

Nonlinear and nonlocal PDEs

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Abstract: We study qualitative properties of solutions of partial differential equations and systems. Useful and preparatory is the action of some integral operators. Then, will be mainly depth regularity properties of the derivatives of solutions of partial differential equations and systems.

Keywords: Partial differential equations, systems, regularity, singular integral operators.

MSC2010: 35J15, 30H35, 42B25

The study of some qualitative properties of solutions of partial differential equations and systems is considered. To do this, it is useful to understand the behavior of some integral operators. Then, will be depth regularity properties of the derivatives of solutions of second order partial differential equations having discontinuous coefficients, and later of systems.

To be more specific let us consider $\Omega \subset \mathbb{R}^n$ a bounded open set with $\partial\Omega$ sufficiently smooth, $f \in L^{p,\lambda}(\Omega)$, $1 < p < +\infty$, $0 < \lambda < n$ and a_{ij} discontinuous functions of the following second order partial differential equation of elliptic type

$$\mathcal{L}u \equiv \sum_{i,j=1}^n a_{ij}u_{x_i x_j} = f$$

Let us now recall the definition of the Morrey spaces $L^{p,\lambda}(\Omega)$. In the sequel we are mainly interested in regularity of the highest order derivatives of u in these spaces.

Definition 1 Let $1 < p < \infty$, $0 < \lambda < n$. A measurable function $f \in L^1_{\text{loc}}(\Omega)$ is in the Morrey class $L^{p,\lambda}(\Omega)$ if the following norm is finite

$$\|f\|_{L^{p,\lambda}(\Omega)} = \sup_{\substack{x \in \Omega \\ 0 < \rho < \text{diam}\Omega}} \frac{1}{\rho^\lambda} \int_{\Omega \cap B(x,\rho)} |f(y)|^p dy,$$

where $B(x,\rho)$ ranges in the class of the balls centered in x with radius ρ .

Let also set, for $f \in L^1(\Omega, \mathbb{R}^n)$, the integral mean $f_{x,\rho}$ by

$$f_{x,\rho} = \frac{1}{|\Omega \cap B(x,\rho)|} \int_{\Omega \cap B(x,\rho)} f(y) dy$$

where $|\Omega \cap B(x,\rho)|$ is the Lebesgue measure of $\Omega \cap B(x,\rho)$.

If we are not interested in specifying which the center is, we only set f_ρ .

Let us now give the definition of bounded mean oscillation function (BMO) that appear at first in the paper by John and Nirenberg [27].

Definition 2 Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. We say that f belongs to $BMO(\mathbb{R}^n)$ if the seminorm

$$\|f\|_* \equiv \sup_{B(x,\rho)} \frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} |f(y) - f_{x,\rho}| dy < \infty.$$

Let us recall the definition of the space of vanishing mean oscillation functions, given at first by Sarason in [52].

Definition 3 Let $f \in BMO(\mathbb{R}^n)$ and

$$\eta(f, R) = \sup_{\rho \leq R} \frac{1}{|B_\rho|} \int_{B_\rho} |f(y) - f_\rho| dy$$

where B_ρ ranges over the class of the balls of \mathbb{R}^n of radius ρ .

A function $f \in VMO(\mathbb{R}^n)$ if

$$\lim_{R \rightarrow 0} \eta(f, R) = 0.$$

Let us make some remarks concerning VMO class. It was at first defined by Sarason in 1975, and later it was considered by many others e.g. Chiarenza, Frasca and Longo in [9], where the authors answer to a question raised thirty years before by Miranda in [34].

Miranda considers a linear elliptic equation having coefficients $a_{ij} \in W^{1,m}(\Omega)$ and asks whether the gradient of the solution is bounded in L^p , if $p > m$.

Chiarenza, Frasca and Longo suppose $a_{ij} \in VMO$ and prove that Du is Hölder continuous, for all $p \in]1, +\infty[$.

We point out that

$$W^{1,m} \subset VMO,$$

it is proved using Poincarè's inequality:

$$\frac{1}{|B|} \int_B |f(x) - f_B| \leq c(m) \left(\int_B |\nabla u|^m dx \right)^{\frac{1}{m}}$$

and observing that the term on the right-hand side tends to zero as $|B| \rightarrow 0$.

Later, interior estimates obtained by Chiarenza, Frasca and Longo were extended by the same authors to boundary estimates in [10]. From these paper on, many authors have used this space VMO to obtain regularity results for P.D.E. and systems with discontinuous coefficients. We recall, for example, the study made by Bramanti and Cerutti [3] for parabolic equations, Polidoro and Ragusa for ultraparabolic equations in nondivergence form [40] and in divergence form [41], Ragusa for elliptic systems in nondivergence form [43] and many others.

With this useful VMO assumption was, also, investigate the regularity of the minimizers for quadratic functionals by Tachikawa and others.

As first equation, let us consider the following second order elliptic one in nondivergence form.

$$\mathcal{L}u \equiv \sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j} = f$$

Regularity results for elliptic equations of this kind, having known term in Morrey spaces $L^{p,\lambda}$, are obtained in the paper [14].

Let us now consider the following second order differential operator

$$L u \equiv \sum_{i,j=1}^{m_0} a_{i,j}(x,t) \partial_{x_i, x_j} u + \sum_{i,j=1}^N b_{i,j} x_i \partial_{x_j} u - \partial_t u$$

where $z = (x, t) \in \mathbb{R}^{N+1}$, $0 < m_0 \leq N$, treated by Polidoro and Ragusa in [40].

This kind of linear operator of Fokker-Plank type is used in probability and in mathematical physics, as for instance in the study of brownian motions of a particle in a fluid.

Let us point out that the natural geometry for the above operator is not euclidean but is given by a suitable groups structure.

Let us suppose that the matrix $A(z)$ is the $N \times N$ matrix

$$A(z) = \begin{pmatrix} A_0(z) & 0 \\ 0 & 0 \end{pmatrix}$$

where $A_0(z) = (a_{i,j}(z))_{i,j=1,\dots,m_0}$ is symmetric and there exists $\Lambda > 0$ such that

$$\Lambda^{-1} |\xi|^2 \leq \langle A_0(z) \xi, \xi \rangle \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^{m_0}, \forall z \in \mathbb{R}^{N+1}.$$

and $B = (b_{i,j})$ is a suitable $N \times N$ constant real matrix.

Polidoro and Ragusa, continue their study in [41], where are obtained interior regularity and local Hölder continuity of the weak solution u of the following equation

$$\sum_{i,j=1}^{m_0} \partial_{x_i} (a_{i,j}(z) \partial_{x_j} u) + \sum_{i,j=1}^N b_{i,j} x_i \partial_{x_j} u - \partial_t u = \sum_{j=1}^{m_0} \partial_{x_j} F_j(z)$$

where F_j belong to a function space of Morrey type, $0 < m_0 \leq N$ and $B = (b_{i,j})_{i,j=1,\dots,N}$ is a constant real matrix that, as in the previous mentioned paper, has the following form

$$B = \begin{pmatrix} 0 & B_1 & 0 & \dots & 0 \\ 0 & 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B_r \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

being each B_j a $m_{j-1} \times m_j$ block matrix of rank m_j , where $j = 1, 2, \dots, r$, and $m_0 \geq m_1 \geq \dots \geq m_r \geq 1$ such that $m_0 + m_1 + \dots + m_r = N$.

The study of this kind of operators arises in stochastic theory, see the book [53], and in the theory of diffusion processes, see [7] and [8]. Let us consider the operator

$$\mathcal{S} \equiv \sum_{j=1}^n \partial_{x_j}^2 + \sum_{j=1}^n x_j \partial_{x_{n+j}} - \partial_t u. \quad (1)$$

This is the linearized prototype of the Fokker-Plank operator that describes, under suitable conditions, the moving of brownian particles in a flow.

Let us observe that the operator \mathcal{S} is of degenerate type because there are only $n = \frac{N}{2}$ second order derivatives. If we set $X_j = \partial_{x_j}$, $j = 1, \dots, n$, and

$$Y = \langle x, BD \rangle - \partial_t$$

the operator \mathcal{S} has the following form

$$\mathcal{S} = \sum_{j=1}^n X_j^2 + Y.$$

Also, let us observe that the above equation is a linearized version of the Landau equation that, in turns, is a simplified model for the Boltzmann equation, see the paper in [28] by Landau.

We are now interested in interior regularity of solutions of these kind of equations.

If A is a **constant matrix** and $F_j \in C^\infty$ then $u \in C^\infty$, it is proved in the note by Lanconelli and Polidoro [29].

If a_{ij} are **Hölder continuous** the above operator L was studied by Polidoro in the papers [37], [38], [39] and also by Manfredini in [32], where the author, also, proves interior Schauder estimates.

Some related results has been obtained by Lunardi and Da Prato in [30] and [12], in the setting of semigroup theory.

If a_{ij} are **not uniformly continuous** the problem has been less studied. In the note [4] Bramanti, Cerutti and Manfredini study interior regularity of strong solutions to the nondivergence form of above equation; the regularity results in the divergence case has been proved by Manfredini and Polidoro in [33].

In the study made by Polidoro and Ragusa the coefficients a_{ij} are supposed discontinuous, precisely belonging to the Sarason class VMO_L of vanishing mean oscillation functions, the above set is considered as a subset of the John-Nirenberg class BMO_L . The bounded and vanishing mean oscillation classes are now indicated with the new symbols BMO_L and VMO_L to point out that are naturally associated with the following group's structure.

Definition 4 Let $(x, t), (\xi, \tau)$ be in \mathbb{R}^{N+1} . We set

$$(x, t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau), \quad E(t) = \exp(-tB^T)$$

and

$$D(\lambda) = \text{diag}(\lambda I_{m_0}, \lambda^3 I_{m_1}, \dots, \lambda^{2r+1} I_{m_r}),$$

where I_{m_j} is the $m_j \times m_j$ identity matrix.

We say that $(\mathbb{R}^{N+1}, \circ)$ is the “translation group associated to L ” and that $(D(\lambda), \lambda^2)_{\lambda>0}$ is the “dilation group associated to L ”.

Definition 5 We also call “homogeneous dimension” of \mathbb{R}^{N+1} the integer $Q + 2$, where

$$Q = m_0 + 3m_1 + \dots + (2r + 1)m_r.$$

In the note by Polidoro and Ragusa [41] the authors improve the results by Manfredini and Polidoro contained in the mentioned note assuming that the term $F = (F_1, F_2, \dots, F_{m_0}, 0, \dots, 0)$ is such that every F_j belongs to a suitable Morrey spaces.

The method used to obtain these results is inspired by a technique introduced by Chiarenza, Frasca and Longo in the papers [9] and [10]. It is based on explicit representation formula for the highest order derivatives of the solutions of the considered equation and on the $L^{p,\lambda}$ estimates of singular integral operators and commutators of non convolution type with Calderón-Zygmund kernel.

Let us now state the theorems obtained in [41] but first, in order to simplify the notation, let us set $Yu = \langle x, BDu \rangle - \partial_t u$.

Then, we can equivalently rewrite the above equation in the form

$$Lu := \operatorname{div}(A(z)Du) + Yu = \operatorname{div}(F). \quad (2)$$

The main results obtained by Polidoro and Ragusa are contained in the following two theorems.

Theorem 1 *Let Ω be a bounded open set in R^{N+1} and u a weak solution in Ω of equation*

$$\operatorname{div}(A(x,t)Du) + Yu = \operatorname{div}(F).$$

Let us suppose that the coefficients a_{ij} and the matrix B satisfy the above conditions.

Let us also set $a_{ij} \in VMO_L$, $i, j = 1, \dots, m_0$, $u \in L^p(\Omega)$ and $F_j \in L^{p,\lambda}(\Omega)$, $\forall j = 1, \dots, m_0$, $0 \leq \lambda < Q + 2$ and $1 < p < \infty$.

Then, for any compact set $K \subset \Omega$ we have that $\partial_{x_j} u \in L^{p,\lambda}(K)$, $\forall j = 1, \dots, m_0$, for every $1 < p < \infty$, $0 \leq \lambda < Q + 2$.

Moreover, there exists a positive constant c depending on p, λ, K, Ω and L such that, $\forall j = 1, \dots, m_0$,

$$\|\partial_{x_j} u\|_{L^{p,\lambda}(K)} \leq c \left(\sum_{k=1}^{m_0} \|F_k\|_{L^{p,\lambda}(\Omega)} + \|u\|_{L^p(\Omega)} \right). \quad (3)$$

Theorem 2 *Let Ω be a bounded open set in R^{N+1} and u a weak solution in Ω of the above equation.*

Let us suppose that the operator L satisfies the same hypotheses of the above theorem. Let us also consider $a_{ij} \in VMO_L$, $i, j = 1, \dots, m_0$, $u \in L^p(\Omega)$ and the known term $F_j \in L^{p,\lambda}(\Omega)$ $\forall j = 1, \dots, m_0$, $0 \leq \lambda < Q + 2$ and $p > Q + 2 - \lambda$.

Then, for any compact $K \subset \Omega$ there exists a positive constant c depending on p, λ, K, Ω and L such that, $\forall z, \zeta \in K$, $z \neq \zeta$,

$$\frac{|u(z) - u(\zeta)|}{\|\zeta^{-1} \circ z\|^{1 - \frac{Q+2}{p} + \frac{\lambda}{p}}} \leq c \left(\sum_{k=1}^{m_0} \|F_k\|_{L^{p,\lambda}(\Omega)} + \|u\|_{L^p(\Omega)} \right). \quad (4)$$

Schetch of the proof of Theorem 1. First of all we give some definitions which are useful in the sequel.

Let $r, s \in R$, with $0 < s < r$ and let $\phi \in C^\infty(\Omega)$ a function such that $\phi(y) = 1$ for $0 \leq y \leq s$ and $\phi(y) = 0$ for $t \geq r$.

For every $\zeta \in \Omega$ and $r > 0$ such that $B_r = B_r(\zeta) \subset \Omega$ we set

$$\eta(z) = \phi(\|\zeta^{-1} \circ z\|). \quad (5)$$

Let u be a solution of the equation, then u satisfies

$$L(\eta u) = \operatorname{div}(G) + g$$

where

$$G = \eta F + u A D \eta, \quad g = \langle A D u, D \eta \rangle - \langle F, D \eta \rangle + u Y^* \eta, \quad (6)$$

and Y^* is the adjoint of the operator Y .

The proof is based on the following representation formula of the first derivatives of the function $v(z) = \eta(x) u(z)$ in terms of singular integral operators and commutators with Calderón-Zygmund kernel.

For a.e. $z \in \mathbb{R}^{N+1}$ we can write

$$\begin{aligned} \partial_{x_j}(\eta u)(z) &= \sum_{h,j=1}^{m_0} \lim_{\epsilon \rightarrow 0} \int_{\|\zeta^{-1} \circ z\| \geq \epsilon} \Gamma_{jh}(z, \zeta^{-1} \circ z) \cdot \\ &\quad \{(a_{hk}(z) - a_{hk}(\zeta)) \partial_{x_k}(\eta u)(z) - G_h(\zeta)\} d\zeta + \\ &\quad + \int_{\mathbb{R}^{N+1}} \Gamma_j(z, \zeta^{-1} \circ z) g(\zeta) d\zeta + \sum_{k=1}^{m_0} c_{jk}(z) G_k(z), \end{aligned}$$

where we denote $c_{jk} = \int_{\|\zeta\|=1} \Gamma_j(z; \zeta) \nu_k(\zeta) d\sigma$ and $(\nu_1, \dots, \nu_{N+1})$ the outer normal at the set Σ_{N+1} .

Let us denote $T_j g(z) = \int \Gamma_j(z, \zeta^{-1} \circ z) g(\zeta) d\zeta$.

Then, we can express the $\partial_{x_j} v(z)$ in the following form

$$\partial_{x_j} v(z) = \sum_{h,k=1}^{m_0} C_{j,h}[a_{h,k}; v_k](z) - \sum_{h=1}^{m_0} T_{j,h}(G_h)(z) + T_j g(z) + \sum_{h=1}^{m_0} c_{j,h} G_h(z).$$

Recalling some Morrey estimates and the *BMO* assumption, we obtain

$$\begin{aligned} \|\partial_{x_j} v\|_{L^{p,\mu}(L, B_r)} &\leq c \left(\sum_{h,k=1}^{m_0} \|a_{h,k}\|_* \cdot \|\partial_{x_k} v\|_{L^{p,\lambda}(L, B_r)} + \right. \\ &\quad \left. + \|G\|_{L^{p,\lambda}(L, B_r)} + \|g\|_{L^{p,\nu}(L, B_r)} \right) \end{aligned}$$

where $0 \leq \nu \leq \lambda < Q + 2$ and $\mu = \min(\lambda, \nu + p)$.

Then, using the definition of G and g and some useful estimates we get the conclusion.

Schetck of proof of Theorem 2. Using the representation formula of ηu instead of that of $\partial_{x_j} v$, and some useful Sobolev Morrey embedding estimates we obtain the requested result.

Let us now continue the use of the vanishing mean oscillation functions, treating the regularity problem for minimizers

$$u(x) : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$$

of quadratic growth functional

$$\int_{\Omega} A(x, u, Du) dx,$$

where $\Omega \subset \mathbb{R}^m$ is a bounded open set and

$$A : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \rightarrow \mathbb{R}.$$

$A(\cdot, u, p)$ is in the *VMO* class, namely, is not assumed the continuity of $A(x, u, p)$ with respect to x .

Both *partial* and *global* regularity of the minimizer u are studied.

At first, let us talk about some *estimates in Morrey Spaces* for the derivatives of “*local minimizers*” of variational integrals of the form

$$\mathcal{A}(u, \Omega) = \int_{\Omega} F(x, u, Du) dx$$

where Ω is a domain in \mathbb{R}^m and the integrand has the following form

$$F(x, u, Du) = A(x, u, g(x) h(u) Du Du).$$

We are *not assuming the continuity* of A and g with respect to x .

A “local minimizer” of the functional \mathcal{A} is a function $u \in W_{loc}^{1,p}(\Omega, \mathbb{R}^n)$ which satisfies

$$\mathcal{A}(u; \text{supp } \varphi) \leq \mathcal{A}(u + \varphi; \text{supp } \varphi)$$

for every $\varphi \in W_0^{1,p}(\Omega, \mathbb{R}^n)$. Partial regularity for solutions of nonlinear elliptic systems has been well studied by Morrey in [36], Giusti in [23], Giusti and Miranda in [24], using an *indirect argument* similar to that one introduced by De Giorgi and Almgren in the regularity theory of parametrix minimal surfaces.

New perturbation method are later considered by Giaquinta and Giusti in [17], Giaquinta and Modica in [19], to study higher integrability of the gradient of the solutions.

Using a *perturbation method, or direct argument*, Tachikawa and Ragusa prove partial regularity for minimizers of the following variational integrals

$$\mathcal{A}(u; \Omega) := \int_{\Omega} F(x, u, Du) dx$$

where $u : \Omega \rightarrow \mathbb{R}^n$, $Du = (D_{\alpha} u^i)$, ($\alpha = 1, \dots, m$, $i = 1, \dots, n$) and the integrand has the following special form

$$F(x, u, Du) = A(x, u, g^{\alpha\beta}(x) h_{ij}(u) D_{\alpha} u^i D_{\beta} u^j).$$

This kind of functionals arises as *p-energy of maps between Riemannian manifolds*. From this point of view, the *geometric interest* may occur on the above functionals. Moreover, some methods of proofs of regularity for solutions of nonlinear elliptic systems can be applied to the equations of nonlinear *Hodge theory*, studied e.g. by Sibner and R. B. Sibner in [54].

Also, Giaquinta and Giusti in [18] consider the quadratic functionals

$$\int_{\Omega} g^{\alpha\beta}(x) h_{ij}(u) D_{\alpha} u^i D_{\beta} u^j dx$$

where $g^{\alpha\beta}$ and h_{ij} are symmetric positive definite matrices having smooth coefficients.

A geometrically useful example is the following

$$\int_{B_1(0)} \frac{|Du|^2}{(1 + |u|^2)} dx$$

that is, in local coordinates, the energy of a map.

Under similar assumptions to that in the paper by Giaquinta and Modica [22], is proved by Giaquinta and Modica in [21] that minimizers have Hölder continuous derivatives in an open set

$$\Omega_0 \subset \Omega : \text{meas}(\Omega \setminus \Omega_0) = 0.$$

The hypothesis considered by Ragusa and Tachikawa has been inspired by Campanato, see [6], where are obtained deep *Hölder regularity* results in $\mathcal{L}^{p,\lambda}$ spaces for solutions of elliptic systems *having nonlinearity greater or equal to 2*.

In the previous mentioned paper the coefficients of the second order elliptic differential operators are supposed *continuous*, the *VMO assumption* is a *more recent idea*.

Tachikawa and Ragusa in [44] investigate the partial regularity of the minimizers of quadratic functionals, whose integrands *have VMO coefficients*, using some majorizations for the functionals, rather than the well known Euler’s equation associated to it. The functional is

$$\int_{\Omega} \left\{ A_{ij}^{\alpha\beta}(x, u) D_{\alpha} u^i D_{\beta} u^j + g(x, u, Du) \right\} dx,$$

where $\Omega \subset \mathbb{R}^m$, $m \geq 3$, is a bounded open set, $u : \Omega \rightarrow \mathbb{R}^n$, $n > 1$,

The assumptions are the following

1. $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$.
2. For every $u \in \mathbb{R}^n$, $A_{ij}^{\alpha\beta}(\cdot, u) \in VMO(\Omega)$.
3. For every $x \in \Omega$ and $u, v \in \mathbb{R}^n$,

$$|A_{ij}^{\alpha\beta}(x, u) - A_{ij}^{\alpha\beta}(x, v)| \leq \omega(|u - v|^2)$$

for some monotone increasing concave function $\omega : \omega(0) = 0$.

4. There exists $\nu > 0$:

$$\nu|\xi|^2 \leq A_{ij}^{\alpha\beta}(x, u)\xi_\alpha^i \xi_\beta^j$$

for a.e. $x \in \Omega$, for all $u \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^{mn}$.

Let us mention that since C^0 is a **proper subset** of VMO , the *continuity* of $A_{ij}^{\alpha\beta}(x, u)$, with respect to x , is *not assumed*.

Let suppose that the function g is a Charathéodory function and has growth less than quadratic, that is

$$(g1) \quad g(\cdot, u, Du) \text{ is measurable in } x \quad \forall u \in \mathbb{R}^N, \forall z \in \mathbb{R}^{nN};$$

$$(g2) \quad g(x, \cdot, \cdot) \text{ is continuous in } (u, z) \text{ a. e. } x \in \Omega.$$

$$(g3)$$

$$|g(x, u, z)| \leq g_1(x) + H|z|^\gamma,$$

$$g_1 \geq 0 \text{ a. e. in } \Omega, g_1 \in L^p(\Omega), 2 < p \leq \infty, H \geq 0, 0 \leq \gamma < 2.$$

Then, let us state the regularity result.

Theorem 3 *Let $\Omega \subset \mathbb{R}^m, m \geq 3$, bounded open set, $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ be a minimum of the functional above defined. Suppose that assumptions on $A_{ij}^{\alpha\beta}(x, u)$ and $g(x, u, Du)$ are satisfied.*

Then, for $\lambda = m(1 - \frac{2}{p})$, we have

$$Du \in L_{loc}^{2,\lambda}(\Omega_0)$$

where

$$\Omega_0 = \{x \in \Omega : \liminf_{R \rightarrow 0} \frac{1}{R^{m-2}} \int_{B(x,R)} |Du(y)|^2 dy = 0\}.$$

As a consequence, for $\alpha \in (0, 1)$,

$$u \in C^{0,\alpha}(\Omega_0).$$

In a subsequent paper Tachikawa and Ragusa extend the above regularity results because is considered the following more general class *Campanato* space.

Let us now give the definition of this class of functions and some basic properties.

Let $1 \leq p < \infty$ and $\lambda \geq 0$.

By *Campanato space* $\mathcal{L}^{p,\lambda}(\Omega)$ we denote the linear space of functions $u \in L^p(\Omega)$:

$$[f]_{p,\lambda} = \left\{ \sup_{\substack{x \in \Omega \\ 0 < \rho < \text{diam} \Omega}} \rho^{-\lambda} \int_{\Omega \cap B(x,\rho)} |f(y) - f_\rho|^p dy \right\}^{\frac{1}{p}} < +\infty.$$

The class $\mathcal{L}^{p,\lambda}(\Omega)$ is a *Banach space* endowed with the *norm*

$$\|f\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{p,\lambda}.$$

A *function*

$$\begin{aligned} u &\in \mathcal{L}^{p,\lambda}(\Omega) \\ &\Downarrow \\ \sup_{x \in \Omega, 0 < \rho < \text{diam } \Omega} \rho^{-\lambda} \inf_{c \in \mathbb{R}} \int_{\Omega \cap B(x,\rho)} |f - c|^p dy &< \infty. \end{aligned}$$

Using *Hölder inequality* :

$$\mathcal{L}^{p_1,\lambda_1}(\Omega) \subset \mathcal{L}^{p,\lambda}(\Omega)$$

where

$$p \leq p_1, \quad \frac{n - \lambda}{p} \geq \frac{n - \lambda_1}{p_1}.$$

For $0 \leq \lambda < m$

$$[f]_{p,\lambda} \leq C \|f\|_{L^{p,\lambda}(\Omega)}$$

is now clear the following relation between *Morrey* and *Campanato spaces*

$$L^{p,\lambda}(\Omega) \subset \mathcal{L}^{p,\lambda}(\Omega).$$

Theorem 4 Let $\Omega \subset \mathbb{R}^m$, $m \geq 3$, $1 < q \leq 2$, $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a minimum of $\mathcal{A}(u, \Omega)$ defined above, satisfying the same assumptions.

Then, for $\lambda = m(1 - \frac{q}{p})$, we have

$$Du \in \mathcal{L}_{\text{loc}}^{q,\lambda}(\Omega_0, \mathbb{R}^{nN})$$

where

$$\Omega_0 = \left\{ x \in \Omega : \liminf_{R \rightarrow 0} \frac{1}{R^{m-2}} \int_{B(x,R)} |Du(y)|^2 dy = 0 \right\}.$$

In both the two mentioned Theorems let us point out the *existence* of Ω_0 . The set Ω_0 is *obligatory*, indeed when we pass from the regularity theory for *scalar* minimizers of solutions of elliptic equations to the regularity theory for *vector-valued* minimizers of solutions of elliptic systems, *the situation changes completely*: regularity is an *exceptional occurrence everywhere*, excluding the two dimensional case.

As example, let us recall the study made by De Giorgi in [13] where he shows that his regularity result for solutions of second order elliptic equations with measurable bounded coefficients *cannot be extended* to solutions of elliptic systems.

He presented the quadratic functional

$$\mathcal{S} = \int_{\Omega} A_{ij}^{\alpha\beta}(x) D_{\alpha} u^i D_{\beta} u^j dx$$

with $A_{ij}^{\alpha\beta} \in L^{\infty}(\Omega)$, such that

$$\exists \nu > 0 : A_{ij}^{\alpha\beta} \chi_{\alpha}^i \chi_{\beta}^j \geq \nu |\chi|^2, \quad \text{a.e. } x \in \Omega, \quad \forall \chi \in \mathbb{R}^{nN}$$

De Giorgi proves that \mathcal{S} has a minimizer that is a function having a *point of discontinuity* in the origin.

Later, Souček in [55] showed that minimizers of functionals of \mathcal{S} type can be discontinuous, *not only in a point*, but on a **dense subset** of Ω . Modifying De Giorgi's example, Giusti and Miranda in [25] showed that solutions of the quasilinear elliptic systems

$$\int_{\Omega} A_{ij}^{\alpha\beta}(u) D_{\alpha} u^i D_{\beta} \varphi^j dx = 0, \forall \varphi \in C_0^{\infty}(\Omega, \mathbb{R}^N)$$

with analytic elliptic coefficients $A_{ij}^{\alpha\beta}$, have *singularities in dimension* $m \geq 3$. Similar examples were presented in the meantime independently by Maz'ya in [31], Giaquinta in [16], Morrey in [35] and others were interested in these problems of solutions of elliptic systems in general *non regular*.

Then, Tachikawa and Ragusa can prove regularity *except on a set*, hopefully *not too large*.

Let us recall that, for linear systems, some useful regularity results. Assuming $A_{ij}^{\alpha\beta}$ **constants** or in $C^0(\bar{\Omega})$, have been obtained by Campanato in [5]; **without** assuming **continuity** of coefficients Acquistapace in [2] refines Campanato's results considering that coefficients $A_{ij}^{\alpha\beta}$ belong to a class *neither* containing *nor* contained in $C^0(\bar{\Omega})$, hence in general *discontinuous*. Moreover, we recall the study made by Huang in [26] for linear elliptic systems with coefficients in the class *VMO*.

Therefore, it seems to be natural to expect partial regularity results under the condition that the coefficients of the principal terms $A_{ij}^{\alpha\beta}$ are in *VMO*, even for *nonlinear cases*. Indeed, the positive answer is given by Daněček and Viszus in [11]. Precisely, they treat the regularity of minimizer for the functional

$$\int_{\Omega} \left\{ A_{ij}^{\alpha\beta}(\mathbf{x}) D_{\alpha} u^i D_{\beta} u^j + g(x, u, Du) \right\} dx,$$

where $g(x, u, Du)$ is a lower order term which satisfies

$$|g(x, u, z)| \leq f(x) + L|z|^{\gamma},$$

being $f \in L^p(\Omega)$, $2 < p \leq \infty$, $f \geq 0$ a.e. on Ω , $L \geq 0$, $0 \leq \gamma < 2$.

They obtain Hölder regularity of mini mizer assuming that $A_{ij}^{\alpha\beta}(x) \in VMO$.

Tachikawa and Ragusa in [44] *extends both* the results by Huang and Daněček and Viszus because they treat the functional whose integrand contains $g(x, u, Du)$ and has coefficients $A_{ij}^{\alpha\beta}$ dependent not only on x but also on u .

Another step in the walk of Tachikawa and Ragusa is the *generalization* of [44] proving $L^{2,\lambda}$ -regularity of minimizers for quadratic functionals whose integrand has a more generalized form depending on x, u, Du , in the case

$$m \leq 4.$$

These results are contained in the paper [45] where is considered and proved what follows. Let $\Omega \subset \mathbb{R}^m$ a domain and

$$\mathcal{A}(u, \Omega) = \int_{\Omega} A(x, u, Du) dx,$$

being $A(x, u, p)$ such that are true the following assumptions

- (A-1) For every $(u, p) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $A(\cdot, u, p) \in VMO(\Omega)$ and the mean oscillation of $A(\cdot, u, p)/|p|^2$ vanishes uniformly with respect to u, p in the following sense:
for some function $\sigma(y, \rho) \geq 0$ with

$$\limsup_{R \rightarrow 0} \sup_{\rho < R} \frac{1}{|B(x, \rho)|} \int_{B(x, \rho)} \sigma(y, \rho) dy = 0,$$

$A(\cdot, u, p)$ satisfies

$$|A(y, u, p) - A_{x,\rho}(u, p)| \leq \sigma(y, \rho)|p|^2, \quad \forall (u, p) \in \mathbb{R}^n \times \mathbb{R}^{mn}$$

(A-2) For every $x \in \Omega$, $p \in \mathbb{R}^{mn}$ and $u, v \in \mathbb{R}^n$,

$$|A(x, u, p) - A(x, v, p)| \leq \omega(|u - v|^2)|p|^2$$

for some monotone increasing concave function $\omega : \omega(0) = 0$.

(A-3) For almost all $x \in \Omega$, $\forall u \in \mathbb{R}^n$, $A(x, u, \cdot) \in C^2(\mathbb{R}^{mn})$.

(A-4) There exist $\mu_1 \geq \mu_0 > 0$:

$$\mu_0|p|^2 \leq A(x, u, p) \leq \mu_1|p|^2, \quad \forall (x, u, p) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}.$$

Theorem 5 *Let $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ be a local minimizer of the functional $\mathcal{A}(u, \Omega)$. Suppose (A-1)-(A-4) be true. Then, for every $0 < \lambda < \min\{2 + \varepsilon, m\}$, for some $\varepsilon > 0$,*

$$Du \in L_{\text{loc}}^{2,\lambda}(\Omega_0, \mathbb{R}^{mn})$$

where

$$\Omega_0 = \{x \in \Omega : \liminf_{R \rightarrow 0} \frac{1}{R^{m-2}} \int_{B(x,R)} |Du(y)|^2 dy = 0\}.$$

Moreover,

$$\mathcal{H}^{m-2-\delta}(\Omega \setminus \Omega_0) = 0$$

for some $\delta > 0$, where \mathcal{H}^r denotes the r -dimensional Hausdorff measure.

The fruitful cooperation Ragusa-Tachikawa allows to obtain some improvements. Precisely, using the perturbation, or direct, method in the papers [46] and [47] is studied partial regularity for minimizers of the following variational integrals

$$\mathcal{A}(u; \Omega) := \int_{\Omega} F(x, u, Du) dx$$

where Ω is a domain of \mathbb{R}^m and the integrand $F(x, u, \xi) : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$ is p -growth, $p \geq 2$, namely, for some $0 < \lambda_0 < \Lambda_0$ and $\mu \in \mathbb{R}$, F satisfies

$$\lambda_0(\mu^2 + |\xi|^2)^{p/2} \leq F(x, u, \xi) \leq \Lambda_0(\mu^2 + |\xi|^2)^{p/2}$$

for all $(x, u, \xi) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$.

Moreover,

$$\lambda_0(\mu^2 + |\xi|^2)^{p/2} |\eta|^2 \leq \frac{\partial^2 F(x, u, \xi)}{\partial \xi_{\beta}^j \partial \xi_{\alpha}^i} \eta_{\alpha}^i \eta_{\beta}^j \leq \Lambda_1(\mu^2 + |\xi|^2)^{p/2} |\eta|^2$$

for all $(x, u, \xi, \eta) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \times \mathbb{R}^{mn}$.

If $\mu = 0$ the variational feature of the functional is very different from that one for $\mu \neq 0$.

If $\mu = 0$, the Euler-Lagrange equation of the functional is degenerate at a point $x \in \Omega$ where $Du(x) = 0$.

In [46], the authors prove partial regularity results for the case that $\mu \neq 0$ while in [47] the integrand of the functional has the special form

$$F(x, u, Du) = A(x, u, g^{\alpha\beta}(x) h_{ij}(u) D_{\alpha} u^i D_{\beta} u^j),$$

and is showed a partial regularity result which holds even if $\mu = 0$.

Let us now state both the results and give an idea of the proof only for the second case $\mu = 0$.

General but non-degenerate case, see [46]:

(F-1) $\exists \Lambda_1 > \lambda_1 > 0, \exists \mu \neq 0 : \forall (x, u, \xi, \eta) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \times \mathbb{R}^{mn},$

$$\lambda_1(\mu^2 + |\xi|)^p \leq F(x, u, \xi) \leq \Lambda_1(\mu^2 + |\xi|)^p,$$

$$\lambda_1(\mu + |\xi|^2)^{p/2-1} |\eta|^2 \leq \frac{\partial^2 F(x, u, \xi)}{\partial \xi_\alpha^i \partial \xi_\beta^j} \eta_\alpha^i \eta_\beta^j \leq \Lambda_1(\mu^2 + |\eta|^2)^{p/2-1} |\xi|^2;$$

(F-2) for every $(u, \xi) \in \mathbb{R}^n \times \mathbb{R}^{mn}, F(\cdot, u, \xi) \in VMO(\Omega)$ and the mean oscillation of $F(\cdot, u, \xi)/(\mu^2 + |\xi|^2)^{2/p}$ vanishes uniformly with respect to u, ξ in the following sense:

$\exists \rho_0 > 0, \exists \sigma(z, \rho) : \mathbb{R}^m \times [0, \rho_0) \rightarrow [0, \infty)$ with

$$\limsup_{R \rightarrow 0} \sup_{\rho < R} \frac{1}{|\Omega \cap B(x, \rho)|} \int_{B(0, \rho) \cap \Omega} \sigma(z, \rho) dz = 0,$$

such that $F(\cdot, u, \xi)$ satisfies, $\forall x \in \bar{\Omega}, y \in B(x, \rho_0) \cap \Omega,$

$$|F(y, u, \xi) - F_{x, \rho}(u, \xi)| \leq \sigma(x - y, \rho)(\mu^2 + |\xi|^2)^{p/2}, \forall (u, \xi) \in \mathbb{R}^n \times \mathbb{R}^{mn},$$

(F-3) for every $x \in \Omega, \xi \in \mathbb{R}^{mn}$ and $u, v \in \mathbb{R}^n$

$$|F(x, u, \xi) - F(x, v, \xi)| \leq (1 + |\xi|^2)^{\frac{p}{2}} \omega(|u - v|^2);$$

(F-4) for almost all $x \in \Omega$ and all $u \in \mathbb{R}^n, F(x, u, \cdot) \in C^2(\mathbb{R}^{mn}).$

Theorem 6 Assume $p \geq 2, \Omega \subset \mathbb{R}^m$ a bounded domain with sufficiently smooth boundary $\partial\Omega,$ $u \in H^{1,p}(\Omega, \mathbb{R}^n)$ a minimizer of

$$\mathcal{F}(u, \Omega) = \int_{\Omega} F(x, u, Du) dx$$

in the class $X_g(\Omega) = \{u \in H^{1,p}(\Omega) : u - g \in H_0^{1,p}(\Omega)\}$ for a given boundary data $g \in H^{1,s}(\Omega),$ $s > p.$ Let (F-1)-(F-4) be true.

Then, for some $\varepsilon > 0, \forall \tau : 0 < \tau < \min\{2 + \varepsilon, m(1 - \frac{p}{s})\},$

$$Du \in L^{p, \tau}(\Omega_0, \mathbb{R}^{mn})$$

where Ω_0 is a relatively open subset of $\bar{\Omega}$ which satisfies

$$\bar{\Omega} \setminus \Omega_0 = \{x \in \Omega : \liminf_{R \rightarrow 0} \frac{1}{R^{m-p}} \int_{\Omega \cap B(x, R)} |Du(y)|^p dy > 0\}.$$

Moreover, $\mathcal{H}^{m-p-\delta}(\bar{\Omega} \setminus \Omega_0) = 0$ for some $\delta > 0.$

As a corollary of the above theorem is true the following partial Hölder regularity result.

Corollary 7 Let g, u and Ω_0 be as in the previous Theorem. Assume that $p + 2 \geq m$ and that $s > \max\{m, p\}.$ Then, for some $\alpha \in (0, 1),$

$$u \in C^{0, \alpha}(\Omega_0, \mathbb{R}^n).$$

Moreover, as a consequence of the proof of the previous Theorem we have the following full-regularity result for the case that F does not depend on $u.$

Corollary 8 *Assume that F and g satisfy all assumptions of the previous Theorem and that F does not depend on u .*

Let u be a minimizer of \mathcal{F} in the class X_g , then

$$Du \in L^{p,\tau}(\Omega, \mathbb{R}^{mn}).$$

Moreover, if $p + 2 \geq m$ and $s > \max\{m, p\}$, we have full-Hölder regularity of u

$$u \in C^{0,\alpha}(\bar{\Omega}, \mathbb{R}^n).$$

Degenerate case, see [47]:

Let us set $\mu \geq 0$, $p \geq 2$. Also, let the integrand function $A(x, u, t)$ be defined on $\Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$, in the sequel we assume that it satisfies the following assumptions

(A-1) There exist positive constants C, λ, Λ , $\lambda \leq \Lambda$ such that

$$\begin{aligned} \lambda(\mu^2 + t)^{\frac{p}{2}} &\leq A(x, u, t) \leq \Lambda(\mu^2 + t)^{\frac{p}{2}}, \\ \lambda(\mu^2 + t)^{\frac{p}{2}-1} &\leq |A_t(x, u, t)| \leq \Lambda(\mu^2 + t)^{\frac{p}{2}-1}, \\ \lambda(\mu^2 + t)^{\frac{p}{2}-2} &\leq A_{tt}(x, u, t) \leq \Lambda(\mu^2 + t)^{\frac{p}{2}-2}, \end{aligned}$$

for all $(x, u, t) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn}$;

(A-2) For every $(u, t) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $A(\cdot, u, t) \in VMO(\Omega)$ and the mean oscillation of $A(\cdot, u, t)/(\mu^2 + |t|)^{(p/2)}$ vanishes uniformly with respect to u, t in the following sense: $\exists \rho_0 > 0$ and a function $\sigma(z, \rho) : \mathbb{R}^m \times [0, \rho_0[\rightarrow [0, +\infty[$ with

$$\limsup_{r \rightarrow 0} \frac{1}{|\Omega \cap Q(x, \rho)|} \int \sigma(z, \rho) dz = 0,$$

such that $A(\cdot, u, t)$ satisfies,

$$|A(y, u, t) - A_{x,\rho}(u, t)| \leq \sigma(x - y, \rho)(\mu^2 + t)^{\frac{p}{2}},$$

for all $(u, t) \in \mathbb{R}^n \times \mathbb{R}^{mn}$, $\forall x \in \Omega$ and $y \in Q(x, \rho_0) \cap \Omega$;

(A-3) For every $x \in \Omega$, $t \in \mathbb{R}^{mn}$ and $u, v \in \mathbb{R}^n$,

$$|A(x, u, t) - A(x, v, t)| \leq \omega(|u - v|^2)(\mu^2 + t)^{\frac{p}{2}};$$

(A-4) For almost all $x \in \Omega$ and all $u \in \mathbb{R}^n$. $A(x, u, \cdot) \in C^2(\mathbb{R}^{mn})$, and A_{tt} satisfies, for some $\alpha > 0$,

$$|A_{tt}(x, u, t)t - A_{tt}(x, u, s)s| \leq c(\mu^2 + t + s)^{\frac{p-2}{2}-\alpha}|t - s|^\alpha;$$

(A-5) There exist constants $\lambda_0, \Lambda_0, \lambda_1, \Lambda_1$,

$\lambda_i < \Lambda_i$, $i = 0, 1$:

$$\begin{aligned} \lambda_0|\zeta|^2 &\leq g^{\alpha\beta}(x)\zeta_\alpha\zeta_\beta \leq \Lambda_0|\zeta|^2, \\ \lambda_1|\eta|^2 &\leq h_{ij}(u)\eta^i\eta^j \leq |\eta|^2, \end{aligned}$$

for all $x \in \Omega$, $u, \zeta \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^n$;

(A-6) For every $u, v \in \mathbb{R}^n$

$$|h(u) - h(v)| \leq \omega(|u - v|^2);$$

(A-7) g is in the class $L^\infty \cap VMO(\Omega)$.

Theorem 9 *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Let also $u \in W^{1,p}(\Omega, \mathbb{R}^n)$, $p \geq 2$, be a minimizer of the functional*

$$\mathcal{A}(u, \Omega) = \int_{\Omega} F(x, u, Du) dx$$

where

$$F(x, u, Du) = A(x, u, g^{\alpha\beta}(x)h_{ij}(u)D_\alpha u^i D_\beta u^j).$$

Suppose that $A(x, u, Du)$ satisfies (A-1) – (A-7).

Then, there exists an open set $\Omega_0 \subset \Omega$:

$$u \in C^{0,\alpha}(\Omega_0), \quad \forall \alpha \in (0, 1).$$

Let us now give an idea of the proof of the above Theorem.

It is proved progressing as in the paper [22]. Let us set $x_0 \in \Omega$, $R > 0$, $Q(2R) = Q(x_0, 2R) \subset\subset \Omega$. For every $(u, t) \in \mathbb{R}^n \times \mathbb{R}^{mn}$ let us define

$$\begin{aligned} A_R(u, t) &= \frac{1}{|\Omega \cap Q(R)|} \int_{\Omega \cap Q(R)} A(y, u, t) dy, \\ u_R &= \frac{1}{|\Omega \cap Q(R)|} \int_{\Omega \cap Q(R)} u(y) dy, \\ g_R &= \frac{1}{|\Omega \cap Q(R)|} \int_{\Omega \cap Q(R)} g(y) dy, \end{aligned}$$

and

$$A_0(\zeta) = A_R(u_R, g_R h(u_R) \zeta \zeta).$$

Now, let us consider the following “frozen functional”

$$\mathcal{A}_0(u) = \int_{Q(R)} A_0(Du) dx = \int_{Q(R)} A_R(u_R, g_R h(u_R) Du) dx.$$

Using formula (4.8) contained in [22] (see also [20] formula (2.9) for $p = 2$), we have

$$\begin{aligned} \int_{Q(R)} |Dw|^p dx &\leq \{ \mathcal{A}_0(u) - \mathcal{A}_0(v) \} = \\ &= \int_{Q(R)} [A_R(u_R, g_R h(u_R) Du) - A_R(u_R, g_R h(u_R) Dv)] dx. \end{aligned}$$

Adding and subtracting the terms

$$\begin{aligned} &A(x, u_R, g_R h(u_R) Du), \quad A(x, u, g_R h(u_R) Du) \\ &A(x, u, g(x) h(u_R) Du), \quad A(x, u, g(x) h(u) Du) \\ &A(x, u, g(x) h(v) Dv), \quad A(x, v, g(x) h(v) Dv) \\ &A(x, v, g(x) h(u_R) Dv), \quad A(x, v, g_R h(u_R) Dv) \end{aligned}$$

we obtain the following four kind of integrals.

$$\begin{aligned} \int_{Q(R)} |Dw|^p dx &\leq c \int_{Q(R)} H(Du) dx \left[\left(\frac{1}{|Q(R)|} \int_{Q(R)} \sigma(x - x_0, R)^{q'} dx \right)^{\frac{1}{q'}} \right. \\ &+ \left(\frac{1}{|Q(R)|} \int_{Q(R)} \omega(|u_R - u|^2)^{q'} dx \right)^{\frac{1}{q'}} + \left(\frac{1}{|Q(R)|} \int_{Q(R)} \omega(|u_R - v|^2)^{q'} dx \right)^{\frac{1}{q'}} \\ &\left. + \left(\frac{1}{|Q(R)|} \int_{Q(R)} |g_R - g(x)|^{q'} dx \right)^{\frac{1}{q'}} \right] = I + II + III + IV. \end{aligned}$$

Let us use the above mentioned regularity theorem by Uhlenbeck in the first part of I, in II and III Hölder, Jensen and Poincarè inequality and in IV the assumption on g , we have

$$\begin{aligned} \int_{Q(R)} |Du|^p dx &\leq C \left\{ \left(\frac{r}{R}\right)^\lambda + \left(\frac{1}{|Q(R)|} \int_{Q(R)} \sigma(x, R) dx\right)^{\frac{q-1}{q}} + \right. \\ &\left. + \omega \left(R^{p-m} \int_{Q(R)} |Du|^p dx \right)^{\frac{q-1}{q}} + \eta(g, R) \right\} \cdot \int_{Q(2R)} H(Du) dx. \end{aligned}$$

Furthermore recalling the VMO assumption we have

$$\frac{1}{|B(R)|} \int_{B(R)} \sigma(x, R) dx \rightarrow 0, \quad \eta(g, R) \rightarrow 0 \quad \text{as } R \rightarrow 0.$$

We conclude using ‘‘A useful lemma’’ and technical results contained in the [15], then the proof is ended.

Last treated argument is the following one that in recent years has been an increase interest in problems related to $p(x)$ -growth. They appear not only in purely mathematical context but also in some physical problems.

Zhikov in [57] treated a $p(x)$ -growth functional related to the thermistor problem, and got some results on the higher integrability of the gradient of minimizers.

Rajagopal and Ružička in [42] (see also [51]) proposed a model of electrorheological fluid, using equations with $p(x)$ -growth term.

Moreover, in the deep study of Acerbi and Mingione in [1] the authors obtain some regularity results for stationary electrorheological fluid.

Let us now consider

$$\mathcal{F}(u) = \int_{\Omega} (g^{\alpha\beta}(x) h_{ij}(u) D_{\alpha} u^i D_{\beta} u^j)^{p(x)/2} dx,$$

where $g^{\alpha\beta}(x)$, $h_{ij}(u)$ and $p(x)$ satisfies

(C1) *There exist positive constants $\lambda_g, \Lambda_g, \lambda_h, \Lambda_h$:*

$$\lambda_g |\zeta|^2 \leq g^{\alpha\beta}(x) \zeta_{\alpha} \zeta_{\beta} \leq \Lambda_g |\zeta|^2, \quad \lambda_h |\eta|^2 \leq h_{ij}(u) \eta^i \eta^j \leq \Lambda_h |\eta|^2$$

for all $x \in \Omega$, $\zeta \in \mathbb{R}^m$ and $u, \eta \in \mathbb{R}^n$;

(C2) *There exist constants $\exists 0 < \tau, \tau' < 1$, $0 < \sigma \leq 1$, L_p, L_g, L_h :*

$$\begin{aligned} |p(x) - p(y)| &\leq L_p |x - y|^{\sigma} / 2 =: \omega_p(|x - y|/2), \quad \forall x, y \in \Omega, \\ |g^{\alpha\beta}(x) - g^{\alpha\beta}(y)| &\leq L_g |x - y|^{\tau} =: \omega_g(|x - y|), \quad \forall x, y \in \Omega, \\ |h_{ij}(u) - h_{ij}(v)| &\leq L_h |u - v|^{\tau'} =: \omega_h(|u - v|^2), \quad \forall u, v \in \mathbb{R}^n; \end{aligned}$$

(C3) $2 \leq \gamma_1 := \inf_{x \in \Omega} p(x) \leq \sup_{x \in \Omega} p(x) =: \gamma_2 < +\infty$.

The research by Ragusa and Tachikawa continue by obtaining the papers [48], [49], [50]. In the first paper $g^{\alpha\beta}(x)$ are in the class VMO instead of assumption on g in (C2), Then, $u \in C^{0,\alpha}(\Omega_0)$ for some $\alpha \in (0, 1)$, where Ω_0 is an open subset of Ω with $\mathcal{H}^{m-\gamma_1}(\Omega \setminus \Omega_0) = 0$. Moreover, if $g^{\alpha\beta}(x)$ and $h_{ij}(u)$ are Hölder continuous, $u \in C^{1,\alpha'}(\Omega_0)$ for some $\alpha' \in (0, 1)$.

In the second paper the authors delete σ and add the ‘‘one-sided condition’’

$$\exists \lambda_h^* > 0 : -\frac{1}{2} u^k h_{ij,k}(u) \eta^i \eta^j \leq \lambda_h^* |\eta|^2$$

for all $u \in B_M(0)$ and $\eta \in \mathbb{R}^n$, where $h_{ij,k} = \partial h_{ij} \partial u^k$,

Then, $u \in C^{1,\alpha}(\Omega)$, for some $\alpha \in (0, 1)$.

In the last paper the authors prove that u is Hölder continuous near the boundary $\partial\Omega$.

For locally bounded minimizers, the $C^{1,\alpha}$ result contained in [48] has been improved by Tachikawa in [56] obtaining

$$\dim_{\mathcal{H}}(\Omega \setminus \Omega_0) < m - [\gamma_1] - 1,$$

where $[\cdot]$ stands for the Gauss symbol.

In [48] the technique to obtain higher integrability properties for local minimizers employs a direct approach as well as classical inequalities by Jensen and Poincarè. In [49] and [50] we gain the results by contradiction.

1 Acknowledgements

The author is supported by the Ministry of Education and Science of the Russian Federation (Agreement number N. 02.a03.21.0008).

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