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ON A THIRD ORDER DIFFERENCE EQUATION

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Abstract. In this paper, we solve the difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{-ax_n + bx_{n-2}}, \quad n = 0, 1, ...,$$

where a and b are positive real numbers and the initial values x_{-2}, x_{-1} and x_0 are real numbers. We find invariant sets and discuss the global behavior of the solutions of that equation. We show that when $a > \frac{4}{27}b^3$, under certain conditions there exist solutions, either periodic or converge to periodic solutions.

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1. Introduction

In their paper [9], the authors studied some special cases of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + Bx_n x_{n-1} + Cx_{n-1}}, \quad n = 0, 1, ...,$$

with nonnegative parameters and with arbitrary nonnegative initial conditions such that the denominator is always positive. In [15], Dehghan, et al. studied the global attractivity of the positive equilibrium of some special cases that contains at least one quaderatic term of the second order rational difference equations

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{\alpha x_n + \beta x_{n-1} + \gamma}, \quad n = 0, 1, ...,$$

which has quadratic terms in their numerators and linear terms in their denominators. In [17], the authors investigated the global behaviour of non-negative solutions of the rational difference equation with arbitrary delay and quadratic terms in its numerator:

$$x_{n+1} = \frac{Ax_n^2 + Bx_nx_{n-k} + Cx_{n-k}^2 + Dx_n + Ex_{n-k}}{\alpha x_n + \beta x_{n-k} + \gamma}, \quad n = 0, 1, ...,$$

with $k \in \{1, 2, ...\}$, where all parameters are non-negative, with $A+B+C+D+E > 0$ and $\gamma > 0$.

In [2], we have studied the behavior of the solutions of the difference equation

$$x_{n+1} = \frac{ax_nx_{n-1}}{-bx_n + cx_{n-2}}, \quad n = 0, 1, ...,$$

where a, b, c are positive real numbers and the initial conditions x_0, x_{-1}, x_{-2} are real numbers. Also, in [6] we have studied the global behavior of the fourth order difference equation

$$x_{n+1} = \frac{ax_nx_{n-2}}{-bx_n + cx_{n-3}}, \quad n = 0, 1, ...,$$

where a, b, c are positive real numbers and the initial conditions x_0, x_{-1}, x_{-2}, x_{-3} are real numbers. For more publications on global behavior of the solutions and forbidden sets, one can see [1]- [23].
In this paper, we shall determine the forbidden set, find the solution and investigate the behavior of the solutions of the equation

$$x_{n+1} = \frac{x_nx_{n-2}}{-ax_n + bx_{n-2}}, \quad n = 0, 1, ..., \tag{1}$$

where a and b are positive real numbers and the initial values x_{-2}, x_{-1} and x_0 are real numbers.

2. Solution of equation (1)

The reciprocal transformation

$$x_n = \frac{1}{y_n}$$

reduces equation (1) into the third order linear homogeneous difference equation

$$y_{n+1} - by_n + ay_{n-2} = 0, \quad n = 0, 1.... \tag{2}$$

The characteristic equation of equation (2) is

$$\lambda^3 - b\lambda^2 + a = 0. \tag{3}$$

Clear that equation (3) has a negative real root λ_0 for all values of $(a, b > 0)$. Therefore, the roots of equation (3) are

$$\lambda_0, \quad \lambda_\pm = -\frac{\lambda_0 - b}{2} \pm \frac{\sqrt{(\lambda_0 - b)^2 - 4\lambda_0(\lambda_0 - b)}}{2}.$$

The roots of equation (3) depends on the relation between a and b.

Lemma 1. *For equation (3), we have the following:*

1. *If $a > \frac{4}{27}b^3$, then equation (3) has one negative real root and two complex conjugate roots.*
2. *If $a = \frac{4}{27}b^3$, then equation (3) has one negative real root and a repeated positive real root.*
3. *If $a < \frac{4}{27}b^3$, then equation (3) has three real different roots, one of them is negative and two positive roots.*

Proof. It is sufficient to see that, the discriminant of the polynomial

$$p(\lambda) = \lambda^3 - b\lambda^2 + a = 0$$

is

$$\triangle = -4b^3 a + 27a^2.$$

We shall consider the three cases given in lemma (1).
Case $a > \frac{4}{27}b^3$:
When $a > \frac{4}{27}b^3$, the roots of equation (3) are

$$\lambda_0 < -\frac{b}{3}, \quad \lambda_\pm = -\frac{\lambda_0 - b}{2} \pm i\frac{\sqrt{4\lambda_0(\lambda_0 - b) - (\lambda_0 - b)^2}}{2}.$$

Then the solution of equation (1) is

$$x_n = \frac{1}{c_1\lambda_0^n + (\frac{-a}{\lambda_0})^{\frac{n}{2}}(c_2\cos n\varphi + c_3\sin n\varphi)}, \tag{4}$$

where

$$|\lambda_\pm| = \sqrt{\lambda_0(\lambda_0 - b)} = \sqrt{\frac{-a}{\lambda_0}} \quad \text{and} \quad \varphi = tan^{-1}(\sqrt{\frac{3\lambda_0 + b}{\lambda_0 - b}}) \in]0, \frac{\pi}{2}[.$$

Using the initials x_{-2}, x_{-1} and x_0, the values of c_1, c_2 and c_3 are:

$$\begin{array}{lll} c_1 & = & \frac{1}{\Delta_1}(c_{11}\frac{1}{x_0}+c_{12}\frac{1}{x_{-1}}+c_{13}\frac{1}{x_{-2}}), \\ c_2 & = & \frac{1}{\Delta_1}(c_{21}\frac{1}{x_0}+c_{22}\frac{1}{x_{-1}}+c_{23}\frac{1}{x_{-2}}) \\ \text{and} & & \\ c_3 & = & \frac{1}{\Delta_1}(c_{31}\frac{1}{x_0}+c_{32}\frac{1}{x_{-1}}+c_{33}\frac{1}{x_{-2}}), \end{array} \tag{5}$$

where

$$\begin{array}{l} c_{11}=\frac{\lambda_0}{a}\sqrt{-\frac{\lambda_0}{a}}\sin\varphi, \quad c_{12}=-\frac{\lambda_0}{a}\sin 2\varphi, \quad c_{13}=-\sqrt{-\frac{\lambda_0}{a}}\sin\varphi, \\ c_{21}=-\frac{1}{a}\sin 2\varphi-\frac{1}{\lambda_0^2}\sqrt{-\frac{\lambda_0}{a}}\sin\varphi, \quad c_{22}=\frac{\lambda_0}{a}\sin 2\varphi, \quad c_{23}=\sqrt{-\frac{\lambda_0}{a}}\sin\varphi, \\ c_{31}=-\frac{1}{a}\cos 2\varphi-\frac{1}{\lambda_0^2}\sqrt{-\frac{\lambda_0}{a}}\cos\varphi, \quad c_{32}=\frac{\lambda_0}{a}\cos 2\varphi+\frac{1}{\lambda_0^2}, \quad c_{33}=\sqrt{-\frac{\lambda_0}{a}}\cos\varphi-\frac{1}{\lambda_0} \end{array} \tag{6}$$

and

$$\Delta_1=\begin{vmatrix} 1 & 1 & 0 \\ \frac{1}{\lambda_0} & \sqrt{-\frac{\lambda_0}{a}}\cos\varphi & -\sqrt{-\frac{\lambda_0}{a}}\sin\varphi \\ \frac{1}{\lambda_0^2} & -\frac{\lambda_0}{a}\cos 2\varphi & \frac{\lambda_0}{a}\sin 2\varphi \end{vmatrix}. \tag{7}$$

By simple calculations, we can write the solution of equation (1) as

$$x_n=\frac{1}{\frac{\alpha_{1n}}{x_0}+\frac{\alpha_{2n}}{x_{-1}}+\frac{\alpha_{3n}}{x_{-2}}}, \tag{8}$$

where

$$\begin{array}{lll} \alpha_{1n} & = & \frac{1}{\Delta_1}(c_{11}\lambda_0^n+c_{21}(\frac{-a}{\lambda_0})^{\frac{n}{2}}\cos n\varphi+c_{31}(\frac{-a}{\lambda_0})^{\frac{n}{2}}\sin n\varphi), \\ \alpha_{2n} & = & \frac{1}{\Delta_1}(c_{12}\lambda_0^n+c_{22}(\frac{-a}{\lambda_0})^{\frac{n}{2}}\cos n\varphi+c_{32}(\frac{-a}{\lambda_0})^{\frac{n}{2}}\sin n\varphi) \\ \text{and} & & \\ \alpha_{3n} & = & \frac{1}{\Delta_1}(c_{13}\lambda_0^n+c_{23}(\frac{-a}{\lambda_0})^{\frac{n}{2}}\cos n\varphi+c_{33}(\frac{-a}{\lambda_0})^{\frac{n}{2}}\sin n\varphi) \end{array} \tag{9}$$

are such that $c_{ij},\ i,j=1,2,3$ are given in (6).

Case $a=\frac{4}{27}b^3$:

When $a=\frac{4}{27}b^3$, equation (3) has a negative root $\lambda_0=-\frac{b}{3}$ and a repeated positive root $\frac{2b}{3}$.

Then the solution of equation (1) is

$$x_n=\frac{1}{c_1(-\frac{b}{3})^n+c_2(\frac{2b}{3})^n+c_3(\frac{2b}{3})^n n}. \tag{10}$$

Using the initials x_{-2}, x_{-1} and x_0, the values of c_1, c_2 and c_3 in this case are:

$$\begin{aligned} c_1 &= \tfrac{1}{\Delta_2}(c_{11}\tfrac{1}{x_0} + c_{12}\tfrac{1}{x_{-1}} + c_{13}\tfrac{1}{x_{-2}}), \\ c_2 &= \tfrac{1}{\Delta_2}(c_{21}\tfrac{1}{x_0} + c_{22}\tfrac{1}{x_{-1}} + c_{23}\tfrac{1}{x_{-2}}) \\ \text{and} & \\ c_3 &= \tfrac{1}{\Delta_2}(c_{31}\tfrac{1}{x_0} + c_{32}\tfrac{1}{x_{-1}} + c_{33}\tfrac{1}{x_{-2}}), \end{aligned} \tag{11}$$

where

$$\begin{aligned} &c_{11} = -\frac{27}{8b^3}, \quad c_{12} = \frac{9}{2b^2}, \quad c_{13} = -\frac{3}{2b}, \\ &c_{21} = -\frac{27}{b^3}, \quad c_{22} = -\frac{9}{2b^2}, \quad c_{23} = \frac{3}{2b}, \\ &c_{31} = -\frac{81}{4b^3}, \quad c_{32} = \frac{27}{4b^2}, \quad c_{33} = \frac{9}{2b} \end{aligned} \tag{12}$$

and

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 0 \\ (-\frac{3}{b}) & (\frac{3}{2b}) & -(\frac{3}{2b}) \\ (-\frac{3}{b})^2 & (\frac{3}{2b})^2 & -2(\frac{3}{2b})^2 \end{vmatrix}.$$

By simple calculations, we can write the solution of equation (1) in this case as

$$x_n = \frac{1}{\frac{\alpha_{1n}}{x_0} + \frac{\alpha_{2n}}{x_{-1}} + \frac{\alpha_{3n}}{x_{-2}}}, \tag{13}$$

where

$$\begin{aligned} \alpha_{1n} &= \tfrac{1}{\Delta_2}(c_{11}(-\tfrac{b}{3})^n + c_{21}(\tfrac{2b}{3})^n + c_{31}(\tfrac{2b}{3})^n n, \\ \alpha_{2n} &= \tfrac{1}{\Delta_2}(c_{12}(-\tfrac{b}{3})^n + c_{22}(\tfrac{2b}{3})^n + c_{32}(\tfrac{2b}{3})^n n \\ \text{and} & \\ \alpha_{3n} &= \tfrac{1}{\Delta_2}(c_{13}(-\tfrac{b}{3})^n + c_{23}(\tfrac{2b}{3})^n + c_{33}(\tfrac{2b}{3})^n n \end{aligned} \tag{14}$$

are such that c_{ij}, $i, j = 1, 2, 3$ are given in (12).

Case $a < \frac{4}{27}b^3$:

When $a < \frac{4}{27}b^3$, the roots of equation (3) are

$$\lambda_0 > -\frac{b}{3}, \quad \lambda_\pm = -\frac{\lambda_0 - b}{2} \pm \frac{\sqrt{(\lambda_0 - b)^2 - 4\lambda_0(\lambda_0 - b)}}{2},$$

where

$$\lambda_+ > \lambda_- > |\lambda_0| > 0.$$

Then the solution of equation (1) is

$$x_n = \frac{1}{c_1\lambda_0^n + c_2\lambda_-^n + c_3\lambda_+^n}. \tag{15}$$

Using the initials x_{-2}, x_{-1} and x_0, the values of c_1, c_2 and c_3 in this case are:

$$\begin{array}{lcl} c_1 & = & \frac{1}{\Delta_3}(c_{11}\frac{1}{x_0} + c_{12}\frac{1}{x_{-1}} + c_{13}\frac{1}{x_{-2}}), \\ c_2 & = & \frac{1}{\Delta_3}(c_{21}\frac{1}{x_0} + c_{22}\frac{1}{x_{-1}} + c_{23}\frac{1}{x_{-2}}) \\ \text{and} & & \\ c_3 & = & \frac{1}{\Delta_3}(c_{31}\frac{1}{x_0} + c_{32}\frac{1}{x_{-1}} + c_{33}\frac{1}{x_{-2}}), \end{array} \tag{16}$$

where

$$\begin{array}{lll} c_{11} = \dfrac{\lambda_- - \lambda_+}{\lambda_-^2\lambda_+^2}, & c_{12} = \dfrac{-\lambda_-^2 + \lambda_+^2}{\lambda_-^2\lambda_+^2}, & c_{13} = \dfrac{\lambda_- - \lambda_+}{\lambda_-\lambda_+}, \\ c_{21} = \dfrac{\lambda_+ - \lambda_0}{\lambda_+^2\lambda_0^2}, & c_{22} = \dfrac{\lambda_0^2 - \lambda_+^2}{\lambda_+^2\lambda_0^2}, & c_{23} = \dfrac{\lambda_+ - \lambda_0}{\lambda_+\lambda_0}, \\ c_{31} = \dfrac{\lambda_0 - \lambda_-}{\lambda_0^2\lambda_-^2}, & c_{32} = \dfrac{\lambda_-^2 - \lambda_0^2}{\lambda_0^2\lambda_-^2}, & c_{33} = \dfrac{\lambda_0 - \lambda_-}{\lambda_0\lambda_-} \end{array} \tag{17}$$

and

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{\lambda_0} & \frac{1}{\lambda_-} & \frac{1}{\lambda_+} \\ \frac{1}{\lambda_0^2} & \frac{1}{\lambda_-^2} & \frac{1}{\lambda_+^2} \end{vmatrix}.$$

By simple calculations, we can write the solution of equation (1) in this case as

$$x_n = \frac{1}{\frac{\alpha_{1n}}{x_0} + \frac{\alpha_{2n}}{x_{-1}} + \frac{\alpha_{3n}}{x_{-2}}}, \tag{18}$$

where

$$\begin{array}{lcl} \alpha_{1n} & = & \frac{1}{\Delta_3}(c_{11}\lambda_0^n + c_{21}\lambda_-^n + c_{31}\lambda_+^n), \\ \alpha_{2n} & = & \frac{1}{\Delta_3}(c_{12}\lambda_0^n + c_{22}\lambda_-^n + c_{32}\lambda_+^n) \\ \text{and} & & \\ \alpha_{3n} & = & \frac{1}{\Delta_3}(c_{13}\lambda_0^n + c_{23}\lambda_-^n + c_{33}\lambda_+^n) \end{array} \tag{19}$$

are such that c_{ij}, $i, j = 1, 2, 3$ are given in (17).

Using equations (8), (13) and (18), we can write the forbidden set of equation (1) as

$$F = \bigcup_{n=-2}^{\infty} \{(x_0, x_{-1}, x_{-2}) \in \mathbb{R}^3 : \frac{\alpha_{1n}}{x_0} + \frac{\alpha_{2n}}{x_{-1}} + \frac{\alpha_{3n}}{x_{-2}} = 0\},$$

where α_{1n}, α_{2n} and α_{3n} are given as follows:

$$\begin{cases} \alpha_{1n}, \alpha_{2n} \text{ and } \alpha_{3n} \text{ are given in (9)}, & a > \frac{4}{27}b^3; \\ \alpha_{1n}, \alpha_{2n} \text{ and } \alpha_{3n} \text{ are given in (14)}, & a = \frac{4}{27}b^3; \\ \alpha_{1n}, \alpha_{2n} \text{ and } \alpha_{3n} \text{ are given in (19)}, & a < \frac{4}{27}b^3. \end{cases}$$

3. Global behavior of equation (1)

Consider the set

$$D = \{(x, y, z) \in \mathbb{R}^3 : \frac{\lambda^2}{x} - \frac{a}{y} - \frac{a\lambda}{z} = 0\},$$

with

$$\begin{cases} \lambda = \lambda_0, & a > \frac{4}{27}b^3 \ ; \\ \lambda = -\frac{b}{3}, & a = \frac{4}{27}b^3. \end{cases}$$

Clear that, when $a = \frac{4}{27}b^3$, the set D can be written as

$$D = \{(x, y, z) \in \mathbb{R}^3 : \frac{9}{x} - \frac{12b}{y} + \frac{4b^2}{z} = 0\}.$$

Note that, for the point $(x, y, z) \in \mathbb{R}^3$, the relation $\frac{\lambda_0^2}{x} + \frac{a}{y} + \frac{a\lambda_0}{z} = 0$ is equivalent to $c_1(x, y, z) = 0$, where c_1 is given by either (5) or (11) according to the relations $a > \frac{4}{27}b^3$ and $a = \frac{4}{27}b^3$ respectively.

Theorem 2. *The set D is an invariant for equation (1).*

Proof. Let $(x_0, x_{-1}, x_{-2}) \in D$. We show that $(x_k, x_{k-1}, x_{k-2}) \in D$ for each $k \in N$. The proof is by induction on k. The point $(x_0, x_{-1}, x_{-2}) \in D$, implies

$$\frac{\lambda_0^2}{x_0} - \frac{a}{x_{-1}} - \frac{a\lambda_0}{x_{-2}} = 0.$$

Now for $k = 1$, we have

$$\frac{\lambda_0^2}{x_1} - \frac{a}{x_0} - \frac{a\lambda_0}{x_{-1}} = \frac{\lambda_0^2}{x_0x_{-2}}(-ax_0 + bx_{-2}) - \frac{a}{x_0} - \frac{a\lambda_0}{x_{-1}}$$

$$= \frac{1}{x_0x_{-1}x_{-2}}(-a\lambda_0^2x_0x_{-1} + b\lambda_0^2x_{-1}x_{-2} - ax_{-1}x_{-2} - a\lambda_0x_0x_{-2})$$

$$= \frac{1}{x_0x_{-1}x_{-2}}(-a\lambda_0^2x_0x_{-1} + (\lambda_0^2b - a)x_{-1}x_{-2} - a\lambda_0x_0x_{-2})$$

$$= \frac{1}{x_0x_{-1}x_{-2}}(-a\lambda_0^2x_0x_{-1} + \lambda_0^3x_{-1}x_{-2} - a\lambda_0x_0x_{-2})$$

$$= \lambda_0(\frac{\lambda_0^2}{x_0} - \frac{a}{x_{-1}} - \frac{a\lambda_0}{x_{-2}}) = 0.$$

This implies that $(x_1, x_0, x_{-1}) \in D$.
Suppose that the $(x_k, x_{k-1}, x_{k-2}) \in D$. That is

$$\frac{\lambda_0^2}{x_k} - \frac{a}{x_{k-1}} - \frac{a\lambda_0}{x_{k-2}} = 0.$$

Then

$$\frac{\lambda_0^2}{x_{k+1}} - \frac{a}{x_k} - \frac{a\lambda_0}{x_{k-1}} = \frac{\lambda_0^2}{x_kx_{k-2}}(-ax_k + bx_{k-2}) - \frac{a}{x_k} - \frac{a\lambda_0}{x_{k-1}}$$

$$= \frac{1}{x_kx_{k-1}x_{k-2}}(-a\lambda_0^2x_kx_{k-1} + b\lambda_0^2x_{k-1}x_{k-2} - ax_{k-1}x_{k-2} - a\lambda_0x_kx_{k-2})$$

$$= \frac{1}{x_kx_{k-1}x_{k-2}}(-a\lambda_0^2x_kx_{k-1} + (\lambda_0^2b - a)x_{k-1}x_{k-2} - a\lambda_0x_kx_{k-2})$$

$$= \frac{1}{x_kx_{k-1}x_{k-2}}(-a\lambda_0^2x_kx_{k-1} + \lambda_0^3x_{k-1}x_{k-2} - a\lambda_0x_kx_{k-2})$$

$$= \lambda_0(\frac{\lambda_0^2}{x_k} - \frac{a}{x_{k-1}} - \frac{a\lambda_0}{x_{k-2}}) = 0.$$

Therefore, $(x_{k+1}, x_k, x_{k-1}) \in D$.
This completes the proof.

Now assume that $a < \frac{4}{27}b^3$. We shall consider the three sets

$$D_i = \{(x, y, z) \in \mathbb{R}^3 : \frac{\lambda^2}{x} - \frac{a}{y} - \frac{a\lambda}{z} = 0\}, \quad i = 1, 2, 3,$$

with

$$\begin{cases} \lambda = \lambda_0, & \text{i=1;} \\ \lambda = \lambda_-, & \text{i=2;} \\ \lambda = \lambda_+, & \text{i=3.} \end{cases}$$

By simple calculations, we can see that:

$$\begin{cases} D_i \text{ is equivalent to } c_1(x, y, z) = 0, & \text{i=1;} \\ D_i \text{ is equivalent to } c_2(x, y, z) = 0, & \text{i=2;} \\ D_i \text{ is equivalent to } c_1(x, y, z) = 0, & \text{i=3,} \end{cases}$$

where c_i, $i = 1$, 2 and 3 are given by (16).

Theorem 3. *Each set of the sets D_i, $i = 1$, 2 and 3 is an invariant for equation (1).*

Proof. The proof is similar to that of theorem (2) and will be omitted.

Theorem 4. *Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}, x_{-2}) \notin F \cup D$. If $a > \frac{4}{27}b^3$, then we have the following:*

1. *If $a \geq b+1$, then $\{x_n\}_{n=-2}^{\infty}$ converges to zero.*
2. *If $a < b+1$, then we have the following:*
   - (a) *If $a \geq 1$, then $\{x_n\}_{n=-2}^{\infty}$ converges to zero.*
   - (b) *If $a < 1$, then we have the following:*
     - i. *If $a^2 + ab - 1 > 0$, then $\{x_n\}_{n=-2}^{\infty}$ converges to zero.*
     - ii. *If $a^2 + ab - 1 = 0$, then $\{x_n\}_{n=-2}^{\infty}$ is bounded.*
     - iii. *If $a^2 + ab - 1 < 0$, then $\{x_n\}_{n=-2}^{\infty}$ is unbounded.*

Proof. The solution of equation (1) when $a > \frac{4}{27}b^3$ is

$$x_n = \frac{1}{c_1\lambda_0^n + (-\frac{a}{\lambda_0})^{\frac{n}{2}}(c_2 \cos n\varphi + c_3 \sin n\varphi))}.$$

1. When $a > b+1$, we have $-a < -\sqrt[3]{a} < \lambda_0 < -1$. That is $(\frac{-a}{\lambda_0})^n \to \infty$ and λ_0^n is unbounded.
   If $a = b+1$, then we have that $-a < -\sqrt[3]{a} < \lambda_0 = -1$. That is $(\frac{-a}{\lambda_0})^n \to \infty$ as $n \to \infty$ and the result follows.
2. When $a < b+1$, we have that $\lambda_0 > -1$.
   - (a) If $a \geq 1$, then $-a \leq -\sqrt[3]{a} \leq -1 < \lambda_0$. That is $(\frac{-a}{\lambda_0})^n \to \infty$, from which the result follows.
   - (b) If $a < 1$, then $a < \sqrt[3]{a}$ and we have the following:
     - i. If $a^2 + ab - 1 > 0$, then $\lambda_0 > -a > -\sqrt[3]{a} > -1$. This implies that $\lambda_0^n \to 0$ and $(\frac{-a}{\lambda_0})^n \to \infty$, from which the result follows.
     - ii. If $a^2 + ab - 1 = 0$, then $\lambda_0 = -a > -\sqrt[3]{a} > -1$. That is $\lambda_0^n \to 0$. But as

$$|c_1\lambda_0^n + c_2 \cos n\varphi + c_3 \sin n\varphi| \neq 0 \text{ for all } n \geq 0, \tag{20}$$

the quantity (20) attains its infemum value say $\epsilon > 0$ and the result follows.

iii. If $a^2+ab-1<0$, then $-a>\lambda_0>-\sqrt[3]{a}>-1$. This implies that $\lambda_0^n \to 0$ and $(\frac{-a}{\lambda_0})^n \to 0$, from which the result follows.

Theorem 5. *Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}, x_{-2}) \notin F \cup D$. If $a=\frac{4}{27}b^3$, then we have the following:*

1. *If $a \geq b+1$, then $\{x_n\}_{n=-2}^{\infty}$ converges to zero.*
2. *If $a < b+1$, then we have the following:*
   (a) *If $0<b<\frac{3}{2}$, then $\{x_n\}_{n=-2}^{\infty}$ is unbounded.*
   (b) *If $\frac{3}{2} \leq b < 3$, then $\{x_n\}_{n=-2}^{\infty}$ converges to zero.*

Proof. The solution of equation (1) when $a=\frac{4}{27}b^3$ is

$$x_n = \frac{1}{c_1(-\frac{b}{3})^n + c_2(\frac{2b}{3})^n + c_3(\frac{2b}{3})^n n}.$$

1. When $a \geq b+1$, it is sufficient to see that $\lambda_0 = -\frac{b}{3} \leq -1$ and the result follows.
2. When $a < b+1$, we have that $\lambda_0 = -\frac{b}{3} > -1$.
   (a) If $0<b<\frac{3}{2}$, then $\frac{b}{3}<\frac{1}{2}$ and $\frac{2b}{3}<1$, from which the result follows.
   (b) If $\frac{3}{2} \leq b < 3$, then $\frac{1}{2} \leq \frac{b}{3} \leq 1$ and $1 \leq \frac{2b}{3} \leq 2$, from which the result follows.

Theorem 6. *Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}, x_{-2}) \notin F \cup D_3$. If $a<\frac{4}{27}b^3$, then we have the following:*

1. *If $a > -1+b$, then we have the following:*
   (a) *If $0<b<\frac{3}{2}$, then $\{x_n\}_{n=-2}^{\infty}$ is unbounded.*
   (b) *If $b>\frac{3}{2}$, then $\{x_n\}_{n=-2}^{\infty}$ converges to zero.*
2. *If $a=-1+b$, then we have the following:*
   (a) *If $1 \leq b < \frac{3}{2}$, then $\{x_n\}_{n=-2}^{\infty}$ converges to the $\frac{1}{c_3}$.*
   (b) *If $b>\frac{3}{2}$, then $\{x_n\}_{n=-2}^{\infty}$ converges to zero.*

3. *If* $a < -1 + b$*, then* $\{x_n\}_{n=-2}^{\infty}$ *converges to zero.*

Proof. Let $f(\lambda) = \lambda^3 - b\lambda^2 + a$. It is clear that $f(\lambda)$ is increasing on $]-\infty[, 0 \cup]\frac{2b}{3}, \infty[$ and decreasing on $]0, \frac{2b}{3}[$. The solution of equation (1) when $a < \frac{4}{27}b^3$ is

$$x_n = \frac{1}{c_1\lambda_0^n + c_2\lambda_-^n + c_3\lambda_+^n}.$$

We have also

$$0 < |\lambda_0| < \lambda_- < \frac{2b}{3} < \lambda_+.$$

The condition $(x_0, x_{-1}, x_{-2}) \notin F \cup D_3$ ensures that $c_3 \neq 0$.

1. When $a > -1 + b$, we have two cases:
   (a) If $0 < b < \frac{3}{2}$, then $\frac{2b}{3} < \lambda_+ < 1$ (otherwise $a < -1 + b$, which is a contradiction). Then $0 < |\lambda_0| < \lambda_- < \frac{2b}{3} < \lambda_+ < 1$, from which the result follows.
   (b) If $b > \frac{3}{2}$, then $1 < \lambda - < \frac{2b}{3} < \lambda_+$ and the result follows.
2. If $a = -1 + b$, then either $\lambda_- = 1$ or $\lambda_+ = 1$.
   (a) If $1 \leq b < \frac{3}{2}$, then $\lambda_+ = 1$. That is $0 < |\lambda_0| < \lambda - < \frac{2b}{3} < \lambda_+ = 1$. Then

$$x_n = \frac{1}{c_1\lambda_0^n + c_2\lambda_-^n + c_3} \to \frac{1}{c_3} \text{ as } n \to \infty.$$

   (b) If $b > \frac{3}{2}$, then we have $0 < |\lambda_0| < \lambda - = 1 < \frac{2b}{3} < \lambda_+$, from which the result follows.
3. If $a < -1 + b$, then $\lambda_- < 1 < \lambda_+$. That is $\lambda_+^n \to \infty$ and the result follows.

In the following results, we show that when $a > \frac{4}{27}b^3$, under certain conditions there exist solutions, either periodic or converge to periodic solutions for equation (1).
Suppose that $\varphi = \frac{p}{q}\pi$, where p and q are positive relatively prime integers such that $0 < p < \frac{q}{2}$.

Theorem 7. *Assume that* $a > \frac{4}{27}b^3$*,* $a < b + 1$*. Let* $\{x_n\}_{n=-2}^{\infty}$ *be a solution of equation (1) such that* $(x_0, x_{-1}, x_{-2}) \notin D \cup F$*. If* $a^2 + ba - 1 = 0$*, then* $\{x_n\}_{n=-2}^{\infty}$ *converges to a periodic solution with prime period* $2q$*.*

Proof. Assume that $\{x_n\}_{n=-2}^{\infty}$ is a solution of equation (1) such that $(x_0, x_{-1}, x_{-2}) \notin D \cup F$ and let the angle $\varphi = \frac{p}{q}\pi \in]0, \frac{\pi}{2}[$.
When $a > \frac{4}{27}b^3$ and $a^2 + ba - 1 = 0$ ($\lambda_0 = -a > -1$), the solution of equation (1) is

$$x_n = \frac{1}{c_1\lambda_0^n + c_2\cos n\varphi + c_3\sin n\varphi}.$$

Then we can write

$$\begin{aligned} x_{2qm+l} &= \frac{1}{c_1\lambda_0^{2qm+l} + c_2\cos(2qm+l)\varphi + c_3\sin(2qm+l)\varphi} \\ &= \frac{1}{c_1\lambda_0^{2qm+l} + c_2\cos l\varphi + c_3\sin l\varphi},\ l = 1, 2, ..., 2q. \end{aligned}$$

As $m \to \infty$, we get

$$x_{2qm+l} \to \mu_l = \frac{1}{c_2\cos l\varphi + c_3\sin l\varphi},\ l = 1, 2, ..., 2q.$$

Therefore, the solution $\{x_n\}_{n=-2}^{\infty}$ converges to

$$\{..., \mu_1, \mu_2, ..., \mu_{2q-1}, \mu_{2q}, \mu_1, \mu_2, ..., \mu_{2q-1}, \mu_{2q}, ...\}. \tag{21}$$

Simple calculations show that the solution (21) is a period-$2q$ solution for equation (1) and will be omitted.
This completes the proof.

Theorem 8. *Assume that $a > \frac{4}{27}b^3$, $a < b+1$ and $a^2 + ba - 1 = 0$. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of equation (1) such that $(x_0, x_{-1}, x_{-2}) \notin F$. If $(x_0, x_{-1}, x_{-2}) \in D$, then $\{x_n\}_{n=-2}^{\infty}$ is a periodic solution with prime period $2q$.*

Proof. Assume that $\{x_n\}_{n=-2}^{\infty}$ is a solution of equation (1) such that $(x_0, x_{-1}, x_{-2}) \notin F$ and let the angle $\varphi = \frac{p}{q}\pi \in]0, \frac{\pi}{2}[$.
When $(x_0, x_{-1}, x_{-2}) \in D$, we have that $c_1 = 0$ and the solution of equation (1) is

$$x_n = \frac{1}{c_2\cos n\varphi + c_3\sin n\varphi}.$$

Then we have

$$\begin{aligned} x_{n+2q} &= \frac{1}{c_2\cos(n+2q)\varphi + c_3\sin(n+2q)\varphi} \\ &= \frac{1}{c_2\cos(n\varphi + 2p\pi) + c_3\sin(n\varphi + 2p\pi)} \\ &= \frac{1}{c_2\cos(n\varphi) + c_3\sin(n\varphi)} \\ &= x_n. \end{aligned}$$

This completes the proof.

Example (1) Figure 1. shows that if $a = b = \frac{1}{\sqrt{2}}$, ($a > \frac{4}{27}b^3$, $a < b+1$, $a^2+ab-1 = 0$ and $\varphi = \frac{1}{4}\pi$), then a solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1) with initial conditions $x_{-2} = 2$, $x_{-1} = 0.1$ and $x_0 = 1$ converges to a period-8 solution.

Example (2) Figure 2. shows that if $a = \frac{1}{\sqrt{3}}$, $b = \frac{2}{\sqrt{3}}$ ($a > \frac{4}{27}b^3$, $a < b+1$, $a^2+ab-1 = 0$ and $\varphi = \frac{1}{6}\pi$), then a solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1) with initial conditions $x_{-2} = -\frac{1}{3}$, $x_{-1} = -\frac{\sqrt{3}}{2}$ and $x_0 = 1$ ($(x_{-2}, x_{-1}, x_0) \in D$) is periodic with prime period-12 solution.

Example (3) Figure 3. shows that if $a = b = 1$, ($a > \frac{4}{27}b^3$, $a < b+1$, $a^2+ab-1 > 0$, then a solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1) with initial conditions $x_{-2} = -0.2$, $x_{-1} = 2.1$ and $x_0 = 2.82$ converges to zero.

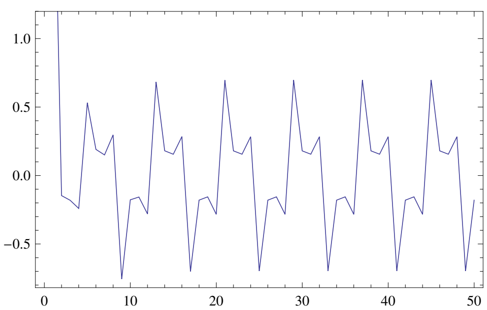

Figure 1: $x_{n+1} = \frac{x_n x_{n-2}}{-\frac{1}{\sqrt{2}}x_n + \frac{1}{\sqrt{2}}x_{n-2}}$

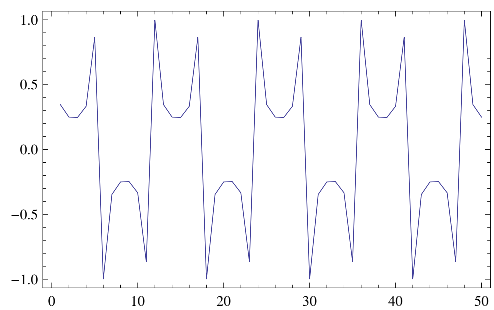

Figure 2: $x_{n+1} = \frac{x_n x_{n-2}}{-\frac{1}{\sqrt{3}}x_n + \frac{2}{\sqrt{3}}x_{n-2}}$

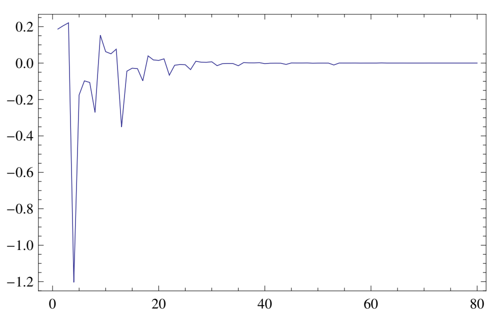

Figure 3: $x_{n+1} = \frac{x_n x_{n-2}}{-x_n + x_{n-2}}$

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