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ON THE EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF SINGULAR NONLINEAR SEMIPOSITIONE SYSTEMS

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Abstract. In this paper we consider the existence of positive solutions for singular nonlinear semipositone system of the form

$$\begin{cases} -div(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u) = |x|^{-(\alpha+1)p+c_1}(a_1 u^{p-1} - f_1(v) - \frac{b_1}{u^{\gamma_1}}), & x \in \Omega, \\ -div(|x|^{-\beta q}|\nabla v|^{q-2}\nabla v) = |x|^{-(\beta+1)q+c_2}(a_2 v^{q-1} - f_2(u) - \frac{b_2}{v^{\gamma_2}}), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with $0 \in \Omega$, $1 < p, q < N$, $0 \leq \alpha < \frac{N-p}{p}$, $0 \leq \beta < \frac{N-q}{q}$, $\gamma_1, \gamma_2 \in (0,1)$, and $a_1, a_2, b_1, b_2, c_1, c_2$ are positive parameters. Here $f_i : [0,\infty) \to \mathbb{R}$ are C^2 functions for $i = 1, 2$. We discuss the existence of positive solution when f_1, f_2 satisfy certain additional conditions. We use the method of sub-super solutions to establish our results.

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1. Introduction

We study the existence of positive solutions to the singular infinite semipositone system

$$\begin{cases} -div(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u) = |x|^{-(\alpha+1)p+c_1}(a_1 u^{p-1} - f_1(v) - \frac{b_1}{u^{\gamma_1}}), & x \in \Omega, \\ -div(|x|^{-\beta q}|\nabla v|^{q-2}\nabla v) = |x|^{-(\beta+1)q+c_2}(a_2 v^{q-1} - f_2(u) - \frac{b_2}{v^{\gamma_2}}), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ with $0 \in \Omega$, $1 < p, q < N$, $0 \leq \alpha < \frac{N-p}{p}$, $0 \leq \beta < \frac{N-q}{q}$, $\gamma_1, \gamma_2 \in (0,1)$, and $a_1, a_2, b_1, b_2, c_1, c_2$ positive parameters. Here $f_i : [0, \infty) \to \mathbb{R}$ are continuous functions for $i = 1, 2$. We make the following assumptions:
(A1) There exist $L > 0$ and $b > 0$ such that $f_i(u) < Lu^b$, for all $u \geq 0$ and $i = 1, 2$.
(A2) There exists a constant $S > 0$ such that $\max\{a_1 u^{p-1} - f_1(v), a_2 v^{q-1} - f_2(u)\} < S$ for all $u, v \geq 0$.
Elliptic problems involving more general operator, such as the degenerate quasilinear elliptic operator given by $-div(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u)$, were motivated by the following Caffarelli, Kohn and Nirenberg's inequality (see [1], [2]). The study of this type of problem is motivated by its various applications, for example, in fluid mechanics, in newtonian fluids, in flow through porous media and in glaciology (see [3], [4]). So, the study of positive solutions of singular elliptic problems has more practical meanings. We refer to [5], [6], [7], [8] for additional results on elliptic problems. Here we focus on further extending the single equation in [9] to the system (1). Our approach is based on the method of sub-super solutions, see [10, 11].

2. Preliminaries and existing result

In this paper, we denote $W_0^{1,p}(\Omega, |x|^{-\alpha p})$, the completion of $C_0^\infty(\Omega)$, with respect to the norm $\|u\| = \left(\int_\Omega |x|^{-\alpha p}|\nabla u|^p dx\right)^{\frac{1}{p}}$. To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} -div(|x|^{-sr}|\nabla \phi|^{r-2}\nabla \phi) = \lambda |x|^{-(s+1)r+t}|\phi|^{r-2}\phi, & x \in \Omega, \\ \phi = 0 & x \in \partial\Omega. \end{cases} \tag{2}$$

For $r = p$, $s = \alpha$ and $t = c_1$, let $\phi_{1,p}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,p}$ of (2) such that $\phi_{1,p}(x) > 0$ in Ω and $\|\phi_{1,p}\|_\infty = 1$ and for $r = q$, $s = \beta$ and $t = c_2$, let $\phi_{1,q}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,q}$ of (2) such that $\phi_{1,q}(x) > 0$ in Ω, and $\|\phi_{1,q}\|_\infty = 1$ (see [12, 13]). It can be shown that $\frac{\partial \phi_{1,r}}{\partial n} < 0$ on $\partial\Omega$ for $r = p, q$. Here n is the outward normal. We will also consider the unique solution $(\zeta_p(x), \zeta_q(x)) \in W_0(\Omega, |x|^{-\alpha p}) \times W_0(\Omega, |x|^{-\beta q})$ for the system

$$\begin{cases} -div(|x|^{-\alpha p}|\nabla \zeta_p|^{p-2}\nabla \zeta_p) = |x|^{-(\alpha+1)p+c_1}, & x \in \Omega, \\ -div(|x|^{-\beta q}|\nabla \zeta_q|^{q-2}\nabla \zeta_q) = |x|^{-(\beta+1)q+c_2}, & x \in \Omega, \\ \zeta_p = \zeta_q = 0, & x \in \partial\Omega, \end{cases}$$

to discuss our existence result. It is well known that $\zeta_r(x) > 0$ in Ω and $\dfrac{\partial \zeta_r(x)}{\partial n} < 0$ on $\partial\Omega$, for $r = p, q$ (see [12]).
A pair of nonnegative functions (ψ_1, ψ_2), (z_1, z_2) are called a sub-solution and super-solution of (1) if they satisfy $(\psi_1, \psi_2) = (0, 0) = (z_1, z_2)$ on $\partial\Omega$ and

$$\begin{aligned}
\int_\Omega |x|^{-\alpha p}|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w dx &\le \int_\Omega |x|^{-(\alpha+1)p+c_1}(a_1\psi_1^{p-1} - f_1(\psi_2) - \frac{b_1}{\psi_1^{\gamma_1}})w dx,\\
\int_\Omega |x|^{-\beta q}|\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla w dx &\le \int_\Omega |x|^{-(\beta+1)q+c_2}(a_2\psi_2^{q-1} - f_2(\psi_1) - \frac{b_2}{\psi_2^{\gamma_2}})w dx,\\
\int_\Omega |x|^{-\alpha p}|\nabla z_1|^{p-2}\nabla z_1\cdot\nabla w dx &\ge \int_\Omega |x|^{-(\alpha+1)p+c_1}(a_1 z_1^{p-1} - f_1(z_2) - \frac{b_1}{z_1^{\gamma_1}})w dx,\\
\int_\Omega |x|^{-\beta q}|\nabla z_2|^{q-2}\nabla z_2\cdot\nabla w dx &\ge \int_\Omega |x|^{-(\beta+1)q+c_2}(a_2 z_2^{q-1} - f_2(z_1) - \frac{b_2}{z_2^{\gamma_2}})w dx.
\end{aligned}$$

for all $w \in W = \{w \in C_0^\infty(\Omega) \mid w \ge 0, x \in \Omega\}$. Then the following result holds:

Lemma 1. *(see [12]). Suppose there exist sub and super- solutions* (ψ_1, ψ_2) *and* (z_1, z_2) *respectively of (1) such that* $(\psi_1, \psi_2) \le (z_1, z_2)$. *Then (1) has a solution* (u, v) *such that* $(u, v) \in [(\psi_1, \psi_2), (z_1, z_2)]$.

Theorem 2. *Assume if* $a_1 > \Big(\dfrac{p}{p-1+\gamma_1}\Big)^{p-1}\lambda_{1,p}$, $a_2 > \Big(\dfrac{q}{q-1+\gamma_2}\Big)^{q-1}\lambda_{1,q}$, *then there exists* $c > 0$ *such that if* $\max\{b_1, b_2\} \le c$, *then the system (1) admits a positive solution.*

Proof. We start with the construction of a positive subsolution for (1). To get a positive subsolution, we can apply an anti-maximum principle (see [14]), from which we know that there exist a $\delta_1 > 0$ and a solution z_λ of

$$\begin{cases} -div(|x|^{-sr}|\nabla z|^{r-2}\nabla z) = |x|^{-(s+1)r+t}(\lambda z^{r-1} - 1), & x \in \Omega,\\ z = 0 & x \in \partial\Omega, \end{cases} \tag{3}$$

for $\lambda \in (\lambda_{1,r}, \lambda_{1,r} + \delta_1)$, for $r = p, q$, $s = \alpha, \beta$ and $t = c_1, c_2$.

Fix $\hat{\lambda}_1 \in \Big(\lambda_{1,p}, \min\Big\{(\dfrac{p-1+\gamma_1}{p})^{p-1}a_1, \lambda_{1,p} + \delta_1\Big\}\Big)$ and
$\hat{\lambda}_2 \in \Big(\lambda_{1,q}, \min\Big\{(\dfrac{q-1+\gamma_2}{q})^{q-1}a_2, \lambda_{1,q} + \delta_1\Big\}\Big)$. Let $\theta_i = \|z_{\hat{\lambda}_i}\|$ for $i = 1, 2$. It is well known that $z_{\hat{\lambda}_1}, z_{\hat{\lambda}_2} > 0$ in Ω and $\dfrac{\partial z_{\hat{\lambda}_1}}{\partial n}, \dfrac{\partial z_{\hat{\lambda}_2}}{\partial n} < 0$ on $\partial\Omega$, where n is the outer unit normal to Ω. Hence there exist positive constants $\epsilon, \delta, \sigma_p, \sigma_q$ such that

$$|x|^{-sr}|\nabla z_{\hat{\lambda}_i}|^r \ge \epsilon, \quad x \in \overline{\Omega_\delta}, \tag{4}$$

$$z_{\hat{\lambda}_i} \geq \sigma_r, \quad x \in \Omega_0 = \Omega \backslash \overline{\Omega_\delta}, \tag{5}$$

with $r = p, q$; $s = \alpha, \beta$; $i = 1, 2$ and $\overline{\Omega_\delta} = \{x \in \Omega \mid d(x, \partial\Omega) \leq \delta\}$. Choose $\eta_1, \eta_2 > 0$ such that $\eta_1 \leq \min |x|^{-(s+1)r+t}$, and $\eta_2 \geq \max |x|^{-(s+1)r+t}$, in $\overline{\Omega}_\delta$, for $r = p, q$, $s = \alpha, \beta$ and $t = c_1, c_2$. We construct a subsolution (ψ_1, ψ_2) of (1) using $z_{\hat{\lambda}_1}, z_{\hat{\lambda}_2}$. Define $(\psi_1, \psi_2) = \Big(M(\frac{p-1+\gamma_1}{p})z_{\hat{\lambda}_1}^{\frac{p}{p-1+\gamma_1}}, M(\frac{q-1+\gamma_2}{q})z_{\hat{\lambda}_2}^{\frac{q}{q-1+\gamma_2}}\Big)$, where

$$M = \min\Bigg\{ \left(\frac{\left(\frac{q}{q-1+\gamma_2}\right)^b \theta_1^{\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}}}{L\theta_2^{\frac{qb}{q-1+\gamma_2}}} \right)^{\frac{1}{b-p+1}}, \left(\frac{\left(\frac{p}{p-1+\gamma_1}\right)^b \theta_2^{\frac{(1-\gamma_2)(q-1)}{q-1+\gamma_2}}}{L\theta_1^{\frac{pb}{p-1+\gamma_1}}} \right)^{\frac{1}{b-q+1}},$$
$$\left(\frac{\left(\frac{p-1}{Lp}\right)\theta_1^{\frac{p(p-1)}{p-1+\gamma_1}}\left[\left(\frac{p-1+\gamma_1}{p}\right)^{p-1}a_1 - \hat{\lambda}_1\right]}{\left(\frac{q-1+\gamma_2}{q}\right)^b \theta_2^{\frac{qb}{q-1+\gamma_2}}} \right)^{\frac{1}{b-p+1}}, \left(\frac{\left(\frac{q-1}{Lq}\right)\theta_2^{\frac{q(q-1)}{q-1+\gamma_2}}\left[\left(\frac{q-1+\gamma_2}{q}\right)^{q-1}a_2 - \hat{\lambda}_2\right]}{\left(\frac{p-1+\gamma_1}{p}\right)^b \theta_1^{\frac{pb}{p-1+\gamma_1}}} \right)^{\frac{1}{b-q+1}} \Bigg\}.$$

Let $w \in W$. Then a calculation shows that

$$\nabla\psi_1 = M z_{\hat{\lambda}_1}^{\frac{1-\gamma_1}{p-1+\gamma_1}} \nabla z_{\hat{\lambda}_1},$$

$$\int_\Omega |x|^{-\alpha p}|\nabla\psi_1|^{p-2}\nabla\psi_1 \cdot \nabla w dx = M^{p-1}\int_\Omega |x|^{-\alpha p} z_{\hat{\lambda}_1}^{\frac{(1-\gamma_1)(p-1)}{p-1+\lambda_1}} |\nabla z_{\hat{\lambda}_1}|^{p-2}\nabla z_{\hat{\lambda}_1}\nabla w dx$$
$$= M^{p-1}\int_\Omega |x|^{-\alpha p}|\nabla z_{\hat{\lambda}_1}|^{p-2}\nabla z_{\hat{\lambda}_1}\left[\nabla\Big(z_{\hat{\lambda}_1}^{\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}} w\Big) - \Big(\nabla z_{\hat{\lambda}_1}^{\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}}\Big) w\right] dx$$

$$= M^{p-1}\int_\Omega \Big[|x|^{-(\alpha+1)p+c_1} z_{\hat{\lambda}_1}^{\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}}(\hat{\lambda}_1 z_{\hat{\lambda}_1}^{p-1} - 1) - |x|^{-\alpha p}\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}\frac{|\nabla z_{\hat{\lambda}_1}|^p}{z_{\hat{\lambda}_1}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}\Big] w dx \tag{6}$$

$$= \int_\Omega \Big[|x|^{-(\alpha+1)p+c_1} M^{p-1}\hat{\lambda}_1 z_{\hat{\lambda}_1}^{\frac{p(p-1)}{p-1+\gamma_1}} - |x|^{-(\alpha+1)p+c_1} M^{p-1} z_{\hat{\lambda}_1}^{\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}}$$
$$- |x|^{-\alpha p} M^{p-1}\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}\frac{|\nabla z_{\hat{\lambda}_1}|^p}{z_{\hat{\lambda}_1}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}\Big] w dx,$$

and

$$\int_\Omega |x|^{-(\alpha+1)p+c_1}\Big[a_1\psi_1^{p-1} - f_1(\psi_2) - \frac{b_1}{\psi_1^{\gamma_1}}\Big] w dx =$$

$$\int_\Omega \Big[|x|^{-(\alpha+1)p+c_1} a_1 M^{p-1}\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1} z_{\hat{\lambda}_1}^{\frac{p(p-1)}{p-1+\gamma_1}} - |x|^{-(\alpha+1)p+c_1} f_1\Big(M(\frac{q-1+\gamma_2}{q}) z_{\hat{\lambda}_2}^{\frac{q}{q-1+\gamma_2}}\Big)$$

$$- |x|^{-(\alpha+1)p+c_1} \frac{b_1}{M^{\gamma_1}\left(\frac{p-1+\gamma_1}{p}\right)^{\gamma_1} z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}\Bigg] w dx. \tag{7}$$

Similarly

$$\begin{aligned}\int_\Omega |x|^{-\beta q}|\nabla\psi_2|^{q-2}\nabla\psi_2\nabla w dx = \int_\Omega \Big[|x|^{-(\beta+1)q+c_2} M^{q-2}\hat{\lambda_2} z_{\hat{\lambda_2}}^{\frac{q(q-1)}{q-1+\gamma_2}} - \\ |x|^{-(\beta+1)q+c_2} M^{q-1} z_{\hat{\lambda_2}}^{\frac{(1-\gamma_2)(q-1)}{q-1+\gamma_2}} - |x|^{-\beta q} M^{q-1}\frac{(1-\gamma_2)(q-1)}{q-1+\gamma_2}\frac{|\nabla z_{\hat{\lambda_2}}|^q}{z_{\hat{\lambda_2}}^{\frac{\gamma_2 q}{q-1+\gamma_2}}}\Big] w dx,\end{aligned} \tag{8}$$

and

$$\begin{aligned}\int_\Omega |x|^{-(\beta+1)q+c_2}\Big[a_2\psi_2^{q-1} - f_2(\psi_1) - \frac{b_2}{\psi_2^{\gamma_2}}\Big] w dx = \int_\Omega \Big[|x|^{-(\beta+1)q+c_2} a_2 M^{q-1}\Big(\frac{q-1+\gamma_2}{q}\Big)^{q-1} z_{\hat{\lambda_2}}^{\frac{q(q-1)}{q-1+\gamma_2}} \\ -|x|^{-(\beta+1)q+c_2} f_2\Big(M\Big(\frac{p-1+\gamma}{p}\Big) z_{\hat{\lambda_1}}^{\frac{p}{p-1+\gamma_1}}\Big) - |x|^{-(\beta+1)q+c_2}\frac{b_2}{M^{\gamma_2}\left(\frac{q-1+\gamma_2}{q}\right)^{\gamma_2} z_{\hat{\lambda_2}}^{\frac{\gamma_2 q}{q-1+\gamma_2}}}\Big] w dx\end{aligned} \tag{9}$$

Let $c = \min\Big\{M^{p-1+\gamma_1}\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}\Big(\frac{p-1+\gamma_1}{p}\Big)^{\gamma_1}\frac{\epsilon}{\eta_2}, M^{q-1+\gamma_2}\frac{(1-\gamma_2)(q-1)}{q-1+\gamma_2}\Big(\frac{q-1+\gamma_2}{q}\Big)^{\gamma_2}\frac{\epsilon}{\eta_2},$ $\frac{M^{p-1+\gamma_1}}{p}\Big(\frac{p-1+\gamma_1}{p}\Big)^{\gamma_1}\sigma_p^p\Big[\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1} a_1 - \hat{\lambda_1}\Big], \frac{M^{q-1+\gamma_2}}{q}\Big(\frac{q-1+\gamma_2}{q}\Big)^{\gamma_2}\sigma_q^q\Big[\Big(\frac{q-1+\gamma_2}{q}\Big)^{q-1} a_2 - \hat{\lambda_2}\Big]\Big\}.$

First we consider the case when $x \in \overline{\Omega}_\delta$. We have $|x|^{-\alpha p}|\nabla\phi_{1,p}| \geq \epsilon$ on $\overline{\Omega}_\delta$. Since $\Big(\dfrac{p}{p-1+\gamma_1}\Big)^{p-1}\hat{\lambda_1} \leq a_1$, we have

$$|x|^{-(\alpha+1)p+c_1} M^{p-1}\hat{\lambda_1} z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}} \leq |x|^{-(\alpha+1)p+c_1} a_1 M^{p-1}\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1} z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}}, \tag{10}$$

and from the choice of M, we know that

$$LM^{b-p+1}\theta_2^{\frac{qb}{q-1+\gamma_2}} \leq \Big(\frac{q}{q-1+\gamma_2}\Big)^b \theta_1^{\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}}. \tag{11}$$

By (11) and (A_1) we have

$$\begin{aligned}-|x|^{-(\alpha+1)p+c_1} M^{p-1} z_{\hat{\lambda_1}}^{\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}} &\leq -|x|^{-(\alpha+1)p+c_1} LM^b\Big(\frac{q-1+\gamma_2}{q}\Big)^b z_{\hat{\lambda_2}}^{\frac{qb}{q-1+\gamma_2}} \\ &\leq -|x|^{-(\alpha+1)p+c_1} f_2\Big(M\Big(\frac{q-1+\gamma_2}{q}\Big) z_{\hat{\lambda_2}}^{\frac{q}{q-1+\gamma_2}}\Big).\end{aligned} \tag{12}$$

Next, from (7) and definition of c, we have

$$|x|^{-\alpha p} M^{p-1}\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}|\nabla z_{\hat{\lambda_1}}|^p \geq |x|^{-(\alpha+1)p+c_1}\frac{b_1}{M^{\gamma_1}\big(\frac{p-1+\gamma_1}{p}\big)^{\gamma_1}},$$

and

$$-|x|^{-\alpha p}M^{p-1}\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}\frac{|\nabla z_{\hat{\lambda_1}}|^p}{z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}} \leq -|x|^{-(\alpha+1)p+c_1}\frac{b_1}{M^{\gamma_1}\left(\frac{p-1+\gamma_1}{p}\right)^{\gamma_1} z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}. \tag{13}$$

Hence by using (10) , (12) and (13) for $b_1 \leq c$, we have

$$\int_{\overline{\Omega}_\delta}|x|^{-\alpha p}|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w dx \leq \int_{\overline{\Omega}_\delta}\Big[|x|^{-(\alpha+1)p+c_1}a_1M^{p-1}\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1}z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}} - |x|^{-(\alpha+1)p+c_1}f_1\Big(M\Big(\frac{q-1+\gamma_2}{q}\Big)z_{\hat{\lambda_2}}^{\frac{q}{q-1+\gamma_2}}\Big) - |x|^{-(\alpha+1)p+c_1}\frac{b_1}{M^{\gamma_1}\left(\frac{p-1+\gamma_1}{p}\right)^{\gamma_1} z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}\Big]wdx$$

$$= \int_{\overline{\Omega}_\delta}|x|^{-(\alpha+1)p+c_1}\Big[a_1\psi_1^{p-1} - f_1(\psi_2) - \frac{b_1}{\psi_1^{\gamma_1}}\Big]wdx. \tag{14}$$

Similarly

$$\int_{\overline{\Omega}_\delta}|x|^{-\beta q}|\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla w dx \leq \int_{\overline{\Omega}_\delta}\Big[|x|^{-(\beta+1)q+c_2}a_2M^{q-1}\Big(\frac{q-1+\gamma_2}{q}\Big)^{q-1}z_{\hat{\lambda_2}}^{\frac{q(q-1)}{q-1+\gamma_2}} - |x|^{-(\beta+1)q+c_2}f_2\Big(M\Big(\frac{p-1+\gamma_1}{p}\Big)z_{\hat{\lambda_1}}^{\frac{p}{p-1+\gamma_1}}\Big) - |x|^{-(\beta+1)q+c_2}\frac{b_2}{M^{\gamma_2}\left(\frac{q-1+\gamma_2}{q}\right)^{\gamma_2} z_{\hat{\lambda_2}}^{\frac{\gamma_2 q}{q-1+\gamma_2}}}\Big]wdx$$

$$= \int_{\overline{\Omega}_\delta}|x|^{-(\beta+1)q+c_2}\Big[a_2\psi_2^{q-1} - f_2(\psi_1) - \frac{b_2}{\psi_2^{\gamma_2}}\Big]wdx. \tag{15}$$

On the other hand, on $\Omega_0 = \Omega\backslash\overline{\Omega}_\delta$, we have $z_{\hat{\lambda_1}} \geq \sigma_p$ and $z_{\hat{\lambda_2}} \geq \sigma_q$, for some $0 < \sigma_p, \sigma_q < 1$, and from the definition of c, for $b_1 \leq c$ we have

$$\frac{b_1}{M^{\gamma_1}\left(\frac{p-1+\gamma_1}{p}\right)^{\gamma_1}} \leq \frac{1}{p}M^{p-1}\sigma_p^p\Big[\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1}a_1 - \hat{\lambda_1}\Big] \leq \frac{1}{p}M^{p-1}z_{\hat{\lambda_1}}^p\Big[\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1}a_1 - \hat{\lambda_1}\Big]. \tag{16}$$

Also from the choice of M, we have

$$LM^{b-p+1}\Big(\frac{q-1+\gamma_2}{q}\Big)^b z_{\hat{\lambda_2}}^{\frac{qb}{q-1+\gamma_2}} \leq z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}}\frac{p-1}{p}\Big[\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1}a_1 - \hat{\lambda_1}\Big]. \tag{17}$$

Hence from (16) and (17) we have

$$\int_{\Omega_0}|x|^{-\alpha p}|\nabla\psi_1|^{p-2}\nabla\psi_1\nabla w dx = \int_{\Omega_0}\Big[|x|^{-(\alpha+1)p+c_1}M^{p-1}\hat{\lambda_1}z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}} - |x|^{-(\alpha+1)p+c_1}M^{p-1}z_{\hat{\lambda_1}}^{\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}} - |x|^{-\alpha p}M^{p-1}\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}\frac{|\nabla z_{\hat{\lambda_1}}|^p}{z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}\Big]wdx$$

$$\leq \int_{\Omega_0}|x|^{-(\alpha+1)p+c_1}M^{p-1}\hat{\lambda_1}z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}}wdx = \int_{\Omega_0}|x|^{-(\alpha+1)p+c_1}\frac{1}{z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}\Big[\frac{1}{p}\hat{\lambda_1}M^{p-1}z_{\hat{\lambda_1}}^p + \frac{p-1}{p}\hat{\lambda_1}M^{p-1}z_{\hat{\lambda_1}}^p\Big]wdx$$

$$\leq \int_{\Omega_0} |x|^{-(\alpha+1)p+c_1} \frac{1}{z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}} \Big[\Big(\frac{1}{p} M^{p-1}\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1} a_1 z_{\hat{\lambda_1}}^{p} - \frac{b_1}{M^{\gamma_1}\big(\frac{p-1+\gamma_1}{p}\big)^{\gamma_1}}\Big)+$$

$$M^{p-1} z_{\hat{\lambda_1}}^{p}\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1}\Big(\frac{(p-1)a_1}{p} - LM^{b-p+1}\Big(\frac{q-1+\gamma_2}{q}\Big)^{b}\Big(\frac{p-1+\gamma_1}{p}\Big)^{1-p}\frac{z_{\hat{\lambda_2}}^{\frac{qb}{q-1+\gamma_2}}}{z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}}}\Big)\Big] wdx$$

$$= \int_{\Omega_0} |x|^{-(\alpha+1)p+c_1}\Big[a_1 M^{p-1}\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1} z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}} - LM^{b}\Big(\frac{q-1+\gamma_2}{q}\Big)^{b} z_{\hat{\lambda_2}}^{\frac{qb}{q-1+\gamma_2}} - \frac{b_1 z_{\hat{\lambda_1}}^{\frac{-\gamma_1 p}{p-1+\gamma_1}}}{M^{\gamma_1}\big(\frac{p-1+\gamma_1}{p}\big)^{\gamma_1}}\Big] wdx$$

$$\leq \int_{\Omega_0} |x|^{-(\alpha+1)p+c_1}\Big[a_1 M^{p-1}\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1} z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}} - f_1\Big(M\Big(\frac{q-1+\gamma_2}{q}\Big) z_{\hat{\lambda_2}}^{\frac{q}{q-1+\gamma_2}}\Big)$$

$$- \frac{b_1}{M^{\gamma_1}\big(\frac{p-1+\gamma_1}{p}\big)^{\gamma_1} z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}\Big] wdx = \int_{\Omega_0} |x|^{-(\alpha+1)p+c_1}\Big[a_1\psi_1^{p-1} - f_1(\psi_2) - \frac{b_1}{\psi_1^{\gamma_1}}\Big] wdx. \qquad (18)$$

Similarly

$$\int_{\Omega_0} |x|^{-\beta q}|\nabla\psi_2|^{q-2}\nabla\psi_2\nabla wdx \leq \int_{\Omega_0} |x|^{-(\beta+1)q+c_2}\Big[a_2\psi_2^{q-1} - f_2(\psi_1) - \frac{b_2}{\psi_2^{\gamma_2}}\Big] wdx. \qquad (19)$$

By using (14), (15), (18) and (19) we see that (ψ_1, ψ_2) is a sub-solution of (1). Next, we construct a super-solution (z_1, z_2) of (1) such that $(z_1, z_2) \geq (\psi_1, \psi_2)$. Let $(z_1, z_2) = [(S^*)^{\frac{1}{p-1}}\zeta_p(x), (S^*)^{\frac{1}{q-1}}\zeta_q(x)]$. By (A_2) and choose a large constant S^*, we shall verify that (z_1, z_2) is a super-solution of (1). To this end, let $w \in W$. Then we have

$$\int_{\Omega} |x|^{-\alpha p}|\nabla z_1|^{p-2}|\nabla z_1|\nabla wdx = S^*\int_{\Omega} |x|^{-(\alpha+1)p+c_1} wdx$$

$$\geq \int_{\Omega} |x|^{-(\alpha+1)p+c_1}\Big[a_1 z_1^{p-1} - f_1(z_2) - \frac{b_1}{z_1^{\gamma_1}}\Big] wdx. \qquad (20)$$

Similarly,

$$\int_{\Omega} |x|^{-\beta q}|\nabla z_2|^{q-2}|\nabla z_2|\nabla wdx \geq \int_{\Omega} |x|^{-(\beta+1)q+c_2}\Big[a_2 z_2^{q-1} - f_2(z_1) - \frac{b_2}{z_2^{\gamma_2}}\Big] wdx. \qquad (21)$$

Thus (z_1, z_2) is a super-solution of (1). Finally, we can choose $S^* \gg 1$ such that $(\psi_1, \psi_2) \geq (z_1, z_2)$ in Ω. Hence, if $\max\{b_1, b_2\} \leq c$, by Lemma 1 there exists a positive solution (u, v) of (1) such that $(\psi_1, \psi_2) \leq (u, v) \leq (z_1, z_2)$. This completes the proof of Theorem 2.

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