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BELLWETHERS OF COMPOSITION OPERATORS ACTING BETWEEN WEIGHTED BERGMAN SPACES AND WEIGHTED BANACH SPACES OF HOLOMORPHIC FUNCTIONS

E. Wolf

Abstract. Let $\mathbb{D}$ denote the open unit disk in the complex plane and $H(\mathbb{D})$ the set of all analytic functions on $\mathbb{D}$. Now, let $v : \mathbb{D} \to (0, \infty)$ be a weight, A_v^2 the weighted Bergman space generated by the weight v and H_v^∞ the weighted Banach space of holomorphic functions f on $\mathbb{D}$ such that $\sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty$.
An analytic self-map ϕ of $\mathbb{D}$ induces the linear composition operator $C_\phi : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto f \circ \phi$. We investigate under which conditions on the symbol ϕ and the weight v the operator $C_\phi : A_v^2 \to H_v^\infty$ is a bellwether for boundedness of composition operators where C_ϕ being a bellwether means that C_ϕ acts boundedly if and only if all composition operators act boundedly.

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1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane and $H(\mathbb{D})$ be the collection of all analytic functions on $\mathbb{D}$. An analytic self-map ϕ of $\mathbb{D}$ induces through composition a linear composition operator

$$C_\phi : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto f \circ \phi.$$

For many reasons these operators play an important role. One of them is that in the classical setting of the Hardy space H^2 they link operator theory with complex analysis. Moreover they provide a large class of operators which gives many examples on basic operator theoretical questions.
For more information on composition operators we refer the reader to the excellent monographs of Cowen and MacCluer [7] and of Shapiro [9].
Now, let $v : \mathbb{D} \to (0, \infty)$ be a continuous and bounded function. Such a map is

called a *weight*. We will study composition operators that act between the *weighted Bergman space*

$$A_v^2 := \left\{ f \in H(\mathbb{D}),\ \|f\|_{v,2} := \left(\int_{\mathbb{D}} |f(z)|^2 v(z)\, dA(z) \right)^{\frac{1}{2}} < \infty \right\}$$

where $dA(z)$ denotes the normalized area measure and the *weighted Banach spaces of holomorphic functions* given by

$$H_v^\infty := \left\{ f \in H(\mathbb{D});\ \|f\|_v = \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty \right\}.$$

Endowed with the weighted sup-norm $\|.\|_v$ these are Banach spaces. They arise naturally in a number of problems related to functional analysis, Fourier analysis, partial differential equations and convolution equations. Later, they became of interest themselves, see [3] and the references therein for further and deeper information. In [6] Bourdon studied composition operators acting on weighted Banach spaces generated by the weights that satisfy the following conditions.
We say that v is *radial*, if $v(z) = v(|z|)$ for every $z \in \mathbb{D}$. A radial weight is called *typical*, if v is radial, non-increasing w.r.t. $|z|$ and satisfies $\lim_{|z| \to 1} v(z) = 0$.
Taking Bourdon's notation we say that C_ϕ is a *bellwether for boundedness of composition operators* if for each typical weight v the boundedness of $C_\phi : A_v^2 \to H_v^\infty$ ensures the boundedness of all composition operators acting between A_v^2 and H_v^∞.
Bourdon analyzed the relation between ϕ having an angular derivative less than 1 and inducing a bellwether for boundedness of composition operators on typical-growth spaces.
More precisely, if ϕ has an angular derivative less than 1, then C_ϕ is a bellwether. Conversely, if all angular derivatives of ϕ exceed 1, C_ϕ is not a bellwether. It is still an open question, if C_ϕ being a bellwether implies that the inducing symbol ϕ must have an angular derivative less than 1 at some point $\xi \in \partial\mathbb{D}$. In [11] we generalized Bourdon's results to the setting of the unit polydisk using two different settings. In this article we remain in the one-dimensional setting but consider operators acting between A_v^2 and H_v^∞. We give a characterization for C_ϕ to be a bellwether and compare our results to those obtained by Bourdon in [6].

2. Background and basics

2.1. Theory of weights

In this part of the article we give some background information on the involved weights. A very important role play the so-called *radial* weights, i.e. weights which satisfy $v(z) = v(|z|)$ for every $z \in \mathbb{D}$. If additionally $\lim_{|z|\to 1} v(z) = 0$ holds, we refer to them as *typical* weights. Examples include all the famous and popular weights, such as

(a) the *standard weights* $v(z) = (1 - |z|)^\alpha$, $\alpha > 0$,

(b) the *logarithmic weights* $v(z) = (1 - \log(1 - |z|))^\beta$, $\beta < 0$,

(c) the *exponential weights* $v(z) = e^{-\frac{1}{(1-|z|)^\gamma}}$, $\gamma > 0$.

In [8] Lusky studied typical weights satisfying the following two conditions

$$(L1) \qquad \inf_{n\in\mathbb{N}} \frac{v(1-2^{-n-1})}{v(1-2^{-n})} > 0$$

and

$$(L2) \qquad \limsup_{n\to\infty} \frac{v(1-2^{-n-j})}{v(1-2^{-n})} < 1 \text{ for some } j \in \mathbb{N}.$$

In fact, weights having $(L1)$ and $(L2)$ are *normal* weights in the sense of Shields and Williams, see [10]. The standard weights are normal weights, the logarithmic weights satisfy $(L1)$, but not $(L2)$ and the exponential weights satisfy neither $(L1)$ nor $(L2)$. In our context $(L2)$ is not of interest, while $(L1)$ will play a secondary role.

The formulation of results on weighted spaces often requires the so-called associated weights. For a weight v its associated weight is given by

$$\tilde{v}(z) := \frac{1}{\sup\{|f(z)|;\ f \in H(\mathbb{D}),\ \|f\|_v \le 1\}},\ z \in \mathbb{D}.$$

See e.g. [2] and the references therein. Associated weights are continuous, $\tilde{v} \ge v > 0$ and for every $z \in \mathbb{D}$ there is $f_z \in H(\mathbb{D})$ with $\|f_z\|_v \le 1$ such that $f_z(z) = \frac{1}{\tilde{v}(z)}$. Since it is quite difficult to really calculate the associated weight we are interested in simple conditions on the weight that ensure that v and $\tilde{v}$ are equivalent weights, i.e. there is a constant $C > 0$ such that

$$v(z) \le \tilde{v}(z) \le Cv(z) \text{ for every } z \in \mathbb{D}.$$

If v and $\tilde{v}$ are equivalent, we say that v is an *essential* weight. By [4] condition $(L1)$ implies the essentiality of v.

2.2. Setting

This section is devoted to the description of the setting we are working in. In the sequel we will consider weighted Bergman spaces generated by the following class of weights. Let ν be a holomorphic function on $\mathbb{D}$ that does not vanish and is decreasing as well as strictly positive on $[0,1)$. Moreover, we assume that $\lim_{r\to 1}\nu(r)=0$. Now, we define the weight as follows:

$$v(z) := \nu(|z|) \text{ for every } z \in \mathbb{D}. \tag{1}$$

Obviously such weights are bounded, i.e. for every weight v of this type we can find a constant $C>0$ such that $\sup_{z\in\mathbb{D}} v(z) \le C$. Moreover, we assume additionally that $|\nu(z)| \ge \nu(|z|)$ for every $z \in \mathbb{D}$.
Now, we can write the weight v in the following way

$$v(z) = \min\{|g(\lambda z)|,\ |\lambda| = 1\},$$

where g is a holomorphic function on $\mathbb{D}$. Since ν is a holomorphic function, we obviously can choose $g=\nu$. Then we arrive at

$$\begin{aligned}\min\{|\nu(\lambda z)|,\ |\lambda|=1\} &= \min\{|\nu(\lambda r e^{i\Theta})|,\ |\lambda|=1\}\\ &\le |\nu(e^{-i\Theta} r e^{i\Theta})| = |\nu(r)| = |\nu(|z|)| = v(z)\end{aligned}$$

for every $z\in\mathbb{D}$. Conversely, by hypothesis, for every $\lambda\in\partial\mathbb{D}$ we obtain for every $z\in\mathbb{D}$

$$|\nu(\lambda z)| \ge \nu(|\lambda z|) \ge \nu(|z|) = v(z).$$

Thus, the claim follows. The standard, logarithmic and exponential weights can all be defined like that.

2.3. Composition operators between weighted Bergman spaces and weighted Banach spaces of holomorphic functions

In the setting of weighted Banach spaces of holomorphic functions the classical composition operator has been studied by Bonet, Domański, Lindström and Taskinen in [4]. Among other things they proved that in case that v and w are arbitrary weights the boundedness of the operator $C_\phi : H_v^\infty \to H_w^\infty$ is equivalent to

$$\sup_{z\in\mathbb{D}} \frac{w(z)}{\tilde{v}(\phi(z))} < \infty.$$

Moreover, they showed that v satisfies condition $(L1)$ if and only if every composition operator $C_\phi : H_v^\infty \to H_v^\infty$ is bounded.

This was the motivation to study the boundedness of composition operators acting between weighted Bergman spaces and weighted Banach spaces of holomorphic functions. Doing this we obtain the following results which we need in the sequel.

Lemma 1 ([13], Lemma 1)**.** *Let $v(z) = \nu(|z|)$ for every $z \in \mathbb{D}$ with $\nu \in H(\mathbb{D})$ be a weight as defined in Section 2.2. Then there is a constant $M > 0$ such that*

$$|f(z)| \leq M \frac{\|f\|_{v,2}}{(1-|z|^2)v(z)^{\frac{1}{2}}}$$

for every $f \in A_v^2$.

The following result is obtained by using the previous lemma and following exactly the proof of [12] Theorem 2.2. We gave the full proof in [14] Theorem 2.2.

Theorem 2 ([14], Theorem 2.2)**.** *Let $v(z) = \nu(|z|)$ for every $z \in \mathbb{D}$ with $\nu \in H(\mathbb{D})$ be a weight as defined in Section 2.2. Then the operator $C_\phi : A_v^2 \to H_v^\infty$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{v(z)}{(1-|\phi(z)|^2)v(\phi(z))^{\frac{1}{2}}} < \infty. \tag{2}$$

Having characterized the boundedness of the composition operator acting between A_v^2 and H_v^∞ we took the second result of Bonet, Domański, Lindström and Taskinen as a motivation to ask the question: For which weights v are all operators $C_\phi : A_v^2 \to H_v^\infty$ bounded? The answer is as follows:

Theorem 3 ([14], Theorem 3.2)**.** *Let $v(z) = \nu(|z|)$ for every $z \in \mathbb{D}$ with $\nu \in H(\mathbb{D})$ be a weight as defined in Section 2.2. Moreover, let $\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{(1-|z|^2)} < \infty$. Then the composition operator $C_\phi : A_v^2 \to H_v^\infty$ is bounded for every analytic self-map ϕ of $\mathbb{D}$ if and only if*

$$\inf_{n\in\mathbb{N}} \frac{(2^{-n}-2^{-n-2})v(1-2^{-n-1})^{\frac{1}{2}}}{v(1-2^{-n})} > 0.$$

2.4. Geometry of the unit disk

Let $p \in \mathbb{D}$ and α_p denote the Möbius transformation that interchanges p with 0, i.e.

$$\alpha_p(z) = \frac{z-p}{1-\overline{p}z} \text{ for every } z \in \mathbb{D}.$$

As a corollary to the Schwarz lemma, every automorphism α of $\mathbb{D}$ is of the form

$$\alpha(z) = \xi\alpha_p(z) \text{ for every } z \in \mathbb{D}, \text{ where } \xi \in \partial\mathbb{D} \text{ and } p \in \mathbb{D}.$$

The following facts are well-known:

1. $\alpha_p^{-1} = \alpha_p$.
2. $1 - |\alpha_p(z)|^2 = \frac{(1-|z|^2)(1-|p|^2)}{|1-\overline{p}z|^2}$.

Moreover, we need the following lemma given by Bourdon in [6].

Lemma 4 (Bourdon)**.** *Suppose that α is a holomorphic automorphism of $\mathbb{D}$. Then for $0 \le r < 1$,*

$$\max_{|z|=r} |\alpha(z)| = \frac{|\alpha(0)| + r}{1 + |\alpha(0)|r}.$$

Next, we need some facts about the behaviour of holomorphic functions in the unit disk. It is well-known that every bounded function in $H(\mathbb{D})$ has non-tangential limits at every point of a subset of $\partial\mathbb{D}$ having full Lebesgue measure. When $f \in H(\mathbb{D})$ has a non-tangential limit at ξ we denote the value of the limit by $f(\xi)$.
Recall that the holomorphic self-map φ of $\mathbb{D}$ has angular derivative at $\xi \in \partial\mathbb{D}$ if there is $\eta \in \partial\mathbb{D}$ such that

$$\prec \lim_{z \to \xi} \frac{\varphi(z) - \eta}{z - \xi} =: \varphi'(z) \tag{3}$$

exists as a complex number, where $\prec \lim_{z\to\xi}$ denotes the non-tangential limit. The limit in (3) is called the *angular derivative of φ at ξ*. Furthermore, for our investigations we need the Julia-Carathéodory-Theorem as well as some of its consequences. For more details including the proofs of the results listed below we refer the reader to [6].

Theorem 5 (Julia-Carathéodory-Theorem)**.** *Suppose that φ is a holomorphic self-map of $\mathbb{D}$ and $\xi \in \partial\mathbb{D}$. The following are equivalent:*

(a) there is $\eta \in \partial\mathbb{D}$ with

$$\varphi'(\xi) = \prec \lim_{z \to \xi} \frac{\varphi(z) - \eta}{z - \xi} < \infty.$$

(b) φ and φ' have finite non-tangential limits at ξ and $\varphi(\xi) = \eta$ has modulus 1.

(c) $\liminf_{z\to\xi} \frac{1-|\varphi(z)|}{1-|z|} = \delta < \infty$.

Moreover, with the given assumptions, $\delta > 0$, $\varphi'(\xi) = \prec \lim_{z\to\xi} \varphi'(z) = \eta\overline{\xi}\delta$ and finally $\prec \lim_{z\to\xi} \frac{1-|\varphi(z)|}{1-|z|} = |\varphi'(\xi)|$.

Corollary 6. *Suppose that φ has no angular derivatives having modulus ≤ 1. Then there is a positive number $r < 1$ such that $|\varphi(z)| < |z|$ whenever $r < |z| < 1$.*

Lemma 7 (Julia-Carathéodory inequality). *Suppose that* $\liminf_{z\to\xi} \frac{1-|\varphi(z)|}{1-|z|} = \delta < \infty$ *and* η *is the non-tangential limit of* φ *at* ξ*, then for all* $z \in \mathbb{D}$*,*

$$\frac{|\eta - \varphi(z)|^2}{1 - |\varphi(z)|^2} \leq \delta \frac{|\xi - z|^2}{1 - |z|^2}.$$

Another important classical result which plays a main role in this article is the Denjoy-Wolff-Theorem. The n-th iterate of an analytic self-map ϕ of $\mathbb{D}$ is denoted by ϕ^n.

Theorem 8 (Denjoy-Wolff Theorem). *Let* ϕ *be an analytic self-map of* $\mathbb{D}$*. If* ϕ *is not the identity and not an automorphism with exactly one fixed point, then there is a unique point* $p \in \overline{\mathbb{D}}$ *such that* $(\phi^n)_n$ *converges to* p *uniformly on the compact subsets of* $\mathbb{D}$*.*

3. Results

Theorem 9. *Let* v *be a weight as defined in Section 2.2. Moreover we assume that* v *satisfies* $(L1)$ *and* $\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{(1-|z|^2)} < \infty$*. Then the following are equivalent:*

(a) $C_\phi : A_v^2 \to H_v^\infty$ *is bounded for every analytic self-map* ϕ *of* $\mathbb{D}$*.*

(b) $C_{\psi_a} : A_v^2 \to H_v^\infty$ *with*

$$\psi_a(z) = az + 1 - a \text{ for every } z \in \mathbb{D}$$

is bounded for every $a \in (0, 1)$*.*

(c) $\inf_{n\in\mathbb{N}} \frac{(1-(1-a^{n+1})^2)v(1-a^{n+1})^{\frac{1}{2}}}{v(1-a^n)} > 0$ *for every* $a \in (0, 1)$*.*

(d) $\inf_{n\in\mathbb{N}} \frac{(1-(1-a^{n+1})^2)v(1-a^{n+1})^{\frac{1}{2}}}{v(1-a^n)} > 0$ *for some* $a \in (0, 1)$*.*

(e) $\inf_{t\in(0,1]} \frac{(1-(1-at)^2)v(1-at)^{\frac{1}{2}}}{v(1-t)} > 0$ *for some* $a \in (0, 1)$*.*

(f) $C_\alpha : A_v^2 \to H_v^\infty$ *is bounded for some automorphism* α *of* $\mathbb{D}$ *with* $\alpha(0) \neq 0$*.*

(g) $C_\alpha : A_v^2 \to H_v^\infty$ *is bounded for every automorphism* α *of* $\mathbb{D}$*.*

Proof. The implication $(a) \Longrightarrow (b)$ is trivial. Next, we show that (c) follows from (b). To do this, we fix $a \in (0,1)$. Then obviously $\psi_a^n(0) = 1 - a^n$. Since the operator C_{ψ_a} is bounded, we can find $M > 0$ such that

$$\sup_{z \in \mathbb{D}} \frac{v(z)}{(1 - |\psi_a(z)|^2) v(\psi_a(z))} < \infty.$$

Now, we pick $z = \psi_a^n(0)$, $n \in \mathbb{N}$ arbitrary, and obtain using the inequality above

$$\frac{v(1 - a^n)}{(1 - (1 - a^{n+1})^2) v(1 - a^{n+1})^{\frac{1}{2}}} < M$$

which is equivalent to

$$\frac{(1 - (1 - a^{n+1})^2) v(1 - a^{n+1})^{\frac{1}{2}}}{v(1 - a^n)} > \frac{1}{M} > 0.$$

The implication (c) $\Longrightarrow$ (d) is obvious. Thus, we proceed with showing that (e) follows from (d). Here, we suppose that $\beta := \inf_{n \in \mathbb{N}} \frac{(1-(1-a^{n+1})^2) v(1-a^{n+1})^{\frac{1}{2}}}{v(1-a^n)} > 0$. We have to distinguish the following cases:
First, we assume that $a \leq t \leq 1$. Then

$$0 < \gamma := \frac{(1 - (1 - a^2)^2) v(1 - a^2)^{\frac{1}{2}}}{v(0)} \leq \frac{(1 - (1 - at)^2) v(1 - at)^{\frac{1}{2}}}{v(1 - t)}.$$

Now, if $a^{k+1} \leq t < a^k$ we obtain

$$\begin{aligned} \frac{(1 - (1 - at)^2) v(1 - at)^{\frac{1}{2}}}{v(1 - t)} &\geq \frac{(1 - (1 - a^{k+2})^2) v(1 - a^{k+2})^{\frac{1}{2}}}{v(1 - a^k)} \\ &= \frac{(1 - (1 - a^{k+2})^2) v(1 - a^{k+2})^{\frac{1}{2}}}{v(1 - a^{k+1})} \frac{v(1 - a^{k+1})}{v(1 - a^k)} > \delta\beta > 0 \end{aligned}$$

where $\inf_{n \in \mathbb{N}} \frac{v(1-a^{k+1})}{v(1-a^k)} > \delta$ since v has $(L1)$.
Next, we prove that (e) implies (f). First, we suppose that (e) holds:
$\lambda := \inf_{t \in (0,1]} \frac{(1-(1-at)^2) v(1-at)^{\frac{1}{2}}}{v(1-t)} > 0$. Now, let $p = \frac{1-a}{1+a}$ so that p is positive and $\frac{1-p}{1+p} = a$. We show that the automorphism $\alpha_p(z) = \frac{z-p}{1-pz}$ induces a bounded composition operator. Note that $\alpha_p(0) \neq 0$ since $p \neq 0$. Using Lemma 1 and the fact that

v is non increasing and radial we find that for $z \in \mathbb{D}$:

$$\begin{aligned}\frac{v(z)}{(1-|\alpha_p(z)|^2)v(\alpha_p(z))^{\frac{1}{2}}} &\leq \frac{v(|z|)}{v\left(\frac{p+|z|}{1+p|z|}\right)^{\frac{1}{2}}\left(1-\left|\frac{p+|z|}{1+p|z|}\right|^2\right)}\\ &= \frac{v(1-(1-|z|))}{v\left(1-\left(1-\frac{p+|z|}{1+p|z|}\right)\right)^{\frac{1}{2}}\left(1-\left|1-\left(1-\frac{p+|z|}{1+p|z|}\right)\right|^2\right)}\\ &= \frac{v(1-(1-|z|))}{v\left(1-\frac{(1-p)(1-|z|)}{1+p|z|}\right)^{\frac{1}{2}}\left(1-\left|1-\frac{(1-p)(1-|z|)}{1+p|z|}\right|^2\right)}\\ &\leq \frac{v(1-(1-|z|))}{v\left(1-\frac{1-p}{1+p}(1-|z|)\right)^{\frac{1}{2}}\left(1-\left|1-\frac{1-p}{1+p}(1-|z|)\right|^2\right)}\\ &= \frac{v(1-(1-|z|))}{v\left(1-a(1-|z|)\right)^{\frac{1}{2}}\left(1-|1-a(1-|z|)|^2\right)} \leq \frac{1}{\lambda}.\end{aligned}$$

Hence the operator C_{α_p} must be bounded.
Next, we show that (f) implies (g). We suppose that (f) holds. $C_\alpha : A_v^2 \to H_v^\infty$ is bounded for $\alpha = \xi\alpha_p$ where $|\xi| = 1$ and $p \in \mathbb{D}\backslash\{0\}$. Since C_α is bounded, we have

$$\sup_{z\in\mathbb{D}} \frac{v(z)}{v(\alpha(z))^{\frac{1}{2}}(1-|\alpha(z)|^2)} = \sup_{z\in\mathbb{D}} \frac{v(z)}{v(\alpha_p(z))^{\frac{1}{2}}(1-|\alpha_p(z)|^2)} < \infty$$

since v is a radial weight. Thus, C_{α_p} is also bounded. Note that α_p has Denjoy-Wolff point $-\frac{p}{|p|} \in \partial\mathbb{D}$. Hence $|\alpha_p^n(0)| \to 1$ as $n \to \infty$. Let $\tau(z) = \xi\frac{z-q}{1-\bar{q}z}$ be an arbitrary disk automorphism. We choose the positive integer n such that $|\alpha_p^n(0)| > |q|$ and let $s = |\alpha_p^n(0)|$. We know that $C_{\alpha_p^n}$ is bounded. Thus, there is a constant C such that for every $z \in \mathbb{D}$:

$$\frac{v(z)}{v(\alpha_p^n(z))^{\frac{1}{2}}(1-|\alpha_p^n(z)|^2)} \leq C.$$

Let $z \in \mathbb{D}$ be arbitrary, let $r = |z|$ and choose z_0 with $|z_0| = r$ such that $|\alpha_p^n(z_0)| = \frac{s+r}{1+sr}$. Since $x \mapsto \frac{x+r}{1+xr}$ is increasing for $-\frac{1}{r} < x < \infty$, $|q| < s$ and v and $1-|z|^2$ are non-increasing

$$\begin{aligned}\frac{v(z)}{v(\tau(z))^{\frac{1}{2}}(1-|\tau(z)|^2)} &\leq \frac{v(z)}{v\left(\frac{|q|+r}{1+|q|r}\right)^{\frac{1}{2}}\left(1-\left|\frac{|q|+r}{1+|q|r}\right|^2\right)} \leq \frac{v(z)}{v\left(\frac{s+r}{1+sr}\right)^{\frac{1}{2}}\left(1-\left|\frac{s+r}{1+sr}\right|^2\right)}\\ &= \frac{v(z_0)}{v(\alpha^n(z_0))^{\frac{1}{2}}(1-|\alpha^n(z_0)|^2)} \leq C\end{aligned}$$

and it follows that C_τ is bounded.
Last, we prove that (g) implies (a). Let $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic map. Let $p = \phi(0)$. Then C_{α_p} is bounded by hypothesis. Next, we consider $\psi = \alpha_p \circ \phi$. Then C_ψ is bounded since $\psi(0) = \alpha_p(\phi(0)) = \alpha_p(0) = 0$. Then, by the Schwarz Lemma we have that $|\psi(z)| \leq |z|$ for every $z \in \mathbb{D}$ and thus

$$\sup_{z\in\mathbb{D}} \frac{v(z)}{(1-|\psi(z)|^2)v(\psi(z))^{\frac{1}{2}}} \leq \sup_{z\in\mathbb{D}} \frac{v(z)}{(1-|z|^2)v(z)^{\frac{1}{2}}} = \sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{(1-|z|^2)} < \infty$$

by hypothesis. Now, $C_\phi = C_\psi C_{\alpha_{-p}}$ must be bounded as composition of bounded operators.

Next, we want to compare condition $(L1)$ with the following condition

$$\inf_{n\in\mathbb{N}} \frac{(1-(1-2^{-n-1})^2)v(1-2^{-n-1})^{\frac{1}{2}}}{v(1-2^{-n})} > 0. \tag{4}$$

First, we reformulate equation (4) and obtain

$$\inf_{n\in\mathbb{N}} \frac{2^{-n-1}v(1-2^{-n-1})^{\frac{1}{2}}}{v(1-2^{-n})} > 0. \tag{5}$$

Now, we will show that $(L1)$ does not imply (5): To do this we consider the standard weight $v(z) = 1-|z|$ for every $z \in \mathbb{D}$. It is well-known that this weight satisfies $(L1)$. But it does not enjoy (5) as the following calculation shows:

$$\inf_{n\in\mathbb{N}} \frac{2^{-n-1}v(1-2^{-n-1})^{\frac{1}{2}}}{v(1-2^{-n})} = \inf_{n\in\mathbb{N}} \frac{2^{-n-1}2^{\frac{-n-1}{2}}}{2^{-n}} = \inf_{n\in\mathbb{N}} 2^{\frac{-n-3}{2}} = 0.$$

Conversely, $(L1)$ does not follow from (5). We pick the weight $v(z) = (1-|z|^2)e^{-\frac{1}{1-|z|}}$ for every $z \in \mathbb{D}$. Then v does not satisfy $(L1)$ since for every $n \in \mathbb{N}$ we have

$$\frac{v(1-2^{-n-1})}{v(1-2^{-n})} = \frac{2^{-2n-2}e^{-2^{n+1}}}{2^{-2n}e^{-2^n}} = \frac{1}{4}e^{-2^n} \to 0$$

if $n \to \infty$. Hence $(L1)$ is not satisfied.
It remains to show that (5) is fulfilled. For every $n \in \mathbb{N}$:

$$2^{-n-1}\frac{v(1-2^{-n-1})^{\frac{1}{2}}}{v(1-2^{-n})} = \frac{2^{-n-1}2^{-n-1}e^{-\frac{2^{n+1}}{2}}}{2^{-2n}e^{-2^n}} = \frac{1}{4} > 0.$$

Thus, the claim follows.

Example 1. *(a) Let $v_\alpha(z) = (1-|z|)^\alpha$, $\alpha \geq 2$. Then v_α satisfies all the required conditions:*

(i) It is well-known, that v_α has condition $(L1)$.

(ii) Moreover,

$$\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{(1-|z|^2)} = \sup_{z\in\mathbb{D}} \frac{(1-|z|)^{\frac{\alpha}{2}-1}}{1+|z|} \leq \sup_{z\in\mathbb{D}} (1-|z|)^{\frac{\alpha}{2}-1} \leq 1$$

since $\frac{\alpha}{2}-1 \geq 0$ by hypothesis.

(iii) $\inf_{n\in\mathbb{N}} \frac{2^{-n-1}v(1-2^{-n-1})^{\frac{1}{2}}}{v(1-2^{-n})} = \inf_{n\in\mathbb{N}} \frac{2^{-n-1}(2^{-n-1})^{\frac{\alpha}{2}}}{(2^{-n})^\alpha} = \inf_{n\in\mathbb{N}} 2^{(\frac{\alpha}{2}-1)n}2^{-\frac{\alpha}{2}-1} = \frac{1}{4} > 0$

(b) We consider the weights $v_{\alpha,\beta}(z) = \frac{(1-|z|)^\alpha}{(1-\log(1-|z|))^\beta}$, $\alpha \geq 3, \beta > 0$. These weights also satisfy all the required conditions.

(i) $v_{\alpha,\beta}$ has the condition $(L1)$ since both, the standard and the logarithmic weights, have this condition.

(ii) $\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{(1-|z|^2)} = \sup_{z\in\mathbb{D}} \frac{(1-|z|)^{\frac{\alpha}{2}-1}}{(1+|z|)(1-\log(1-|z|))^\beta} < \infty$, *since* $\frac{\alpha}{2} > 0$.

(iii) $\inf_{n\in\mathbb{N}} \frac{2^{-n-1}v(1-2^{-n-1})^{\frac{1}{2}}}{v(1-2^{-n})} = \inf_{n\in\mathbb{N}} \frac{2^{n(\frac{\alpha}{2}-1)}2^{-1-\frac{\alpha}{2}}(1+n\log 2)^\beta}{(1+(n+1)\log 2)^{\frac{\beta}{2}}} > 0$ *since* $\frac{\alpha}{2} - 1 > 0$ *by hypthesis.*

(c) Finally, we look at the exponential weights $v_\gamma(z) = e^{-\frac{1}{(1-|z|)^\gamma}}$, $\gamma > 0$.

(i) It is well-known that the exponential weights do not satisfy condition $(L1)$.

(ii) We have

$$\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{(1-|z|^2)} = \sup_{z\in\mathbb{D}} \frac{e^{-\frac{1}{(1-|z|)^\gamma}}}{(1-|z|^2)} < \infty.$$

(iii) $\inf_{n\in\mathbb{N}} \frac{2^{-n-1}v(1-2^{-n-1})^{\frac{1}{2}}}{v(1-2^{-n})} = \inf_{n\in\mathbb{N}} 2^{-n-1}e^{-\gamma 2^n}e^{\gamma 2^n} = \inf_{n\in\mathbb{N}} 2^{-n-1} = 0.$

Theorem 10. *Let v be a weight as defined in Section 2.2. Moreover we assume that v satisfies $(L1)$ and $\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{(1-|z|^2)} < \infty$. Suppose that $C_\phi : A_v^2 \to H_v^\infty$ is bounded and that ϕ has an agular derivative less than 1 at some point $\xi \in \partial\mathbb{D}$. Then every composition operator acting between A_v^2 and H_v^∞ is bounded.*

Proof. Applying the Julia-Carathéodory theorem we can find $\eta \in \partial\mathbb{D}$ such that ϕ has nontangential limit η at ξ. By Theorem 2 the composition operators $C_{\overline{\eta}z}$ and $C_{\xi z}$ are bounded since

$$\|C_{\overline{\eta}z}\| = \sup_{z\in\mathbb{D}} \frac{v(z)}{(1-|\overline{\eta}z|^2)v(\overline{\eta}z)^{\frac{1}{2}}} = \sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{(1-|z|^2)} < \infty$$

resp.

$$\|C_{\xi z}\| = \sup_{z\in\mathbb{D}} \frac{v(z)}{(1-|\xi z|^2)v(\xi z)^{\frac{1}{2}}} = \sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{(1-|z|^2)} < \infty$$

by hypothesis. Hence the composition operator with the symbol ψ defined by $\psi(z) = \overline{\eta}\phi(\xi z)$ is also bounded. Moreover, $\psi(1) = \overline{\eta}\phi(\xi) = 1$ and $\psi'(1) = \overline{\eta}\phi'(\xi) = |\phi'(\xi)| < 1$. Hence ψ has Denjoy-Wolff point 1. We put $a := \psi'(1) < 1$. Using the Julia-Carathéodory inequality inductively we obtain for every $n \in \mathbb{N}$

$$\frac{|1-\psi^n(z)|^2}{1-|\psi^n(z)|^2} \leq a^n \frac{|1-z|^2}{1-|z|^2}.$$

Hence, for every $n \in \mathbb{N}$: $|1-\psi^n(0)| \leq 2a^n$. Thus, for every $n \in \mathbb{N}$ we arrive at

$$|1-\psi^n(0)| = g(n)a^n$$

where g is a positive bounded function on $\mathbb{N}$. Since $(\psi^n(0))_n$ converges non tangentially to 1 we can apply the Julia-Carathéodory theorem to conclude that $\lim_{n\to\infty} \frac{1-|\psi^{n+1}(0)|}{1-|\psi^n(0)|} = a$ or, equivalently, $\lim_{n\to\infty} \frac{g(n+1)}{g(n)} = 1$. For each $n \in \mathbb{N}$ set $e_n = n + \log_a(g(n))$. Then

$$e_{n+1} - e_n = 1 + \log_a\left(\frac{g(n+1)}{g(n)}\right).$$

Since $\left(\log_a\left(\frac{g(n+1)}{g(n)}\right)\right)_n$ tends to 0 as $n \to \infty$ we can find $K \in \mathbb{N}$ such that $e_{n+1}-e_n \geq \frac{1}{2}$ for every $n \geq K$. Because C_ψ is bounded, $(C_\psi)^3 = C_{\psi^3}$ is also bounded and there is a constant M such that

$$\frac{v(z)}{(1-|\psi^3(z)|^2)v(\psi^3(z))^{\frac{1}{2}}} \leq M \text{ for every } z \in \mathbb{D}.$$

Thus, for every $n \in \mathbb{N}$,

$$M \geq \frac{v(1-(1-|\psi^n(0)|))}{(1-(1-|\psi^{n+3}(0)|)^2)v(1-|\psi^{n+3}(0)|)^{\frac{1}{2}}} = \frac{v(1-a^{e_n})}{(1-(1-a^{e_{n+3}})^2)v(1-a^{e_{n+3}})^{\frac{1}{2}}}.$$

Let $j \in \mathbb{N}$ such that $j \geq K + \log_a(g(K)) = e_K$. Now, let $n_0 \geq K$ be the greatest positive integer such that $e_{n_0} \leq j$. Note that $e_{n_0+1} > j$. Since $e_{n+1} - e_n \geq \frac{1}{2}$ for every $n \geq K$ we have that

$$e_{n_0+3} > e_{n_0+1} \geq j+1.$$

Since v and the standard weights are decreasing we arrive at

$$\frac{v(1-a^j)}{(1-(1-a^{j+1})^2)v(1-a^{j+1})^{\frac{1}{2}}} \leq \frac{v(1-a^{e_{n_0}})}{(1-(1-a^{e_{n_0+3}})^2)v(1-a^{e_{n_0+3}})^{\frac{1}{2}}} \leq M.$$

Since $j \geq e_K$ is arbitrary, it follows that

$$\inf_{n\in\mathbb{N}} \frac{v(1-a^{n+1})^{\frac{1}{2}}(1-(1-a^{n+1})^2)}{v(1-a^n)} > 0.$$

Thus, all composition operators acting between A_v^2 and H_v^∞ are bounded.

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Elke Wolf
Institut für Mathematik,
Universität Paderborn,
Paderborn, Germany
email: *lichte@math.uni-paderborn.de*