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FRACTIONAL DERIVATIVE OPERATOR FOR P-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. In this paper, a differential operator $D^m_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z)$ defined in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ is introduced. By using this operator, we introduce a new subclass of analytic functions $G^p_n(\alpha,\beta,\mu,\lambda,\gamma,\delta)$. Moreover, we discuss coefficient inequality, Hadamand product, growth and distortion theorems, closure theorems, radii of close-to-convexity, starlikeness, convexity and integral operators. Furthermore, we give an application involving fractional calculus for functions in $G^p_n(\alpha,\beta,\mu,\lambda,\gamma,\delta)$.

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1. Introduction and preliminaries

Let $\mathcal{A}_p$ be the class of function $f(z)$ of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \qquad (a_k \geq 0, z \in \mathbb{U},\ p \in \mathbb{N} = 1,2,3,...) \tag{1}$$

which are analytic in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For a function f in $\mathcal{A}_p$, we define the following differential operator

$$D^0_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z) = f(z), \tag{2}$$

$$D^1_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z) = \left(\frac{\mu+\lambda-(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right)f(z) + \left(\frac{(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right)zf'(z), \tag{3}$$

and for $m = 1,2,3,\cdots$

$$D^m_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z) = D_{p,\mu,\lambda,\sigma}(D^{M-1}_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z)). \tag{4}$$

If f is given by (1), then from (4) we get

$$D^m_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z) = z^p + \sum_{k=p+1}^{\infty}\left(\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right)^m a_k z^k, \qquad (5)$$

for $f \in \mathcal{A}_p$, $\sigma,\alpha \geq 0$, $\beta,\mu,\lambda > 0, \lambda \neq \alpha$, $p \in \mathbb{N}$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We observe that the generalized differential operator $D^m_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z)$ reduces to several other differential operators considered earlier for different choices of $\mu,\lambda,\sigma,\alpha$ and β:

(i)$D^m_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z)$ when $p = 1$, we have the operator introduced and studied by Amourah and Yousef [1].

(ii)$D^m_{1,0,1,\sigma}(\alpha,\beta)f(z)$, we have the operator introduced and studied by Aljarah and Darus [2].

(iii) $D^m_{1-\lambda,\lambda,\sigma}(\alpha,\beta)f(z) = z + \sum\limits_{n=2}^{\infty}[1+(n-1)(\lambda-\alpha)(\beta-\sigma)]^m a_n z^n$ was introduced and studied by Ramadan and Darus [4];

(iv) $D^m_{1-\lambda,\lambda,0}(\alpha,\beta)f(z) = z + \sum\limits_{n=2}^{\infty}[1+(n-1)(\lambda-\alpha)\beta]^m a_n z^n$ was introduced and studied by Darus and Ibrahim [3];

(v) $D^m_{\mu,\lambda,0}(0,1)f(z) = z + \sum\limits_{n=2}^{\infty}\left[\frac{\mu+\lambda n}{\mu+\lambda}\right]^m a_n z^n$ was introduced and studied by Swamy [7];

(vi) $D^m_{1-\lambda,\lambda,0}(0,1)f(z) = z + \sum\limits_{n=2}^{\infty}[1+(n-1)\lambda]^m a_n z^n$ was introduced by Al-Oboudi [8];

(vii)$D^m_{0,1,0}(0,1)f(z) = z + \sum\limits_{n=2}^{\infty} n^m a_n z^n$ was introduced and studied by Salagean [6].

Another differential that might be of interests can be seen in [5].

Let $G^p_n(\alpha,\beta,\mu,\lambda,\gamma,\delta)$ denote the subclass of $\mathcal{A}_p$ consisting of functions f which satisfy

$$\operatorname{Re}\left\{z\frac{\gamma z^2(D^m_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z))''' + (2\gamma+1)z(D^m_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z))'' + (D^m_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z))'}{\gamma z^2(D^m_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z))'' + z(D^m_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z))'}\right\} > \delta, \qquad (6)$$

where $D^m_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z)$ is given by (5), $\gamma(0 \leq \gamma \leq 1)$, $\delta(0 \leq \delta < 1)$ and for all $z \in U$.

We note that

(i) $G_n^p(\alpha,\beta,0,0,\gamma,\delta)$ was introduced by Xiao-Fei and An-Ping [9];
(ii) $G_n^1(\alpha,\beta,0,0,\gamma,\delta)$ was introduced by Kamali and Akbulut [10].

2. Coefficient Inequality for the Class $G_n^p(\alpha,\beta,\mu,\lambda,\gamma,\delta)$

We begin with our first result as follows:

Theorem 1. *A function $f \in \mathcal{A}$ is in the class $G_n^p(\alpha,\beta,\mu,\lambda,\gamma,\delta)$ if and only if*

$$\sum_{k=p+1}^{\infty} k\,(k-\delta)\,[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m a_k \le p(p-\delta)\,[\gamma p-\gamma+1]\,. \tag{7}$$

The result (7) is sharp.

Proof. Assume that $f \in G_n^p(\alpha,\beta,\mu,\lambda,\gamma,\delta)$. Then we find from (6) that

$$\operatorname{Re}\left\{\frac{p^2\,[\gamma p-\gamma+1]-\sum\limits_{k=p+1}^{\infty} k^2\,[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m a_k z^{k-p}}{p\,[\gamma p-\gamma+1]-\sum\limits_{k=p+1}^{\infty} k\,[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m a_k z^{k-p}}\right\} > \delta.$$

If we choose z to be the real and let $z \to 1^-$, we get

$$p^2\,[\gamma p-\gamma+1]-\sum_{k=p+1}^{\infty} k^2\,[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m a_k$$

$$\ge \delta p\,[\gamma p-\gamma+1]-\sum_{k=p+1}^{\infty} \delta k\,[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m a_k$$

or

$$\sum_{k=p+1}^{\infty} k\,(k-\delta)\,[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m a_k \le p(p-\delta)\,[\gamma p-\gamma+1]\,,$$

which is equivalent to (7). Conversely, assume that (7) is true. Then we have

$$\left| z\frac{\gamma z^2(D^m_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z))''' + (2\gamma+1)z(D^m_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z))'' + (D^m_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z))'}{\gamma z^2(D^m_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z))'' + z(D^m_{p,\mu,\lambda,\sigma}(\alpha,\beta)f(z))'} - 1\right| =$$

$$\left|\frac{p(p-1)\,[\gamma p-\gamma+1]-\sum\limits_{k=p+1}^{\infty} k(k-1)\,[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m a_k z^{k-p}}{p\,[\gamma p-\gamma+1]-\sum\limits_{k=p+1}^{\infty} k\,[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m a_k z^{k-p}}\right| \le$$

$$\frac{\sum\limits_{k=p+1}^{\infty} k(k-1)\left[\gamma k-\gamma+1\right]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m} a_k - p(p-1)\left[\gamma p-\gamma+1\right]}{p\left[\gamma p-\gamma+1\right]-\sum\limits_{k=p+1}^{\infty} k\left[\gamma k-\gamma+1\right]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m} a_k} \leq 1-\delta.$$

This implies that $f \in G_n^p(\alpha, \beta, \mu, \lambda, \gamma, \delta)$. The result (7) is sharp for the function

$$f(z) = z^p - \frac{p(p-\delta)\left[\gamma p-\gamma+1\right]}{k\left(k-\delta\right)\left[\gamma k-\gamma+1\right]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m}} z^k (k \geq p+1).$$

3. Hadamand Product

Definition 1. *Let $(f*g)(z)$ denote the Hadamand product (Convolution) of two functions*

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \ \ (a_k \geq 0,\ \ n, p \in \mathbb{N}.)$$

and

$$g(z) = z^p - \sum_{k=p+1}^{\infty} b_k z^k \ \ (b_k \geq 0,\ \ n, p \in \mathbb{N})$$

that is

$$(f*g)(z) = z^p - \sum_{k=p+1}^{\infty} a_k b_k z^k.$$

Theorem 2. *If $f(z), g(z) \in G_n^p(\alpha, \beta, \mu, \lambda, \gamma, \delta)$, then $(f*g)(z) \in G_n^p(\alpha, \beta, \mu, \lambda, \gamma, \chi)$ where*

$$\chi = p\left[1 - \frac{(p-\delta)^2\left[\gamma p-\gamma+1\right]}{(p+1)\left(p+1-\delta\right)^2\left[\gamma k-\gamma+1\right]\left[\frac{\mu+\lambda+(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m} - p(p-\delta)^2\left[\gamma p-\gamma+1\right]}\right]. \tag{8}$$

Proof. From Theorem 1, we have

$$\sum_{k=p+1}^{\infty} \frac{k\left(k-\delta\right)\left[\gamma k-\gamma+1\right]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m}}{p(p-\delta)\left[\gamma p-\gamma+1\right]} a_k \leq 1 \tag{9}$$

and

$$\sum_{k=p+1}^{\infty}\frac{k\left(k-\delta\right)\left[\gamma k-\gamma+1\right]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m}}{p(p-\delta)\left[\gamma p-\gamma+1\right]}b_k\leq 1. \tag{10}$$

We have to find the largest χ such that

$$\sum_{k=p+1}^{\infty}\frac{k\left(k-\chi\right)\left[\gamma k-\gamma+1\right]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m}}{p(p-\chi)\left[\gamma p-\gamma+1\right]}a_kb_k\leq 1. \tag{11}$$

From (10) and by means of Cauch-Schwarz inequality, we have

$$\sum_{k=p+1}^{\infty}\frac{k\left(k-\delta\right)\left[\gamma k-\gamma+1\right]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m}}{p(p-\delta)\left[\gamma p-\gamma+1\right]}\sqrt{a_kb_k}\leq 1. \tag{12}$$

Therefore (11) holds true if

$$\sqrt{a_kb_k}\leq\frac{\left(k-\delta\right)\left(p-\chi\right)}{\left(p-\delta\right)\left(k-\chi\right)} \tag{13}$$

for each $k\geq p+1,\ k,p\in\mathbb{N}$.

But (13) is satisfied if

$$\frac{p(p-\delta)\left[\gamma p-\gamma+1\right]}{k\left(k-\delta\right)\left[\gamma k-\gamma+1\right]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m}}\leq\frac{\left(k-\delta\right)\left(p-\chi\right)}{\left(p-\delta\right)\left(k-\chi\right)}$$

or

$$\chi\leq\frac{kp\left[\left(k-\delta\right)^{2}\left[\gamma k-\gamma+1\right]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m}-(p-\delta)^{2}\left[\gamma p-\gamma+1\right]\right]}{k\left(k-\delta\right)^{2}\left[\gamma k-\gamma+1\right]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m}-p(p-\delta)^{2}\left[\gamma p-\gamma+1\right]}$$

$$=p\left[1-\frac{(k-p)(p-\delta)^{2}\left[\gamma p-\gamma+1\right]}{k\left(k-\delta\right)^{2}\left[\gamma k-\gamma+1\right]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m}-p(p-\delta)^{2}\left[\gamma p-\gamma+1\right]}\right].$$

Letting

$$\Phi(k)=p\left[1-\frac{(k-p)(p-\delta)^{2}\left[\gamma p-\gamma+1\right]}{k\left(k-\delta\right)^{2}\left[\gamma k-\gamma+1\right]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m}-p(p-\delta)^{2}\left[\gamma p-\gamma+1\right]}\right]$$

we see that $\Phi(k)$ is increasing in k. This gives that

$$\chi\leq p\left[1-\frac{(p-\delta)^{2}\left[\gamma p-\gamma+1\right]}{(p+1)\left(p+1-\delta\right)^{2}\left[\gamma k-\gamma+1\right]\left[\frac{\mu+\lambda+(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m}-p(p-\delta)^{2}\left[\gamma p-\gamma+1\right]}\right].$$

4. Growth and Distortion Theorems

A growth and distortion property for function f to be in the class $G_n^p(\alpha,\beta,\mu,\lambda,\gamma,\delta)$ is contained in the following theorem.

Theorem 3. *If the function f defined by (1) is in the class $G_n^p(\alpha,\beta,\mu,\lambda,\gamma,\delta)$, then for $|z|=r<1$, we have*

$$r-\frac{|p(p-\delta)\left[\gamma p-\gamma+1\right]|}{(p+1)\,(p+1-\delta)\left[\gamma p+1\right]\left[\frac{\mu+\lambda+(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}r^2\leq|f(z)|\leq$$

$$r+\frac{|p(p-\delta)\left[\gamma p-\gamma+1\right]|}{(p+1)\,(p+1-\delta)\left[\gamma p+1\right]\left[\frac{\mu+\lambda+(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}r^2$$

and

$$1-\frac{(p+1)\,|b|}{(p+1)\,(p+1-\delta)\left[\gamma p+1\right]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}r\leq\left|f^{'}(z)\right|\leq$$

$$1+\frac{(p+1)\,|b|}{(p+1)\,(p+1-\delta)\left[\gamma p+1\right]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}r.$$

Proof. Since $f\in G_n^p(\alpha,\beta,\mu,\lambda,\gamma,\delta)$, from Theorem 1 readily yields the inequality

$$\sum_{k=p+1}^{\infty}a_k\leq\frac{|p(p-\delta)\left[\gamma p-\gamma+1\right]|}{(p+1)\,(p+1-\delta)\left[\gamma p+1\right]\left[\frac{\mu+\lambda+(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}. \tag{14}$$

Thus, for $|z|=r<1$, and making use of (14) we have

$$|f(z)|\leq|z|+\sum_{k=p+1}^{\infty}a_k\left|z^k\right|\leq r+r^2\sum_{k=p+1}^{\infty}a_k\leq r+\frac{|p(p-\delta)\left[\gamma p-\gamma+1\right]|}{(p+1)\,(p+1-\delta)\left[\gamma p+1\right]\left[\frac{\mu+\lambda+(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}r^2$$

and

$$|f(z)|\geq|z|-\sum_{k=p+1}^{\infty}a_k\left|z^k\right|\geq r-r^2\sum_{k=p+1}^{\infty}a_k\geq r-\frac{|p(p-\delta)\left[\gamma p-\gamma+1\right]|}{(p+1)\,(p+1-\delta)\left[\gamma p+1\right]\left[\frac{\mu+\lambda+(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}r^2.$$

Also from Theorem 1, it follows that

$$\frac{(p+1)\,(p+1-\delta)\left[\gamma p+1\right]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}{p+1}\sum_{k=p+1}^{\infty}ka_k\leq$$

$$\sum_{k=p+1}^{\infty}k\,(k-\delta)\left[\gamma k-\gamma+1\right]]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m a_k\leq|b|\,.$$

Hence

$$\left|f^{'}(z)\right| \leq 1+\sum_{k=p+1}^{\infty} ka_k \left|z^k\right| \leq 1+r\sum_{k=p+1}^{\infty} ka_k \leq 1+\frac{(p+1)\,|b|}{(p+1)\,(p+1-\delta)\,[\gamma p+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}r$$

and

$$\left|f^{'}(z)\right| \geq 1-\sum_{k=p+1}^{\infty} ka_k \left|z^k\right| \geq 1-r\sum_{k=p+1}^{\infty} ka_k \geq 1-\frac{(p+1)\,|b|}{(p+1)\,(p+1-\delta)\,[\gamma p+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}r.$$

This completes the proof of Theorem 3.

5. Closure Theorem

Let the functions $f_j(z)$, $j = 1, 2, \cdots, I$ be defined by

$$f_j(z) = z - \sum_{k=p+1}^{\infty} a_{k,j} z^k, \quad a_{k,j} \geq 0 \tag{15}$$

for $z \in U$.

Closure theorems for the class $G_n^p(\alpha, \beta, \mu, \lambda, \gamma, \delta)$ are given by the following:

Theorem 4. *Let the functions $f_j(z)$ defined by (15) be in the class $G_n^p(\alpha, \beta, \mu, \lambda, \gamma, \delta)$, $\alpha, \sigma \geq 0$, $\beta, \lambda, \mu > 0$, $\lambda \neq \alpha$ and $m \in \mathbb{N}_0$, for every $j = 1, 2, \cdots, I$. Then the function $G(z)$ defined by*

$$G(z) = z - \sum_{k=p+1}^{\infty} p_k z^k, \quad p_k \geq 0 \tag{16}$$

is a member of the class $G_n^p(\alpha, \beta, \mu, \lambda, \gamma, \delta)$, where

$$p_k = \frac{1}{I}\sum_{j=1}^{I} a_{k,j} \quad (k \geq 2).$$

Proof. Since $f_j(z) \in G_n^p(\alpha, \beta, \mu, \lambda, \gamma, \delta)$, it follows from Theorem 1 that

$$\sum_{k=p+1}^{\infty} k\,(k-\delta)\,[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m a_{k,j} \leq p(p-\delta)\,[\gamma p-\gamma+1]$$

for every $j = 1, 2, \cdots, I$.

Hence,

$$\sum_{k=p+1}^{\infty} k\,(k-\delta)\,[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m} p_k$$

$$=\sum_{k=p+1}^{\infty} k\,(k-\delta)\,[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m}\left\{\frac{1}{I}\sum_{j=1}^{I} a_{k,j}\right\}$$

$$=\frac{1}{I}\sum_{j=1}^{I}\left(\sum_{k=p+1}^{\infty} k\,(k-\delta)\,[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m} a_{k,j}\right)$$

$$\leq\frac{1}{I}\sum_{j=1}^{I}|p(p-\delta)\,[\gamma p-\gamma+1]| = |p(p-\delta)\,[\gamma p-\gamma+1]|$$

which implies that $G(z)\in G_n^p(\alpha,\beta,\mu,\lambda,\gamma,\delta)$.

Theorem 5. *The class $G_n^p(\alpha,\beta,\mu,\lambda,\gamma,\delta)$ is closed under convex linear combination, where $\alpha,\sigma\geq 0$, $\beta,\lambda,\mu>0$, $\lambda\neq\alpha$ and $m\in\mathbb{N}_0$.*

Proof. Suppose that the functions $f_j(z)$ $(j=1,2)$ defined by (15) are in the class $G_n^p(\alpha,\beta,\mu,\lambda,\gamma,\delta)$. It is suffices to prove that the function

$$H(z)=\varphi f_1(z)+(1-\varphi)f_2(z)\quad (0\leq\varphi\leq 1) \tag{17}$$

is also in the class $G_n^p(\alpha,\beta,\mu,\lambda,\gamma,\delta)$.

Since, for $0\leq\varphi\leq 1$,

$$H(z)=z+\sum_{k=p+1}^{\infty}\{\varphi a_{k,1}+(1-\varphi)a_{k,2}\}\,z^k,$$

we observe that

$$\sum_{k=p+1}^{\infty} k\,(k-\delta)\,[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m}\{\varphi a_{k,1}+(1-\varphi)a_{k,2}\}$$

$$=\varphi\sum_{k=p+1}^{\infty} k\,(k-\delta)\,[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m} a_{k,1}$$

$$+(1-\varphi)\sum_{k=p+1}^{\infty} k\,(k-\delta)\,[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^{m} a_{k,2}$$

$$\leq\varphi\,|b|+(1-\varphi)\,|p(p-\delta)\,[\gamma p-\gamma+1]| = |p(p-\delta)\,[\gamma p-\gamma+1]|\,.$$

Hence $H(z)\in G_n^p(\alpha,\beta,\mu,\lambda,\gamma,\delta)$. This completes the proof of Theorem 5.

6. Radii of Close-To-Convexity, Starlikeness and Convexity

A function $f(z) \in \mathcal{A}$ is said to be close-to-convex of order η if it satisfies

$$\operatorname{Re}\left\{f^{'}(z)\right\} > \eta \tag{18}$$

for some $\eta(0 \leq \eta \leq 1)$ and for all $z \in U$. Also a function $f(z) \in \mathcal{A}$ is said to be starlike of order η if it satisfies

$$\operatorname{Re}\left\{\frac{zf^{'}(z)}{f(z)}\right\} > \eta \tag{19}$$

for some $\eta(0 \leq \eta \leq 1)$ and for all $z \in U$. Further, a function $f(z) \in \mathcal{A}$ is said to be convex of order η, if and only if $zf^{'}(z)$ is starlike of order η, that is if

$$\operatorname{Re}\left\{1 + \frac{zf^{''}(z)}{f^{'}(z)}\right\} > \eta \tag{20}$$

for some $\eta(0 \leq \eta \leq 1)$ and for all $z \in U$.

Theorem 6. *If $f(z) \in G_n^p(\alpha, \beta, \mu, \lambda, \gamma, \delta)$, then $f(z)$ is close-to-convex of order η in $|z| < h_1(p, \gamma, \delta, \eta)$, where*

$$h_1(p, \gamma, \delta, \eta) = \inf_k \left\{ \frac{(p-\eta)(k-\delta)[\gamma k - \gamma + 1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}{p(p-\delta)[\gamma p - \gamma + 1]} \right\}^{\frac{1}{k}} \quad (k, p \in \mathbb{N}).$$

Proof. It is suffices to show that

$$\left|\frac{f^{'}(z)}{z^k} - p\right| \leq \sum_{k=p+1}^{\infty} k a_k |z|^k < p - \eta \tag{21}$$

and

$$\sum_{k=p+1}^{\infty} k(k-\delta)[\gamma k - \gamma + 1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m a_k \leq p(p-\delta)[\gamma p - \gamma + 1].$$

observe that (21) is true if

$$\frac{k|z|^k}{p-\eta} \leq \frac{k(k-\delta)[\gamma k - \gamma + 1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}{p(p-\delta)[\gamma p - \gamma + 1]}. \tag{22}$$

Solving (22) for $|z|$, we obtain

$$|z| \leq \left\{ \frac{(p-\eta)(k-\delta)[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}{p(p-\delta)[\gamma p-\gamma+1]} \right\}^{\frac{1}{k}}, (k,p \in \mathbb{N}).$$

Theorem 7. *If* $f(z) \in G_n^p(\alpha,\beta,\mu,\lambda,\gamma,\delta)$, *then* $f(z)$ *is starlike of order* η *in* $|z| < h_2(p,\gamma,\delta,\eta)$, *where*

$$h_2(p,\gamma,\delta,\eta) = \inf_k \left\{ \frac{k(p-\eta)(k-\delta)[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}{p(k-\eta)(p-\delta)[\gamma p-\gamma+1]} \right\}^{\frac{1}{k}} (k,p \in \mathbb{N}).$$

Proof. We must show that $\left|\frac{zf'(z)}{f(z)} - p\right| < p-\eta$ for $|z| < h_2(p,\gamma,\delta,\eta)$. Since

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq \frac{\sum_{k=p+1}^{\infty}(k-p)a_k|z|^k}{1-\sum_{k=p+1}^{\infty}a_k|z|^k}$$

if $\frac{(k-\eta)|z|^k}{p-\eta} \leq \frac{k(k-\delta)[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}{p(p-\delta)[\gamma p-\gamma+1]}$, $f(z)$ is starlike of order η.

Corollary 8. *If* $f(z) \in G_n^p(\alpha,\beta,\mu,\lambda,\gamma,\delta)$, *then* $f(z)$ *is convex of order* η *in* $|z| < h_3(p,\gamma,\delta,\eta)$,

where

$$h_3(p,\gamma,\delta,\eta) = \inf_k \left\{ \frac{(p-\eta)(k-\delta)[\gamma k-\gamma+1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}{p(k-\eta)(p-\delta)[\gamma p-\gamma+1]} \right\}^{\frac{1}{k}} (k,p \in \mathbb{N}).$$

7. Integral Operators

In this section, we consider integral transforms of functions f in the class $G_n^{p*}(\alpha,\beta,\mu,\lambda,\gamma,\delta)$.

Theorem 9. *If the function f defined by (1) is in the class $G_n^{p*}(\alpha,\beta,\mu,\lambda,\gamma,\delta)$, where $\alpha,\sigma \geq 0$, $\beta,\lambda,\mu > 0$, $\lambda \neq \alpha$, $m \in \mathbb{N}_0$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt, \quad (c > -1) \tag{23}$$

also belongs to the class $G_n^{p*}(\alpha,\beta,\mu,\lambda,\gamma,\delta)$.

Proof. From (23), it follows that $F(z) = z - \sum_{k=p+1}^{\infty} r_k z^n$, where $r_k = \left(\frac{c+1}{c+k}\right) a_k$. Therefore

$$\sum_{k=p+1}^{\infty} k\,(k-\delta)\,[\gamma k - \gamma + 1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m r_k$$

$$= \sum_{k=p+1}^{\infty} k\,(k-\delta)\,[\gamma k - \gamma + 1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m \left(\frac{c+1}{c+k}\right) a_k$$

$$\leq \sum_{k=p+1}^{\infty} k\,(k-\delta)\,[\gamma k - \gamma + 1]\left[\frac{\mu+\lambda+(k-p)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m a_k = |p(p-\delta)\,[\gamma p - \gamma + 1]|\,,$$

since $f(z) \in G_n^{p*}(\alpha,\beta,\mu,\lambda,\gamma,\delta)$. Hence by Theorem 1, $F(z) \in G_n^{p*}(\alpha,\beta,\mu,\lambda,\gamma,\delta)$.

8. Application in the Fractional Calculus

Owa [11] gave the following definitions for the fractional calculus. For other difinitions, see([12],[13],[14]).

Definition 2. *The fractional integral of order ϑ is defined by*

$$D_z^{-\vartheta} f(z) = \frac{1}{\Gamma(\vartheta)}\int_0^z \frac{f(t)}{(z-t)^{1-\vartheta}}dt$$

where $\vartheta > 0$. $f(z)$ is analytic function in a simply connected region of the $z-$plane containing the origin and multiplicity of $(z-t)^{\vartheta-1}$ is remove by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Theorem 10. *Let the function $f(z)$ be in the class $G_n(\mu, \delta, b)$. Then*

$$\left|D_z^{-\vartheta} f(z)\right| \leq \frac{p!}{\Gamma(p+\vartheta+1)} |z|^{p+\vartheta} \tag{24}$$

$$\left[1 + \frac{p(p-\delta)[\gamma p - \gamma + 1]\Gamma(p+2)\Gamma(p+\vartheta+1)}{\Gamma(p+1)\Gamma(p+\vartheta+2)(p+1)(p+1-\delta)[\gamma p+1]\left[\frac{\mu+\lambda+(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m} |z|^k\right] \tag{25}$$

and

$$\left|D_z^{-\vartheta} f(z)\right| \geq \frac{p!}{\Gamma(p+\vartheta+1)} |z|^{p+\vartheta} \tag{26}$$

$$\left[1 - \frac{p(p-\delta)[\gamma p - \gamma + 1]\Gamma(p+2)\Gamma(p+\vartheta+1)}{\Gamma(p+1)\Gamma(p+\vartheta+2)(p+1)(p+1-\delta)[\gamma p+1]\left[\frac{\mu+\lambda+(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m} |z|^k\right]. \tag{27}$$

Proof. Using Theorem 1 we have

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{p(p-\delta)[\gamma p - \gamma + 1]}{(p+1)(p+1-\delta)[\gamma p+1]\left[\frac{\mu+\lambda+(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}. \tag{28}$$

From Definition 2 we get

$$\frac{D_z^{-\vartheta} f(z)\Gamma(p+\vartheta+1)}{p!} = z^p - \sum_{k=p+1}^{\infty} \frac{k!\Gamma(p+\vartheta+1)}{p!\Gamma(k+\vartheta+1)} a_k z^k$$

and

$$\frac{D_z^{-\vartheta} f(z)\Gamma(p+\vartheta+1)}{p!} = z^p - \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)\Gamma(p+\vartheta+1)}{\Gamma(p+1)\Gamma(k+\vartheta+1)} a_k z^k \tag{29}$$

$$= z^p - \sum_{k=p+1}^{\infty} \Psi(k) a_k z^k \tag{30}$$

where $\Psi(k) = \frac{\Gamma(k+1)\Gamma(p+\vartheta+1)}{\Gamma(p+1)\Gamma(k+\vartheta+1)}$.

We know that $\Psi(k)$ is a decreasing function of k and $0 < \Psi(k) \leq \Psi(p+1) = \frac{\Gamma(p+2)\Gamma(p+\vartheta+1)}{\Gamma(p+1)\Gamma(p+\vartheta+2)}$. Using (28) and (29) we have

$$\left|\frac{\Gamma(p+\vartheta+1)z^{-\vartheta} D_z^{-\vartheta} f(z)}{p!}\right| \leq |z|^p + \Psi(p+1)|z|^{p+1} \sum_{k=p+1}^{\infty} a_k \leq$$

$$|z|^p + \frac{p(p-\delta)\left[\gamma p-\gamma+1\right]\Gamma(p+2)\Gamma(p+\vartheta+1)}{\Gamma(p+1)\Gamma(p+\vartheta+2)\,(p+1)\,(p+1-\delta)\left[\gamma p+1\right]\left[\frac{\mu+\lambda+(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}\,|z|^k\,,$$

which gives (2). We also have

$$\left|\frac{\Gamma(p+\vartheta+1)z^{-\vartheta}D_z^{-\vartheta}f(z)}{p!}\right| \geq |z|^p - \Psi(p+1)\,|z|^{p+k}\sum_{k=p+1}^{\infty} a_k \geq$$

$$|z|^p - \frac{p(p-\delta)\left[\gamma p-\gamma+1\right]\Gamma(p+2)\Gamma(p+\vartheta+1)}{\Gamma(p+1)\Gamma(p+\vartheta+2)\,(p+1)\,(p+1-\delta)\left[\gamma p+1\right]\left[\frac{\mu+\lambda+(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right]^m}\,|z|^k$$

which gives (10).

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