorem 2.10] have shown that for any abstract local cohomology functor δ from the category of homologically left bounded complexes of R-modules to itself, there is an *stable under specialization* subset $\mathcal{Z}$ of $SpecR$ such that $\delta \cong \mathbf{R}\Gamma_{\mathcal{Z}}$. Thus, all of these generalizations, with respect to an stable under specialization subset $\mathcal{Z}$ of $SpecR$, may be considered as the largest possible generalization.

Throughout this paper, $\mathcal{Z}$ to a subset of $SpecR$ to be stable under specialization. A subset $\mathcal{Z}$ of $SpecR$ is said to be stable under specialization if $V(p) \subseteq \mathcal{Z}$ for all $p \in \mathcal{Z}$. Let M and N be two R-modules. For notations and terminologies not given in this paper, the reader is referred to [3, 4, 12] if, necessary. The notion of the local cohomology modules with respect to a subset of $SpecR$ to be stable under specialization is introduced in [11] and for complexes see [10]. To be more precise, for any R-module M, set

$$\Gamma_{\mathcal{Z}}(M) := \{x \in M | \mathrm{Supp}_R Rx \subseteq \mathcal{Z}\}.$$

The right derived functor of the functor $\Gamma_{\mathcal{Z}}(-)$ in $C(R)$, $R\Gamma_{\mathcal{Z}}(M)$, exists and is denoted by $R\Gamma_{\mathcal{Z}}(M) := \Gamma_{\mathcal{Z}}(I)$, where I is any injective resolution of M. Also, for any integer i, the i-th local cohomology module of M with respect to $\mathcal{Z}$ is denoted by

$$\mathrm{H}^i_{\mathcal{Z}}(M) := \mathrm{H}_{-i}(R\Gamma_{\mathcal{Z}}(M)).$$

To comply with the usual notation, for $\mathcal{Z} := V(\mathfrak{a})$, we denote $R\Gamma_{\mathcal{Z}}(-)$ and $\mathrm{H}^i_{\mathcal{Z}}(M)$, by $R\Gamma_{\mathfrak{a}}(M)$ and $\mathrm{H}^i_{\mathfrak{a}}(M)$, respectively. Denote the set of all ideals $\mathfrak{b}$ of R such that $V(b) \subseteq \mathcal{Z}$ by $F(\mathcal{Z})$. Since for any R-module M, $\Gamma_{\mathcal{Z}}(M) = \bigcup_{\mathfrak{b} \in F(\mathcal{Z})} \Gamma_{\mathfrak{b}}(M)$, for any integer i, one can easily check that

$$\mathrm{H}^i_{\mathcal{Z}}(M) := \varinjlim_{\mathfrak{b} \in F(\mathcal{Z})} \mathrm{H}^i_{\mathfrak{b}}(M).$$

In this paper, we introduce a generalization of the notion of generalized local cohomology module, which we say a generalized local cohomology module with respect to a subset of $SpecR$ to be stable under specialization. For each integer $i \geq 0$, we define the functor $\mathrm{H}_{\mathcal{Z}}(-,-) : C(R) \to C(R)$ by

$$\mathrm{H}^i_{\mathcal{Z}}(M, N) := \varinjlim_{\mathfrak{a} \in F(\mathcal{Z})} Ext^i_R(\frac{M}{\mathfrak{a}M}, N),$$

for all M and $N \in C(R)$ (where $C(R)$ denotes the category of R-modules and R-homomorphisms). Then $\mathrm{H}_{\mathcal{Z}}(-,-)$ is an additive, R-linear functor which is contravariant in the first variable and covariant in the second variable. This functor do indeed generalize all the functors described in [6, 9, 12]. One of our main goals is to give criteria for the vanishing and non-vanishing of $\mathrm{H}^i_{\mathcal{Z}}(M, N)$, by using $\mathcal{Z}$-$grade_N M$. The organization of this paper is as follows.

We introduce the notion of filter regular sequence with respect to a subset of $SpecR$ to be stable under specialization. Some their characterizations are presented in Section 2. In Section 3, a generalization of generalized local cohomology modules is defined and their basic properties are studied. In the final section we discuss the vanishing and non-vanishing of generalized local cohomology.

2. Preliminaries and Definitions

We start with the following new definition that is a generalization of $\mathfrak{a}$-filter regular M-sequence, where $\mathfrak{a}$ is an ideal of R and M is a finitely generated R-module.

Definition 1. *Let $x_1, x_2, \ldots, x_r$ be a sequence of R and M be a finitely generated R-module. We say that $x_1, x_2, \ldots, x_r$ is an $\mathcal{Z}$-filter regular M-sequence, if $Supp(\dfrac{(x_1, x_2, \ldots, x_{i-1})M :_M x_i}{(x_1, x_2, \cdots, x_{i-1})M}) \subseteq \mathcal{Z}$, for all $i = 1, 2, \ldots, r$.*
In addition, if $x_1, x_2, \ldots, x_r$ belong to $\mathfrak{b}$, then we say that $x_1, x_2, \ldots, x_r$ is an $\mathcal{Z}$-filter regular M-sequence in $\mathfrak{b}$.

Note that as a special case, if $\mathcal{Z} = \emptyset$, then $x_1, x_2, \ldots, x_r$ is an $\mathcal{Z}$-filter regular M-sequence if and only if it is a weak M-sequence in the sense of [9, Definition 1.1.1], if $\mathcal{Z} = V(\mathfrak{a})$ then $x_1, x_2, \ldots, x_r$ is called an $\mathfrak{a}$-filter regular M-sequence in sense of [7, Definition 2.1], and $\mathcal{Z} = W(I, J)$ then $x_1, x_2, \ldots, x_r$ is called an (I, J)-filter regular M-sequence in sense of [5, Definition 2.1].

For a system of elements $\underline{x} = \{x_1, x_2, \cdots, x_r\}$ of R and an integer $0 \leq i \leq r$, let $\underline{x_i} = \{x_1, x_2, \cdots, x_i\}$. Note that $\underline{x}_0$ is the empty set. The following Proposition gives an equivalent condition for the existence of $\mathcal{Z}$-filter regular M-sequence.

Proposition 1. *Let M be an finitely generated module over a local ring R with maximal ideal $\mathfrak{m}$. Then the following condition are equivalent:*

(i) $x_1, x_2, \ldots, x_r$ *is an $\mathcal{Z}$-filter regular M-sequence.*

(ii) $x_i \notin \bigcup p_{p \in Ass \frac{M}{\underline{x}_{i-1}M} \setminus \mathcal{Z}}$ *for all $i = 1, 2, \cdots, r$.*

(iii) $\frac{x_1}{1}, \frac{x_2}{1}, \ldots, \frac{x_r}{1}$ *is a poor M_p-sequence for all $p \in \mathrm{Supp}\ M \setminus \mathcal{Z}$.*

(iv) for all $i = 1, 2, \ldots, r$; $x_1, x_2, \ldots, x_i$ is $\mathcal{Z}$-filter regular M-sequence and $x_{i+1}, x_{i+2}, \ldots, x_r$ is $\mathcal{Z}$-filter regular $\frac{M}{\underline{x}_{i-1}M}$-sequence.

Proof. (ii)$\Rightarrow$ (i) Suppose the contrary and let $1 \leq i \leq r$ be such that $\mathrm{Supp}(\frac{\underline{x}_{i-1}M :_M x_i}{\underline{x}_{i-1}M}) \nsubseteq \mathcal{Z}$. Then there is $q \in \mathrm{Supp}(\frac{\underline{x}_{i-1}M :_M x_i}{\underline{x}_{i-1}M}) \setminus \mathcal{Z}$. Thus there exist $p \subseteq q$, which $p \in Ass((\frac{\underline{x}_{i-1}M :_M x_i}{\underline{x}_{i-1}M})$, then there is $y \in \underline{x}_{i-1}M :_M x_i$ such that $p = 0 : \underline{x}_{i-1}M + y$, therefore $x_i \in p \subseteq \bigcup q_{q \in Ass(\frac{M}{\underline{x}_{i-1}M})} \setminus \mathcal{Z}$. This is a contradiction and the proof is complete.

(i) $\Rightarrow$ (ii) Suppose that contrary. Let $1 \leq i \leq r$ be such that $x_i \in \bigcup p_{p \in Ass \frac{M}{\underline{x}_{i-1}M} \setminus \mathcal{Z}}$. Then there is $x_i \in p$ for some $p \in \bigcup Ass \frac{M}{\underline{x}_{i-1}M} \setminus \mathcal{Z}$, thus $p = (0 : \underline{x}_{i-1}M + y)$

for some $y \in M$, so $p \in Ass\frac{\underline{x}_{i-1}M:_M x_i}{\underline{x}_{i-1}M} \setminus \mathcal{Z}$. Thus is a contradiction. Therefor $x_i \notin \bigcup p_{p\in Ass\frac{M}{\underline{x}_{i-1}M}} \setminus \mathcal{Z}$ for all $i = 1, 2, \ldots, r$ and proof is complete.

(iii)⇒ (i) Let Supp$(\frac{\underline{x}_{i-1}M:_M x_i}{\underline{x}_{i-1}M}) \nsubseteq \mathcal{Z}$, then there is $p \in$ Supp$(\frac{\underline{x}_{i-1}M:_M x_i}{\underline{x}_{i-1}M}) \setminus \mathcal{Z}$, hence $p \in$ Supp $M \setminus \mathcal{Z}$, it follows from (iii) that $(\frac{x_1}{1}, \cdots, \frac{x_{i-1}}{1})M_p = (\frac{x_1}{1}, \cdots, \frac{x_{i-1}}{1})M_p :_{M_p} \frac{x_i}{1}$. Thus $p \notin$ Supp$(\frac{\underline{x}_{i-1}M:_M x_i}{\underline{x}_{i-1}M})$, which is a contradiction.

The equivalence of (iii) and (iv), and (i) ⇒ (iii) are clear.

The following Theorem characterizes the existence of a $\mathcal{Z}$-filter regular M-sequence of length n in $\mathfrak{b}$.

Theorem 1. *Let $n \in \mathbb{N}$. Then the following statements are equivalent.*

(i) $\mathfrak{b}$ *contains a $\mathcal{Z}$-filter regular M-sequence of length n.*

(ii) Any $\mathcal{Z}$-filter regular M-sequence in $\mathfrak{b}$ of length less than n can be extended to a $\mathcal{Z}$-filter regular M-sequence of length n in $\mathfrak{b}$.

(iii) Supp $\mathrm{Ext}^i_R(\frac{R}{\mathfrak{b}}, M) \subseteq \mathcal{Z}$ *for all $i < n$.*

(iv) If Supp $N \subseteq V(\mathfrak{b})$, *then* Supp $\mathrm{Ext}^i_R(N, M) \subseteq \mathcal{Z}$ *for all $i < n$.*

(v) Supp $H^i_{\mathfrak{b}}(M) \subseteq \mathcal{Z}$ *for all $i < n$.*

(vi) If Ann $N \subseteq \mathfrak{b}$, then Supp $H^i_{\mathfrak{b}}(N, M) \subseteq \mathcal{Z}$ *for all $i < n$.*

Proof. The proof is similar to proof of Theorem 2.2 of [7].

Remark 1. *(i) We denote the set of all ideal $\mathfrak{a}$ of R such that $V(\mathfrak{a}) \subseteq \mathcal{Z}$ by $F(\mathcal{Z})$, if for every $\mathfrak{a} \in F(\mathcal{Z})$, Supp$\frac{M}{\mathfrak{a}M} \nsubseteq \mathcal{Z}$, Proposition 2.2 and Theorem 2.3 case (iii) $\Longrightarrow$ (ii) imply that every two maximal $\mathcal{Z}$-filter regular M-sequence in $\mathfrak{a}$ have the same length. We denote the length of a maximal $\mathcal{Z}$-filter regular M-sequence in $\mathfrak{a}$ by $g(\mathfrak{a}, M)$.*

(ii) We define $\mathfrak{a} \leq \mathfrak{b}$ if $\mathfrak{a} \supseteq \mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in F(\mathcal{Z})$. $F(\mathcal{Z})$ is non-empty we shall apply Zorn's Lemma to this partially ordered set, and so it follows from Zorn's Lemma that $F(\mathcal{Z})$ has at least one maximal element.

Definition 2. *We denoted $\mathcal{Z}$-filter regular M-sequence by $fg(\mathcal{Z}, M)$, and define*

$$\begin{aligned} fg(\mathcal{Z}, M) &= \inf\{fg(\mathfrak{a}, M) \mid \mathfrak{a} \in F(\mathcal{Z})\} \\ &= \inf\{fg(\mathfrak{a}, M) \mid \mathfrak{a} \textit{ is maximal element of direct set } F(\mathcal{Z})\}. \end{aligned}$$

As an important special case of the previous remark we have, if $\mathrm{Supp}(\frac{\underline{x}_{i-1}M:x_i}{\underline{x}_{i-1}M}) = \emptyset$, then $x_1, x_2, \ldots, x_i$ is poor $\mathcal{Z}$-regular M-sequence and if, in addition, $\underline{x}_r M \neq M$, we call $x_1, x_2, \ldots, x_r$ a $\mathcal{Z}$-regular M-sequence.

Remark 2. *Let R be a Notherian ring, M a finitely generated R-module, and $\mathfrak{a}$ an ideal of R such that $\mathfrak{a}M \neq M$. Then all maximal M-regular sequence in a have the same length and the common Length of the maximal M-regular sequence in $\mathfrak{a}$ called the grade of $\mathfrak{a}$ on M, denoted by $grade(\mathfrak{a}, M)$, see more details [9].*

Definition 3. *Suppose that M finitely generated R-module and $\mathcal{Z}$ be a subset of SpecR to be stable under specialization. We define the grade of $\mathcal{Z}$ on M, denoted by $grade(\mathcal{Z}, M)$, as $grade(\mathcal{Z}, M) = \inf\{grade(\mathfrak{a}, M) | \mathfrak{a} \in F(\mathcal{Z})\}$*
$= \inf\{grade(\mathfrak{a}, M) |\ \mathfrak{a}$ is maximal element of direct set $F(\mathcal{Z})\}$.

3. Generalized Local Cohomology Modules Defined By $\mathcal{Z}$

In the section, we investigate the basic properties of generalized local cohomology modules defined by a subset of Spec$R$ to be stable under specialization.

Let M and N be finitely generated R-module over a local ring $(R, \mathfrak{m})$ and let $\mathcal{Z}$ to a subset of Spec$R$ to be stable under specialization. For each integer $i \geq 0$, we define the

$$H^i_{\mathcal{Z}}(M, N) := \varinjlim_{\mathfrak{a} \in F(\mathcal{Z})} Ext^i_R(\frac{M}{\mathfrak{a}M}, N)$$

for all $M,\ N \in \mathcal{C}(R)$. Then $H^i_{\mathcal{Z}}(-,-)$ is an additive, R-linear functor which is contravariant in the first variable and covariant in the second variable.

Theorem 2. *Let M be a fixed R-module. Then for each $i \geq 0$, the functors $\varinjlim_{\mathfrak{a} \in F(\mathcal{Z})} Ext^i_R(\frac{M}{\mathfrak{a}M}, -)$ and $\varinjlim_{\mathfrak{a} \in F(\mathcal{Z})} H^i_{\mathfrak{a}}(M, -)$ (from $\mathcal{C}(R) \longrightarrow \mathcal{C}(R)$) are naturally equivalent.*

Proof. We explain the construction of the functor $\varinjlim_{\mathfrak{a} \in F(\mathcal{Z})} \mathrm{H}^i_{\mathfrak{a}}(M, -)$, let $\mathfrak{a}, \mathfrak{b} \in F(\mathcal{Z})$ with $\mathfrak{a} \leq \mathfrak{b}$ ($\mathfrak{a} \supseteq \mathfrak{b}$). Thus the natural homomorphism $\frac{M}{\mathfrak{b}^n M} \longrightarrow \frac{M}{\mathfrak{a}^n M}$ induces the homomorphism $Ext^i_R(\frac{M}{\mathfrak{a}^n M}, N) \longrightarrow Ext^i_R(\frac{M}{\mathfrak{b}^n M}, N)$, for any integear $i \geq 0$ and any R-module N. Also if $n \leq m$, then the diagram

$$\begin{array}{ccc} Ext^i_R(\frac{M}{\mathfrak{a}^n M}, N) & \longrightarrow & Ext^i_R(\frac{M}{\mathfrak{b}^n M}, N) \\ \downarrow & & \downarrow \\ Ext^i_R(\frac{M}{\mathfrak{a}^m M}, N) & \longrightarrow & Ext^i_R(\frac{M}{\mathfrak{b}^m M}, N) \end{array}$$

commutes. Thus we have a homomorphism

$$\lambda_{\mathfrak{a}}^{\mathfrak{b}}: \varinjlim_{\mathfrak{a}\in F(\mathcal{Z})} Ext_R^i(\frac{M}{\mathfrak{a}^n M}, N) \longrightarrow \varinjlim_{\mathfrak{a}\in F(\mathcal{Z})} Ext_R^i(\frac{M}{\mathfrak{b}^n M}, N)$$

that is $\lambda_{\mathfrak{a}}^{\mathfrak{b}}: \mathrm{H}_{\mathfrak{a}}^i(M,N) \longrightarrow \mathrm{H}_{\mathfrak{b}}^i(M,N)$. So that, these homomorphisms together with the modules $\mathrm{H}_{\mathfrak{a}}^i(M,N)$ from the direct system of R-modules and R-homomorphisms over the directed set $F(\mathcal{Z})$.

Since $\varinjlim_{\mathfrak{a}\in F(\mathcal{Z})} \mathrm{H}_{\mathfrak{a}}^i(M,-)$ and $\varinjlim_{\mathfrak{a}\in F(\mathcal{Z})} Hom_R^i(\frac{M}{\mathfrak{a}M},-)$ are naturally equivalent functors (from $\mathcal{C}(R) \longrightarrow \mathcal{C}(R)$) and the sequences, $(\varinjlim_{\mathfrak{a}\in F(\mathcal{Z})} \mathrm{H}_{\mathfrak{a}}^i(M,-))_{i\in\mathbb{N}}$ and $(\varinjlim_{\mathfrak{a}\in F(\mathcal{Z})} Ext_R^i(\frac{M}{\mathfrak{a}M},-))_{i\in\mathbb{N}}$, are negative strongly connected sequences of functors, these two sequences are isomorphic. In particular

$$\varinjlim_{\mathfrak{a}\in F(\mathcal{Z})} \mathrm{H}_{\mathfrak{a}}^i(M,N) \cong \varinjlim_{\mathfrak{a}\in F(\mathcal{Z})} Ext_R^i(\frac{M}{\mathfrak{a}M}, N) = \mathrm{H}_{\mathcal{Z}}(M,N),$$

for any integer $i \geq 0$ and any R-module N.

In this part, we investigate some basic properties of generalized local cohomology modules defined by a subset of stable under specialization of $SpecR$.

Remark 3. *(i) For an R-module M, we denote by $\Gamma_{\mathcal{Z}}(M)$ the set of elements x of M such that $SuppRx \subseteq \mathcal{Z}$.*

(ii) We say that M is $\mathcal{Z}$-torsion (res. $\mathcal{Z}$-torsion free) precisely when $\Gamma_{\mathcal{Z}}(M) = M$ (res. $\Gamma_{\mathcal{Z}}(M) = 0$). It is clear that if $M = R$, then $H_{\mathcal{Z}}^i(M,N)$ is converted to $H_{\mathcal{Z}}^i(N)$. In addition, if $\mathcal{Z} = V(\mathfrak{a})$ then $H_{\mathcal{Z}}^i(N)$, coincides with $H_{\mathfrak{a}}^i(N)$.

Lemma 3. *let M and N are finitely generated R-modules. Then*

(i) $SuppN \subseteq \mathcal{Z}$ if and only if $\Gamma_{\mathcal{Z}}(N) = N$.

(ii) $H_{\mathcal{Z}}^0(M,N) = Hom(M, \Gamma_{\mathcal{Z}}(N))$.

(iii) if $SuppM \cap SuppN \subseteq \mathcal{Z}$, then $H_{\mathcal{Z}}^i(M,N) = Ext_R^i(M,N)$, for all $i \geq 0$.

Proof. (i) It is clear. (ii)

$$\begin{aligned}
\mathrm{H}_{\mathcal{Z}}^0(M,N) &= \varinjlim_{\mathfrak{a}\in F(\mathcal{Z})} \mathrm{H}_{\mathfrak{a}}^0(M,N) \\
&\cong \varinjlim_{\mathfrak{a}\in F(\mathcal{Z})} Hom(M, \Gamma_{\mathfrak{a}}(N)) \\
&= Hom(M, \varinjlim_{\mathfrak{a}\in F(\mathcal{Z})} \Gamma_{\mathfrak{a}}(N)) = Hom(M, \Gamma_{\mathcal{Z}}(N)).
\end{aligned}$$

(iii) There is a minimal injective resolution E^* of N, such that Supp $E^i \subseteq$ Supp N for all $i \geq 0$. Since

$$\text{Supp } (Hom(M, E^i)) \subseteq \text{Supp } M \cap \text{Supp } N \subseteq \mathcal{Z},$$

$Hom(M, E^i)$ is $\mathcal{Z}$-torsion. Therefore, for all $i \geq 0$,

$$\begin{aligned} \mathrm{H}^i_{\mathcal{Z}}(M,N) &\cong \varinjlim_{\mathfrak{a}\in F(\mathcal{Z})} \mathrm{H}^i_{\mathfrak{a}}(M,N) \cong \varinjlim_{\mathfrak{a}\in F(\mathcal{Z})} \mathrm{H}^i(\Gamma_{\mathfrak{a}}(Hom(M,E^*)) \\ &\cong \mathrm{H}^i(\varinjlim_{\mathfrak{a}\in F(\mathcal{Z})} \Gamma_{\mathfrak{a}}(Hom(M,E^*)) \cong \mathrm{H}^i(\Gamma_{\mathcal{Z}}(Hom(M,E^*))) \\ &\cong \mathrm{H}^i(Hom(M,E^*) \cong Ext^i_R(M,N). \end{aligned}$$

It is obvious that if $\mathcal{Z} = V(\mathfrak{a})$, then $\mathrm{H}^i_{\mathcal{Z}}(M,N)$ coincide with $\mathrm{H}^i_a(M,N)$ the generalized local cohomology module was introduced by Herzog [9].

4. Vanishing and Non-Vanishing of $\mathrm{H}^i_{\mathcal{Z}}(M,N)$

Lemma 4. *Suppose that subset $\mathcal{Z}$ be stable under specialization of SpecR, $M(\neq 0)$ finitely generated R-module of finite projective dimension, and N an R-module of finite kurll dimension. Then $H^i_{\mathcal{Z}}(M,N) = 0$ for all $i > pd(M) + \dim N$.*

Proof. Suppose that $\mathfrak{a} \in F(\mathcal{Z})$. Then in view of [1], $\mathrm{H}^i_{\mathfrak{a}}(M,N) = 0$ for all $i > pd(M) + \dim N$. By the Theorem (3.1), we have

$$\mathrm{H}^i_{\mathcal{Z}}(M,N) = \varinjlim_{\mathfrak{a}\in F(\mathcal{Z})} \mathrm{H}^i_{\mathfrak{a}}(M,N),$$

so that $\mathrm{H}^i_{\mathcal{Z}}(M,N) = 0$ for all $i > pd(M) + \dim N$.

Remark 4. *Suppose that M and N are finitely generated R-modules and that $(o : M)N \neq N(M \otimes N \neq 0)$. Recall that the N-grade of M, $grade_N M$, is the length of any maximal N-sequence contained in $(0 : M)$. Then $grade_N M$ is equal to the least integer s such that $Ext^s_R(M,N) \neq 0$.*

For any ideal $\mathfrak{a}$ of R for which $\mathfrak{a}N \neq N$, we define the grade of $\mathfrak{a}$ on N as $grad_N \frac{R}{\mathfrak{a}}$ (Remark 2.5).

Definition 4. *Let M and N be finitely generated R-modules and the subset $\mathcal{Z}$ of SpecR be stable under specialization. We define N-grade of M with respect to $\mathcal{Z}$, denoted by $\mathcal{Z}$-$grade_N M$, as*

$$\begin{aligned} \mathcal{Z} - grade_N M &= \inf\{grade_N \frac{M}{\mathfrak{a}M} | \mathfrak{a} \in F(\mathcal{Z})\} \\ &= \inf\{grade_N \frac{M}{\mathfrak{a}M} | \mathfrak{a} \textit{ is maximal element of directed set } F(\mathcal{Z})\}. \end{aligned}$$

Note: If every $\mathfrak{a} \in F(\mathcal{Z})$, $\frac{M}{\mathfrak{a}M} = 0$, then $\mathcal{Z}$-$grade_N M = \infty$, other wise we have $\mathcal{Z}$-$grade_N M < \infty$.

Theorem 5. *Let M and N finitely generated R-modules and subset of $\mathcal{Z}$ of $SpecR$ be stable under specialization. If $\mathcal{Z}$-$grade_N M = r < \infty$, then $H^i_{\mathcal{Z}}(M,N) = 0$ for all $i < r$, and $H^r_{\mathcal{Z}}(M,N) \neq 0$.*

Proof. By Theorem 2, $\mathrm{H}^i_{\mathcal{Z}}(M,N) = \varinjlim_{\mathfrak{a}\in F(\mathcal{Z})} Ext^i_R(\frac{M}{\mathfrak{a}M}, N)$, for all i. Let $i < grade_N \frac{M}{\mathfrak{a}M}$ for all $\mathfrak{a} \in F(\mathcal{Z})$, this implies that $\mathrm{H}^i_{\mathcal{Z}}(M,N) = 0$. Next there is an ideal $\mathfrak{b}$ of $F(\mathcal{Z})$ for which $grade_N \frac{M}{\mathfrak{b}M} = r$. Let $\mathfrak{a} \in F(\mathcal{Z})$ such that $\mathfrak{b} \geq \mathfrak{a}$ ($\mathfrak{a} \subseteq \mathfrak{b}$). Since $grade_N \frac{M}{\mathfrak{a}M} \geq r$, there is an N-sequence $x_1, x_2, \ldots, x_r$ which is contained in $Ann(\frac{M}{\mathfrak{a}M})$. Consider the natural epimorphism $\mu : \frac{M}{\mathfrak{a}M} \longrightarrow \frac{M}{\mathfrak{b}M}$. Let $A = \ker \mu$, so that the sequence

$$0 \longrightarrow A \longrightarrow \frac{M}{\mathfrak{a}M} \longrightarrow \frac{M}{\mathfrak{b}M} \longrightarrow 0,$$

is exact. Thus induces the long exact sequence

$$\cdots \longrightarrow Ext^{r-1}_R(A,N) \longrightarrow Ext^r_R(\frac{M}{\mathfrak{b}M}, N) \longrightarrow Ext^r_R(\frac{M}{\mathfrak{a}M}, N) \longrightarrow .$$

We know that $(0 : \frac{M}{\mathfrak{a}M}) \subseteq (0 : A)$, so that $x_1, x_2, \ldots, x_r$ is an N-sequence contained in $(0 : A)$. thus $Ext^{r-1}_R(A,N) = 0$. Therefore for every $\mathfrak{a} \in F(\mathcal{Z})$ with $\mathfrak{b} \geq \mathfrak{a}$, the map $Ext^r_R(\frac{M}{\mathfrak{b}M}, N) \longrightarrow Ext^r_R(\frac{M}{\mathfrak{a}M}, N)$ is monomorphism. Since $Ext^r_R(\frac{M}{\mathfrak{b}M}, N) \neq 0$, it follows that $\varinjlim_{\mathfrak{a}\in F(\mathcal{Z})} Ext^i_R(\frac{M}{\mathfrak{a}M}, N) \neq 0$, so that the proof is completed.

Corollary 6. *Suppose that N is finitely generated R-module and subset of $\mathcal{Z}$ of $SpecR$ is stable under specialization. Then*

$$\inf\{i | H^i_{\mathcal{Z}}(N) \neq 0\} = \inf\{depth N_p | p \in \mathcal{Z}\}.$$

Proof. By Theorem 5, $\inf\{i|\mathrm{H}^i_{\mathcal{Z}}(N) \neq 0\} = grade(\mathcal{Z}, N)$. It is clear from the definition that $grade(\mathcal{Z}, N) \leq grade(p, N)$ for all $p \in \mathcal{Z}$, and it follows from Proposition 1, that $grade(p, N) \leq depth N_p$. Further more, if $grade(\mathcal{Z}, N) = \infty$, then $\mathfrak{a}N = N$, for all $a \in F(\mathcal{Z})$. This shows that $depth M_p = \infty$ for all $p \in \mathcal{Z}$. Now, suppose that $N \neq \mathfrak{a}N$ for some $\mathfrak{a} \in F(\mathcal{Z})$ and choose a maximal $\mathfrak{a}$-filter raqular N-sequence $\underline{x}$ in $\mathfrak{a}$. By Proposition 1, there exists $p \in Ass(\frac{M}{\underline{x}M}) \setminus \mathcal{Z}$, and $\mathfrak{a} \subseteq p$. Since $pR_p \in Ass(\frac{M}{\underline{x}M})_p$, pR_p consists of zero-divisors of $\frac{M_p}{\underline{x}M_p}$. Therefore $\underline{x}$ is a maximal M_p-sequence. Hence, the proof is complete.

As a generalization of the usual local cohomology modules, Takahashi, Yoshino and Yoshizawa [12] introduced the local cohomology modules with respect to a pair of ideals (I, J). Let I and J be two ideals of R. We are concerned with the subset $W(I, J) = \{p \in \mathrm{Spec}R \mid I^n \subseteq p + J$ for an integer $n \geq 1\}$ of $\mathrm{Spec}R$. For an R-module M we denoted $\Gamma_{I,J}(M) = \{x \in M \mid \mathrm{Supp}(Rx) \subseteq W(I, J)\}$. Indeed, for $Z := W(I, J)$ one can deduce that $\mathbf{R}\Gamma_{\mathcal{Z}}(-)$ and $\mathrm{H}^i_{\mathcal{Z}}(-)$ are $\mathbf{R}\Gamma_{I,J}(-)$ and $\mathrm{H}^i_{I,J}(-)$, respectively.

Corollary 7. *Suppose that N is finitely generated R-module and that I and J are ideals of R. Then*

$$\inf\{i|H^i_{I,J}(N) \neq 0\} = \inf\{depth N_p|p \in W(I, J)\}.$$

Thus the result coincides with [12, Theorem4.1].

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