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# NOTES ON COMPLEX HYPERBOLIC FUNCTIONS

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Abstract. Noting the derivatives for functions sinh$z$ and cosh$z$, we assume the fractional derivatives for sinh$z$ and cosh$z$. Applying the fractional calculus (fractional integrals and fractional derivatives), we consider generalized expansions for functions sinh$z$ and cosh$z$. Also, the generalized expansion for $f(z) = e^z$ is considered.

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## 1. Introduction

Let $\mathcal{A}(\alpha)$ be the class of functions $f(z)$ of the form

$$f(z) = a_0 z^{\alpha} + a_1 z^{\alpha+1} + a_2 z^{\alpha+2} + \cdots = \sum_{n=0}^{\infty} a_n z^{\alpha+n} \tag{1.1}$$

for $0 \leqq \alpha < 1$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If $\alpha = 0$ in (1.1), then $f(z) \in \mathcal{A}(0)$ becomes

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{n=0}^{\infty} a_n z^n, \tag{1.2}$$

and that

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n. \tag{1.3}$$

This is the orginary Taylor expansion for $f(z) \in \mathcal{A}(0)$. Therefore, for $f(z) \in \mathcal{A}(\alpha)$, we have to consider the generalized Taylor expansion of $f(z)$.

To consider this problem, we have to use the fractional calculus (fractional integrals and fractional derivatives) defined by Owa [1], Owa and Srivastava [2], and Srivastava and Owa [3].

**Definition 1.1** The fractional integral of order $\alpha$ is defined, for an analytic function $f(z)$ in $\mathbb{U}$, by

$$D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(t)}{(z-t)^{1-\alpha}} dt \qquad (\alpha > 0), \tag{1.4}$$

where the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $\log(z-t)$ to be real when $z-t>0$.

**Definition 1.2** The fractional derivative of order $\alpha$ is defined, for an analytic function $f(z)$ in $\mathbb{U}$, by

$$D_z^{\alpha} f(z) = \frac{d}{dz}\left(D_z^{\alpha-1} f(z)\right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \left( \int_0^z \frac{f(t)}{(z-t)^{\alpha}} dt \right), \tag{1.5}$$

where $0 \leqq \alpha < 1$ and the multiplicity of $(z-t)^{-\alpha}$ is removed as Definition 1.1 above.

**Definition 1.3** Under the hypotheses of Definition 1.2, the fractional derivative of order $n+\alpha$ is defined by

$$D_z^{n+\alpha} f(z) = \frac{d^n}{dz^n}\left(D_z^{\alpha} f(z)\right), \tag{1.6}$$

where $0 \leqq \alpha < 1$ and $n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$.

By means of Definition 1.2, we have that

$$\begin{aligned} D_z^{\alpha} z^{\alpha+n} &= \frac{d}{dz}\left(D_z^{\alpha-1} z^{\alpha+n}\right) \\ &= \frac{d}{dz}\left\{ \frac{1}{\Gamma(1-\alpha)} \int_0^z \frac{t^{\alpha+n}}{(z-t)^{\alpha}} dt \right\} \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \left\{ z^{n+1} \int_0^1 \frac{(1-\zeta)^{\alpha+n}}{\zeta^{\alpha}} d\zeta \right\} \qquad (z-t = z\zeta) \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz}\left(z^{n+1} B(1-\alpha, \alpha+n+1)\right) = \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)} z^n, \end{aligned} \tag{1.7}$$

where $B(x,y)$ is the beta function. Therefore, we know, for $f(z) \in \mathcal{A}(\alpha)$, that

$$D_z^{\alpha} f(z) = D_z^{\alpha} \left( \sum_{n=0}^{\infty} a_n z^{\alpha+n} \right) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)} a_n z^n. \tag{1.8}$$

With the above, we know that

$$a_0 = \frac{D_z^\alpha f(0)}{\Gamma(\alpha+1)}. \tag{1.9}$$

Using

$$D_z^{\alpha+1} f(0) = \Gamma(\alpha+2) a_1, \tag{1.10}$$

we can write that

$$a_1 = \frac{D_z^{\alpha+1} f(0)}{\Gamma(\alpha+2)}. \tag{1.11}$$

In general, we see that

$$a_n = \frac{D_z^{\alpha+n} f(0)}{\Gamma(\alpha+n+1)} \tag{1.12}$$

for $0 \leqq \alpha < 1$ and $n \in \mathbb{N}_0$. Consequently, we have that

$$f(z) = \sum_{n=0}^{\infty} \frac{D_z^{\alpha+n} f(0)}{\Gamma(\alpha+n+1)} z^{\alpha+n} \tag{1.13}$$

for $f(z) \in \mathcal{A}(\alpha)$ with $z \neq 0$. Therefore, we use this expansion (1.13) for $f(z) \in \mathcal{A}(\alpha)$.

## 2. Expansions for trigonometric functions

We discuss the expansions for complex hyperbolic funtions sinh$z$ and cosh$z$ for $z \in \mathbb{U}$. Noting that

$$\text{sinh}z = \frac{e^z - e^{-z}}{2} \tag{2.1}$$

and

$$\text{cosh}z = \frac{e^z + e^{-z}}{2}, \tag{2.2}$$

we obtain that

$$\text{sin}iz = \frac{e^{-z} - e^z}{2i} = i\text{sinh}z, \tag{2.3}$$

that is, that

$$\sinh z = \frac{1}{i}\sin(iz). \tag{2.4}$$

Therefore, if $f(z) = \sinh z$, then

$$f'(z) = \cos(iz) = \sin\left(iz + \frac{\pi}{2}\right), \tag{2.5}$$

$$f''(z) = i\cos\left(iz + \frac{\pi}{2}\right) = i\sin(iz + \pi), \tag{2.6}$$

and that

$$f^{(n)}(z) = i^{n-1}\sin\left(iz + \frac{n}{2}\pi\right) \qquad (n \in \mathbb{N}_0). \tag{2.7}$$

With the above, we may assume that

$$D_z^{\alpha} f(z) = i^{\alpha-1}\sin\left(iz + \frac{\alpha}{2}\pi\right) \qquad (0 \leqq \alpha < 1) \tag{2.8}$$

and

$$D_z^{\alpha+n} f(z) = i^{\alpha+n-1}\sin\left(iz + \frac{\alpha+n}{2}\pi\right) \tag{2.9}$$

for $0 \leqq \alpha < 1$ and $n \in \mathbb{N}_0$.

**Remark 2.1** Applying the formula (2.8), we see that

$$f^{(n)}(z) = D_z^{n-\alpha}\left(D_z^{\alpha} f(z)\right) = D_z^{n-\alpha}\left(i^{\alpha-1}\sin\left(iz + \frac{\alpha}{2}\pi\right)\right) = i^{n-1}\sin\left(iz + \frac{n}{2}\pi\right) \tag{2.10}$$

for $0 \leqq \alpha < 1$ and $n \in \mathbb{N}_0$.

Now, we derive

**Theorem 2.1** *If the equation (2.8) is satisfied for $f(z) = \sinh z$, then*

$$\sinh z = \sum_{n=0}^{\infty} \frac{i^{\alpha+n-1}\sin\left(\frac{\alpha+n}{2}\pi\right)}{\Gamma(\alpha+n+1)} z^{\alpha+n} \qquad (z \in \mathbb{U} - \{0\}) \tag{2.11}$$

*where* $0 \leqq \alpha < 1$.

*Proof* By means of (2.8) and (2.9), we know that

$$D_z^{\alpha} f(0) = i^{\alpha-1}\sin\left(\frac{\alpha}{2}\pi\right) \qquad (0 \leqq \alpha < 1) \tag{2.12}$$

and

$$(2.13)\qquad D_z^{\alpha+n} f(0) = i^{\alpha+n-1} \sin\left(\frac{\alpha+n}{2}\pi\right) \qquad (0 \leqq \alpha < 1, n \in \mathbb{N}_0)$$

eachother. Therefore, using (1.13), we complete the proof of the theorem.

Making $\alpha = \frac{1}{2}$ in Theorem 2.1, we have

**Corollary 2.1** *If the equation (2.8) is satisfied for* $f(z) = \mathrm{sinh} z$ *with* $\alpha = \frac{1}{2}$, *then*

$$(2.14)\qquad \mathrm{sinh} z = \sum_{n=0}^{\infty} \frac{i^{n-\frac{1}{2}} \sin\left(\frac{2n+1}{4}\pi\right)}{\Gamma\left(n+\frac{3}{2}\right)} z^{n+\frac{1}{2}}$$

$$= -\frac{\sqrt{2}\sqrt{z}}{\sqrt{\pi}}\left(1 + \frac{2i}{3}z - \frac{2^2}{3\cdot 5}z^2 - \frac{2^3 i}{3\cdot 5\cdot 7}z^3 + \frac{2^4}{3\cdot 5\cdot 7\cdot 9}z^4 + \frac{2^5 i}{3\cdot 5\cdot 7\cdot 9\cdot 11}z^5 - \cdots\right)$$

*for* $z \in \mathbb{U} - \{0\}$.

Further, we consider the function $\mathrm{cosh} z$ for $\mathbb{U}$. Note that

$$(2.15)\qquad f(z) = \mathrm{cosh} z = \frac{e^z + e^{-z}}{2} = \cos(iz).$$

This gives us that

$$(2.16)\qquad f'(z) = -i \sin(iz) = i \cos\left(iz + \frac{\pi}{2}\right),$$

$$(2.17)\qquad f''(z) = -i^2 \sin\left(iz + \frac{\pi}{2}\right) = i^2 \cos(iz + \pi),$$

and that

$$(2.18)\qquad f^n(z) = i^n \cos\left(iz + \frac{n}{2}\pi\right) \qquad (n \in \mathbb{N}_0).$$

From the above, we can assume that

$$(2.19)\qquad D_z^{\alpha} f(z) = i^{\alpha} \cos\left(iz + \frac{\alpha}{2}\pi\right) \qquad (0 \leqq \alpha < 1)$$

and

$$(2.20)\qquad D_z^{\alpha+n} f(z) = i^{\alpha+n} \cos\left(iz + \frac{\alpha+n}{2}\pi\right)$$

for $0 \leqq \alpha < 1$ and $n \in \mathbb{N}_0$.

**Remark 2.2** Using the formula (2.19), we have that

$$f^{(n)}(z) = D_z^{n-\alpha}\left(D_z^{\alpha} f(z)\right) = D_z^{n-\alpha}\left(i^{\alpha}\cos\left(iz + \frac{\alpha}{2}\pi\right)\right) = i^{n}\cos\left(iz + \frac{n}{2}\pi\right) \tag{2.21}$$

for $0 \leqq \alpha < 1$ and $n \in \mathbb{N}_0$.

Now, we have

**Theorem 2.2** *If the equation (2.19) is satisfied for $f(z) = \cosh z$, then*

$$\cosh z = \sum_{n=0}^{\infty} \frac{i^{\alpha+n}\cos\left(\frac{\alpha+n}{2}\pi\right)}{\Gamma(\alpha+n+1)} z^{\alpha+n} \qquad (z \in \mathbb{U} - \{0\}) \tag{2.22}$$

*with $0 \leqq \alpha < 1$.*

*Proof* Note that

$$D_z^{\alpha} f(0) = i^{\alpha}\cos\left(\frac{\alpha}{2}\pi\right) \qquad (0 \leqq \alpha < 1) \tag{2.23}$$

and

$$D_z^{\alpha+n} f(0) = i^{\alpha+n}\cos\left(\frac{\alpha+n}{2}\pi\right) \qquad (0 \leqq \alpha < 1), n \in \mathbb{U} - \{0\}). \tag{2.24}$$

Therefore, the formula (1.13) implies the equality (2.22).

Putting $\alpha = \frac{1}{2}$ in Theorem 2.2, we have

**Corollary 2.2** *If the equation (2.19) is satisfied for $\alpha = \frac{1}{2}$, then*

$$\begin{aligned} \cosh z &= \sum_{n=0}^{\infty} \frac{i^{n+\frac{1}{2}}\cos\left(\frac{2n+1}{4}\pi\right)}{\Gamma\left(n+\frac{3}{2}\right)} z^{n+\frac{1}{2}} \\ &= \frac{\sqrt{2}\sqrt{z}i}{\sqrt{\pi}}\left(1 - \frac{2i}{3}z + \frac{2^2}{3\cdot 5}z^2 - \frac{2^3 i}{3\cdot 5\cdot 7}z^3 \right. \\ &\quad \left. + \frac{2^4}{3\cdot 5\cdot 7\cdot 9}z^4 - \frac{2^5}{3\cdot 5\cdot 7\cdot 9\cdot 11}z^5 - \cdots\right) \end{aligned} \tag{2.25}$$

*for $z \in \mathbb{U} - \{0\}$.*

Finally, we derive

**Theorem 2.3** *If the equations (2.8) for $\sinh z$ and (2.19) for $\cosh z$ are satisfied, then*

$$e^{z} = \sum_{n=0}^{\infty} \frac{i^{\alpha+n}\cos\left(\frac{\alpha+n}{2}\pi\right) - i\sin\left(\frac{\alpha+n}{2}\pi\right)}{\Gamma(\alpha+n+1)} z^{\alpha+n} \qquad (z \in \mathbb{N} - \{0\}) \tag{2.26}$$

*with* $0 \leqq \alpha < 1$ *for* $z \in \mathbb{U} - \{0\}$.

*Proof* Since

$$e^z = \cosh z - \sinh z, \tag{2.27}$$

using Theorem 2.1 and Theorem 2.2, we see (2.26).

If we take $\alpha = \frac{1}{2}$ in Theorem 2.3, we obtain

**Corollary 2.3** *If the equations (2.8) and (2.19) are satisfied for* $\alpha = \frac{1}{2}$, *then*

$$e^z = \frac{\sqrt{2}\sqrt{z}\sqrt{i}}{\sqrt{\pi}} \left\{ (1-i) - \frac{2}{3}(1+i)z - \frac{2^2}{3 \cdot 5}(1-i)z^2 + \frac{2^3}{3 \cdot 5 \cdot 7}(1+i)z^3 \right. \\ \left. + \frac{2^4}{3 \cdot 5 \cdot 7 \cdot 9}(1-i)z^4 - \frac{2^5}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}(1+i)z^5 + \cdots \right) \tag{2.28}$$

*for* $z \in \mathbb{U} - \{0\}$.

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